Key

King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 260
Final Exam, Semester II, 2008-2009

Duration: 180 minutes

Name:____________________________________
ID:____________________________________
Section:____________________________________

Answer the questions in the space provided. You must show your work or explain your solution otherwise points may be deducted. If you make an unnecessary approximation in your solution to a problem, your answer will be judged on its accuracy. Points may be deducted for poor or inappropriate approximation .**Write clearly**. **Show all your steps**. No credits will be given to wrong steps. **Calculators and mobile phones are NOT allowed in this exam**.

Q#	Marks	Maximum Marks
1		4
2		4
3		4
4		4
5		4
6		4
7		12
Total		36

Problem1.
Solve the homogeneous equation

$$xy' = y(\ln x - \ln y)$$

$$y' = \left(\frac{y}{x}\right)\ln\left(\frac{x}{y}\right) \iff y' = -\frac{y}{x}\ln\left(\frac{y}{x}\right).$$

Put $u = \frac{y}{x}$, $y = xu$. $y' = u + xu' \iff$

$$-u\ln u = u + xu' \iff x\frac{du}{dx} = -u\left[\ln u + 1\right] \iff$$

2 $$\boxed{-\frac{du}{u(\ln u+1)} = \frac{dx}{x}} \implies -\int \frac{du}{u(\ln u+1)} = \int\frac{dx}{x}$$

$$\iff -\int \frac{1}{v}\,dv = \int \frac{dx}{x} \quad \text{where } v = \ln u + 1$$

Put $K = \ln c \quad (c>0)$

$$\implies \boxed{-\ln|v|} = \boxed{\ln|x| + K}.$$

1 1

We have

$$-\ln|v| = \ln(c|x|) \iff |v| = \frac{1}{c|x|} \implies$$

$$v = \pm\frac{1}{c}\cdot\frac{1}{|x|}. \quad \text{Put } A = \pm\frac{1}{c}. \text{ We have}$$

$$v = \frac{A}{|x|} \iff \ln\left(\frac{y}{x}\right) + 1 = \frac{A}{|x|} \iff$$

$$\frac{y}{x} = e^{A/|x| - 1} \quad \text{and}$$

$$y = x e^{A/|x| - 1}$$

Problem 3. Solve the following problems

(1) Let λ be an eigenvalue of the $n \times n$ matrix A. Prove that λ^k is an eigenvalue of A^k. (k is positive an integer)

Let x be an eigenvector associated with λ. We have

$AX = \lambda X \implies A^2X = \lambda AX = \lambda^2 X$

Similarly $A^3x = \lambda^3 x$, ..., and so on

$A^k x = \lambda^k x$.

(2) Use Cayley-Hamilton theorem to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

The characteristic polynomial is

$p(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 5 \end{vmatrix}$

$= (\lambda - 1)(\lambda - 5) - 6 = \lambda^2 - 6\lambda - 1$

Using Cayley-Hamilton theorem, we have

$A^2 - 6A + I = 0 \iff$

$A(A - 6I) = I$ and $(A - 6I)A = I \implies$

$A^{-1} = A - 6I = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$

Problem 4. Solve the differential equation by the method of variation of parameters.

$$y'' - 2y' + y = \frac{e^t}{t^2+1}$$

The Associated homogeneous equation is

$y'' - 2y' + y = 0$. $y_H = c_1 e^t + c_2 t e^t$

let $y_1 = e^t$ and $y_2 = te^t$

$y_p = u_1 y_1 + u_2 y_2$ where

$u_1 = \int \frac{W_1}{W} dt$, $u_2 = \int \frac{W_2}{W} dt$

$$W_1 = \begin{vmatrix} 0 & te^t \\ \frac{e^t}{t^2+1} & e^t + te^t \end{vmatrix} = \frac{-te^{2t}}{1+t^2}$$

$$W_2 = \begin{vmatrix} e^t & 0 \\ e^t & \frac{e^t}{1+t^2} \end{vmatrix} = \frac{e^{2t}}{1+t^2}$$

$$W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t}$$

$$y_p = -e^t \int \frac{te^{2t}}{e^{2t}(1+t^2)} dt + te^t \int \frac{e^{2t}}{e^{2t}(1+t^2)} d$$

$$= -\frac{1}{2} e^t \ln(1+t^2) + te^t \tan^{-1}(t)$$

The General Solution is

Problem 5
Solve the system

$$y_1' = y_1 - 2y_2 + 2y_3$$
$$y_2' = -2y_1 + y_2 - 2y_3$$
$$y_3' = 2y_1 - 2y_2 + y_3$$

$$Y' = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} Y$$

The characteristic Equation is

$$-(\lambda+1)^2(\lambda-5) = 0$$

The eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -1$, a

$\lambda_3 = 5$

$$E_{-1} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$E_5 = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

The three linearly independent solutions ar

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-t}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{-t}, \quad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{5t}$$

The general solution is

$$Y = c_1 \begin{bmatrix} e^{-t} \\ e^{-t} \\ \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{-t} \\ e^{-t} \end{bmatrix} + c_3 \begin{bmatrix} e^{5t} \\ -e^{5t} \\ e^{5t} \end{bmatrix}.$$

Problem6 .

