Math 202

6.2

3) $(x^2-16)^2 y'' + (x+4)y' + 2y = 0$

Write in standard form

$$y'' + \frac{(x+4)}{(x+4)^2(x-4)^2} y' + \frac{2}{(x+4)^2(x-4)^2} y = 0$$

$x = 4, -4$ are singular points

For $x = -4$:

$(x+4)P(x) = \frac{1}{(x-4)^2}$ is analytic at $x=-4$

$(x+4)^2 Q(x) = \frac{2}{(x-4)^2}$, analytic at $x=-4$

$x = -4$ is a regular singular point

Now for $x = 4$:

$$(x-4)P(x) = \frac{(x+4)(x-4)}{(x+4)^2(x-4)^2} = \frac{1}{(x+4)(x-4)}, \text{ not analytic at } x=+4$$

Thus $x = 4$ is an irregular singular point

10) Using factors, we write DE as

$$y'' + \frac{x(x-3)^2}{x^2(x^2-2x+3)^2} y' - \frac{(x+1)}{x^2(x^2-2x+3)^2} y = 0$$

$P(x) = \frac{(x-3)^2}{x(x^2-2x+3)^2}$, $x=0$ is singular point

$Q(x) = \frac{-(x+1)}{x^2(x^2-2x+3)^2}$

$x P(x) = \frac{(x-3)^2}{(x^2-2x+3)^2}$, analytic at $x=0$

$x^2 Q(x) = \frac{-(x+1)}{(x^2-2x+3)^2}$, analytic at $x=0$

$x=0$ is a regular singular point

19) $3xy'' + (2-x)y' - y = 0$

$x=0$ is a regular singular point

Assume $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

Put in D.E.

$$3x \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} + (2-x)\sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 3a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0} 2a_n (n+r) x^{n+r-1}$$

$$- \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Change index in last two Σ's.

$$\sum_{n=0}^{\infty} [3a_n (n+r)(n+r-1) + 2a_n (n+r)] x^{n+r-1}$$

$$- \sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$\Rightarrow (3a_0 r(r-1) + 2a_0 r) x^{r-1}$$

$$+ \sum_{n=1}^{\infty} \{[3a_n (n+r)(n+r-1) + 2a_n(n+r)] - a_{n-1}[(n+r-1)+1]\} x^{n+r-1} = 0$$

coeff $x^{r-1} \Rightarrow 3a_0 r(r-1) + 2a_0 r = 0$

$a_0 \neq 0 \Rightarrow \boxed{r = \frac{1}{3}, 0}$

coeff $x^{n+r-1} \Rightarrow$

$$(n+r)(3n+3r-1)a_n = (n+r)a_{n-1}$$

$$\boxed{a_n = \frac{a_{n-1}}{3n+3r-1}}, \quad n = 1, 2, \ldots$$

$r = 0$	$r = \frac{1}{3}$
$a_n = \frac{a_{n-1}}{3n-1}$	$a_n = \frac{a_{n-1}}{3n}$
$a_1, a_2, a_3, \ldots$ in terms of a_0	$a_1, a_2, a_3, \ldots$ in terms of a_0

y_1 and y_2 can be written from here.

Math 202
6.2 Contd

20) $x^2y'' - (x - \frac{2}{9})y = 0$

$x=0$ is a regular singular point.

Assume $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

Put in D.E (and simplify)

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} + \sum_{n=0}^{\infty} \frac{2}{9} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} a_n \left((n+r)(n+r-1) + \frac{2}{9}\right) x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0$$

or $a_0\left[r(r-1) + \frac{2}{9}\right] x^r + \sum_{n=1}^{\infty} \left[a_n\left\{(n+r)(n+r-1) + \frac{2}{9}\right\} - a_{n-1}\right] x^{n+r} = 0$

coeff x^r gives $\Rightarrow a_0\left[r(r-1) + \frac{2}{9}\right] = 0$

$a_0 \neq 0 \Rightarrow r^2 - r + \frac{2}{9} = 0 \Rightarrow (r - \frac{1}{3})(r - \frac{2}{3}) = 0$

$\Rightarrow \boxed{r_1 = \frac{2}{3}, \quad r_2 = \frac{1}{3}}$, $r_1 - r_2 \neq$ integer

Coeff $x^{n+r} = 0 \Rightarrow$

$$a_n = \frac{a_{n-1}}{\left[(n+r)(n+r-1) + \frac{2}{9}\right]}, \quad n = 1, 2, \ldots$$

$r = \frac{2}{3}$: $a_n = \dfrac{a_{n-1}}{(n+\frac{2}{3})(n-\frac{1}{3}) + \frac{2}{9}}$

$$a_n = \frac{a_{n-1}}{n(n+\frac{1}{3})}, \quad n = 1, 2, 3 \ldots$$

$a_1 = \dfrac{a_0}{4/3}$

$a_2 = \dfrac{a_1}{2(\frac{7}{3})} = \frac{3}{4} \cdot \frac{3}{14} a_0 = \frac{9}{56} a_0$

$a_3 = \dfrac{a_2}{10} = \frac{9}{560} a_0$

$y = a_0 \left(x^{2/3} + \frac{3}{4} x^{2/3+1} + \frac{9}{56} x^{2/3+2} + \cdots \right)$

$r = 1/3$: $a_n = \dfrac{a_{n-1}}{(n+\frac{1}{3})(n-\frac{2}{3}) + 9}$

or $a_n = \dfrac{a_{n-1}}{n^2 - \frac{1}{3}n} = \dfrac{a_{n-1}}{n(n-\frac{1}{3})}$

$a_1 = \dfrac{a_0}{1(\frac{2}{3})} = \frac{3}{2} a_0$

$a_2 = \dfrac{a_1}{2(\frac{5}{3})} = \frac{3}{10} a_1 = \frac{9}{20} a_0$

$a_3 = \dfrac{a_2}{3(\frac{8}{3})} = \dfrac{a_2}{8} = \frac{9}{160} a_0$

$\ldots$

$y_2 = a_0 \left(x^{1/3} + \frac{3}{2} x^{1/3+1} + \frac{9}{20} x^{1/3+2} + \cdots \right)$

27) (Indicial roots only)

$xy'' - xy' + y = 0$

Assume $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$

$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

DE $\Rightarrow \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$

or $\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} - \sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$

or $a_0 r(r-1) x^{r-1} + \sum_{n=1}^{\infty} \left\{ a_n (n+r)(n+r-1) - a_{n-1}[(n+r-1) - 1] \right\} x^{n+r-1} = 0$

coeff $x^{r-1} \Rightarrow a_0 r(r-1) = 0$

$a_0 \neq 0$ gives $r = 0, 1$

Coeff x^{n+r-1} gives

$$a_n = \frac{a_{n-1}(n+r-2)}{(n+r)(n+r-1)}$$

This can be used to find y_1.