# 19/248) Find two power series solutions of the given differential equation about the point $x=0$

$$\boxed{y'' - 2xy' + y = 0} \quad \text{——— (1)}$$

$$y = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \quad \text{—(2)}$$

Use (2) in (1):

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - 2x \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

First term $x^0$     First term $x^1$     First term $x^0$

$$\overbrace{2c_2 + \sum_{n=3}^{\infty} n(n-1) c_n x^{n-2}} - \sum_{n=1}^{\infty} 2n c_n x^n + \overbrace{c_0 + \sum_{n=1}^{\infty} c_n x^n} = 0$$

$$(2c_2 + c_0) + \sum_{n=3}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=1}^{\infty} c_n x^n = 0$$

($k = n-2$, $k+2 = n$)     ($k = n$)     ($k = n$)

$$(2c_2 + c_0) + \sum_{k=1}^{\infty} (k+2)(k+1) C_{k+2} x^k - \sum_{k=1}^{\infty} 2k C_k x^k + \sum_{k=1}^{\infty} C_k x^k = 0$$

Observe that the three series on the left-hand side start with the same power of $x$ and the same index.

$$(2c_2 + c_0) + \sum_{k=1}^{\infty} \left[ (k+1)(k+2) C_{k+2} + (1-2k) C_k \right] x^k = 0$$

By identity property, the coefficient of each power of $x$ equals zeros. That is,

$$2c_2 + c_0 = 0$$

$$(k+1)(k+2)\, c_{k+2} + (1-2k)\, c_k = 0 \qquad k = 1, 2, 3, \ldots$$

[recurrence relation]

Since, $(k+1)(k+2) \neq 0$ for all values of $k$ $(k = 1, 2, 3, \ldots)$

$$\boxed{\begin{aligned} c_2 &= -\tfrac{1}{2}\, c_0 \\ c_{k+2} &= \frac{2k-1}{(k+1)(k+2)}\, c_k \end{aligned}} \qquad k = 1, 2, 3, 4, \ldots$$

index diff $= (k+2) - k = 2$. We need 2 Columns.

$$c_2 = -\tfrac{1}{2}\, c_0$$

$$c_4 = \frac{3}{3 \cdot 4}\, c_2$$

$$c_6 = \frac{7}{5 \cdot 6}\, c_4$$

$$\vdots$$

$$c_{2m-2} = \frac{4m-9}{(2m-3)(2m-2)}\, c_{2m-4}$$

$$c_{2m} = \frac{4m-5}{(2m-1)(2m)}\, c_{2m-2}$$

$$c_{2m} = \frac{1 \cdot 3 \cdot 7 \cdots (4m-9)(4m-5)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots 2m}\, c_0$$

$$\boxed{c_{2m} = \frac{-1 \cdot 3 \cdot 7 \cdots (4m-9)(4m-5)}{(2m)!}\, c_0} \quad (*)$$

$m = 1, 2, 3, \ldots$

$$c_3 = \frac{1}{2 \cdot 3}\, c_1$$

$$c_5 = \frac{5}{4 \cdot 5}\, c_3$$

$$c_7 = \frac{9}{6 \cdot 7}\, c_5$$

$$\vdots$$

$$c_{2m+1} = \frac{4m-3}{(2m)(2m+1)}\, c_{2m-1}$$

$$c_{2m+1} = \frac{1 \cdot 5 \cdot 9 \cdots (4m-3)}{2 \cdot 3 \cdot 4 \cdots (2m)(2m+1)}\, c_1$$

$$\boxed{c_{2m+1} = \frac{1 \cdot 5 \cdot 9 \cdots (4m-3)}{(2m+1)!}\, c_1} \quad (**)$$

$m = 1, 2, 3, \ldots$

Solution Can be expressed as

$$y = \sum_{n=0}^{\infty} c_n x^n = \sum_{m=0}^{\infty} C_{2m} x^{2m} + \sum_{m=0}^{\infty} C_{2m+1} x^{2m+1}$$

$$= c_0 + \sum_{m=1}^{\infty} C_{2m} x^{2m} + c_1 x + \sum_{m=1}^{\infty} C_{2m+1} x^{2m+1} \quad \text{—— (2)}$$

Use (*) and (**) in (2):

Since, the DE has no singular points $\Longrightarrow R = +\infty$

$$y = C_0 - \sum_{m=1}^{\infty} \frac{1 \cdot 3 \cdot 7 \cdots (4m-5)}{(2m)!} C_0 x^{2m}$$

$$+ C_1 x + \sum_{m=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdots (4m-3)}{(2m+1)!} C_1 x^{2m+1}$$

$$= C_0 \left[ 1 - \sum_{m=1}^{\infty} \frac{1 \cdot 3 \cdot 7 \cdots (4m-5)}{(2m)!} x^{2m} \right]$$

$$+ C_1 \left[ x + \sum_{m=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdots (4m-3)}{(2m+1)!} x^{2m+1} \right]$$

$$= C_0 \, y_1(x) + C_1 \, y_2(x)$$

Hence, the two linearly Independent Solutions are:

$$y_1(x) = 1 - \sum_{m=1}^{\infty} \frac{1 \cdot 3 \cdot 7 \cdots (4m-5)}{(2m)!} x^{2m} \qquad |x| < \infty$$

$$y_2(x) = x + \sum_{m=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdots (4m-3)}{(2m+1)!} x^{2m+1} \qquad |x| < \infty$$

Solution $y'' - 2xy' + y = 0$

* $y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \cdots$

$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + 6c_6 x^5 + \cdots$

$y'' = 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + 30c_6 x^4 + \cdots$

* $-2xy' = -2c_1 x - 4c_2 x^2 - 6c_3 x^3 - 8c_4 x^4 - \cdots$

$y'' - 2xy' + y = 0$

$$(2c_2 + c_0) + (6c_3 - c_1)x + (12c_4 - 3c_2)x^2 + (20c_5 - 5c_3)x^3$$
$$+ (30c_6 - 7c_4)x^4 + \cdots = 0$$

By identity property:

$2c_2 + c_0 = 0 \Rightarrow c_2 = -\frac{1}{2}c_0$

$6c_3 - c_1 = 0 \Rightarrow c_3 = \frac{1}{6}c_1$

$12c_4 - 3c_2 = 0 \Rightarrow c_4 = \frac{1}{4}c_2 = \frac{1}{4}\left(-\frac{1}{2}c_0\right) = -\frac{1}{8}c_0$

$20c_5 - 5c_3 = 0 \Rightarrow c_5 = \frac{1}{4}c_3 = \frac{1}{4}\left(\frac{1}{6}c_1\right) = \frac{1}{24}c_1$

$30c_6 - 7c_4 = 0 \Rightarrow c_6 = \frac{7}{30}c_4 = \frac{7}{30}\left(-\frac{1}{8}c_0\right) = -\frac{7}{240}c_0$

Now, $y = (c_0 + c_2x^2 + c_4x^4 + c_6x^6 + \cdots) + (c_1x + c_3x^3 + c_5x^5 + \cdots)$

$$y = c_0\left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6 + \cdots\right) + c_1\left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \cdots\right)$$

Hence, the two linearly indep. solutions are:

$y_1(x) = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6 + \cdots$, $y_2(x) = x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \cdots$

$|x| < \infty$ $|x| < \infty$