Determine whether the following statements are **True** or **False.**

1. If g is a solution to a homogeneous linear equation, then for any constant k, kg is also a solution to the equation. (T / F)

2. The matrix $A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$ is diagonalizable (T / F)

3. The order of the differential equation $y^4 \dfrac{d^2y}{dt^2} + t^6 \dfrac{dy}{dt} + y = e^t$ is six (T / F)

4. The differential equation $\dfrac{dy}{dt} + ty = e^{t^2}$ is nonlinear in y (T / F)

5. Let W be the set of all continuous functions f such that $f(6) = 10$. W is a subspace of the set of all continuous functions on $(-\infty, \infty)$. (T / F)

6. If $y_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^t$, $y_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^t$, and $y_3 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} e^t$

are solutions of a third order homogeneous linear differential then the general solution is $y = c_1 y_1 + c_2 y_2 + c_3 y_3$ for some constants c_1, c_2, and c_3. (T / F)

7. $\left\{ \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$. (T / F)

8. The initial value problem $\begin{cases} y'' - \dfrac{y}{x} = 2 \\ y(2) = 1, \quad y'(2) = 1 \end{cases}$

has a unique solution on the interval $(1,3)$. (T / F)

(4) The system $\begin{array}{l} y_1'' = y_1 + y_2 + e^t \\ y_2'' = y_1 + y_1' + y_2 + y_2' + e^t \end{array}$ can be converted to

(a) $$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} 0 \\ e^t \\ 0 \\ e^t \end{bmatrix}$$

(b) $$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

(c) $$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} 0 \\ e^t \\ 0 \\ e^t \end{bmatrix}$$

(d) $$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

(5) Determine the general solution to $Y' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} Y$

(a) $$\begin{bmatrix} c_1 e^x(\cos 2x + \sin 2x) + c_2 e^x(\cos 2x - \sin 2x) \\ 2c_1 e^x \cos 2x + 2c_2 e^x \sin 2x \end{bmatrix}$$

(b) $$\begin{bmatrix} c_1 e^x + c_2 e^x(\cos 2x - \sin 2x) \\ 2c_1 e^x \cos 2x + 2c_2 e^x \sin 2x \end{bmatrix}$$

(c) $$\begin{bmatrix} c_1 e^x(\cos 2x + \sin 2x) \\ 2c_2 e^x \sin 2x \end{bmatrix}$$

(d) $$\begin{bmatrix} c_1 e^x + c_2 e^x \cos 2x \\ 2c_2 e^x \sin 2x \end{bmatrix}$$

(6) Show that the given differential equation is exact and solve the initial value problem.

$$2x + y^2 + 2xyy' = 0, \qquad y(2) = \sqrt{\frac{5}{2}}.$$

Find $y(3)$

(a) $y(3) = -2$
(b) $y(3) = -1$
(c) $y(3) = 0$
(d) $y(3) = 1$
(e) $y(3) = 2$

(7) The eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{bmatrix}$$

are $\lambda_1 = 3$, $\lambda_2 = 2$, and $\lambda_3 = 1$, and the corresponding eigenvectors are respectively

$$v_1 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Find A^k. The product of the diagonal entries is

(a) 2^k
(b) 5^k
(c) 3^k
(d) 1
(e) 6^k

Problem 7.

(1) Consider the set $S = \left\{ \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \right\}$ of vectors in $\mathcal{M}_{2,2}$ (the set of all real 2×2 matrices). Then

(a) S is a basis for $\mathcal{M}_{2,2}$

(b) S is linearly independent

(c) S spans $\mathcal{M}_{2,2}$

(d) S is not linearly independent, and S does not span $\mathcal{M}_{2,2}$.

(2) Give the general solution of the homogeneous linear differential equation whose characteristic equation has the following roots:

$$0, 4, 4, 4, \pm 2i, \pm 2i$$

(a) $y = Ae^{4x} + (Dx + E)\cos 2x + (Fx + G)\sin 2x$

(b) $y = (Ax^2 + Bx + C)e^{4x} + (Dx + E)\cos 2x + (Fx + G)\sin 2x$

(c) $y = (Ax^2 + Bx + C)e^{4x} + Dx\cos 2x + E\sin 2x + F$

(d) $y = (Ax^2 + Bx + C)e^{4x} + (Dx + E)\cos 2x + (Fx + G)\sin 2x + K$

(e) None of these

(3) Determine a form for the particular solution y_p of the given differential equation.

$$y''' + y'' + 9y' + 9y = 4e^{-x} + 3x\sin 3x - 6\cos 3x$$

(a) $y_p = Axe^{-x} + (Bx^2 + Cx)\sin 3x + (Dx^2 + Ex)\cos 3x$

(b) $y_p = Axe^{-x} + (Bx + C)\sin 3x + (Dx + E)\cos 3x$

(c) $y_p = Ae^{-x} + (Bx^2 + C)\sin 3x + (Dx^2 + E)\cos 3x$

(d) $y_p = A + (Bx^2 + Cx)e^{-x}\sin 3x + (Dx^2 + Ex)e^{-x}\cos 3x$

(e) None of these

Problem2. Find the general solution of the Bernoulli equation

$$\frac{dy}{dx} + y = xy^3$$

$n = 3$, $\quad y^{-3}\dfrac{dy}{dx} + y^{-2} = x$. $\quad$ Put $\quad v = y^{1-n} = y^{-2}$

then $\quad \dfrac{dv}{dx} = -2y^{-3}\left(\dfrac{dy}{dx}\right) \Rightarrow$

$$-\frac{1}{2}\frac{dv}{dx} + v = x \iff \boxed{\frac{dv}{dx} - 2v = -2x}$$

An integrating factor is

$$e^{\int P(x)\,dx} = \boxed{e^{-2x}} \Rightarrow$$

$$\frac{d}{dx}\left[e^{-2x}v\right] = -2xe^{-2x}.$$

Integrating, we find that

$$e^{-2x}v = \frac{1}{2}e^{-2x}(2x+1) + C$$

$$\boxed{v = x + \frac{1}{2} + Ce^{2x}}$$

But $\quad v = \dfrac{1}{y^2} \Rightarrow$

$$\boxed{y^2 = \frac{1}{x + \frac{1}{2} + Ce^{2x}}}$$