



Second solution using see 4.2

Consider the DE $xy'' + y = 0$, using Frobenius Theorem we found the first solution

$$y_1(x) = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \dots$$

We can construct the second solution using see 4.2

$$y_2(x) = y_1(x) \int \frac{k(x)}{[y_1(x)]^2} dx \text{ where } k(x) = e^{-\int p(x) dx}$$

$$xy'' + y = 0 \Rightarrow y'' + \frac{1}{x}y = 0 \Rightarrow p(x) = 0 \Rightarrow k(x) = 1$$

Now,
$$y_2(x) = y_1(x) \int \frac{dx}{[y_1(x)]^2}$$

Step 1 $[y_1(x)]^2 = (x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \dots)(x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \dots)$

~~...~~ $= (0) + (0)x + (1)x^2 + (-\frac{1}{2} - \frac{1}{2})x^3 + (\frac{1}{12} + \frac{1}{4} + \frac{1}{12})x^4 + (-\frac{1}{144} - \frac{1}{24} - \frac{1}{24} - \frac{1}{144})x^5 + \dots$

$$y_1^2 = x^2 - x^3 + \frac{5}{12}x^4 - \frac{7}{72}x^5 + \dots$$

Step 2: Find $1/y_1^2$ (long division)

$$\begin{array}{r}
\frac{1}{x^2} + \frac{1}{x} + \frac{7}{12} + \frac{19}{12}x + \dots \\
\hline
x^2 - x^3 + \frac{5}{12}x^4 - \frac{7}{72}x^5 + \dots \quad | \quad 1 \\
\underline{+1 \quad -x \quad + \frac{5}{12}x^2 - \frac{7}{72}x^3 + \dots} \\
x - \frac{5}{12}x^2 + \frac{7}{72}x^3 + \dots \\
\underline{+7x \quad -x^2 \quad + \frac{5}{12}x^3 + \dots} \\
\frac{7}{12}x^2 - \frac{23}{72}x^3 + \dots \\
\underline{+ \frac{7}{12}x^2 \quad - \frac{7}{12}x^3 + \dots} \\
\frac{19}{12}x^3 + \dots \\
\underline{+ \frac{19}{12}x^3 + \dots} \\
0 + \dots
\end{array}$$

$$\text{So } \frac{1}{[y_1(x)]^2} = x^{-2} + x^{-1} + \frac{7}{12} + \frac{19}{12}x + \dots$$

Step 3:

$$\int \frac{dx}{y_1^2} = \int (x^{-2} + x^{-1} + \frac{7}{12} + \frac{19}{12}x + \dots) dx$$

$$= -x^{-1} + \ln x + \frac{7}{12}x + \frac{19}{24}x^2 + \dots$$

Step 4:

$$y_2 = y_1 \int \frac{dx}{y_1^2} = y_1 \left(-x^{-1} + \ln x + \frac{7}{12}x + \frac{19}{24}x^2 + \dots \right)$$

$$= y_1 \ln x + y_1 \left(-x^{-1} + \frac{7}{12}x + \frac{19}{24}x^2 + \dots \right)$$

$$= y_1 \ln x + \left(x - \frac{1}{2}x^2 + \frac{1}{12}x^3 + \frac{1}{144}x^4 + \dots \right) \left(-x^{-1} + \frac{7}{12}x + \frac{19}{24}x^2 + \dots \right)$$

$$= y_1 \ln x + (-1) + \left(-\frac{1}{2}\right)x + \left(\frac{7}{12} - \frac{1}{12}\right)x^2 + \left(\frac{19}{24} - \frac{7}{24} - \frac{1}{144}\right)x^3 + \dots$$

$$= y_1 \ln x + \left(-1 - \frac{1}{2}x + \frac{1}{2}x^2 - \frac{71}{144}x^3 + \dots \right)$$

$$y_2(x) = C y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^n \quad \text{Note: } r_2 = 0$$

$$\text{Hence, } C = 1, \quad b_0 = -1, \quad b_1 = -\frac{1}{2}, \quad b_2 = \frac{1}{2}, \quad b_3 = -\frac{71}{144}$$