Second solution using sec 4.2

Consider the DE $xy'' + y = 0$. using Frobenius Theorem we found the first solution

$$y_1(x) = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots$$

We can construct the second solution using sec 4.2

$$y_2(x) = y_1(x)\int \frac{K(x)}{[y_1(x)]^2}\,dx \quad \text{where } K(x) = e^{-\int P(x)dx}$$

$$xy'' + y = 0 \Rightarrow y'' + \frac{1}{x}y = 0 \Rightarrow P(x) = 0 \Rightarrow K(x) = 1$$

Now,

$$\boxed{y_2(x) = y_1(x)\int \frac{dx}{[y_1(x)]^2}}$$

Step1 $[y_1(x)]^2 = (x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots)(x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots)$

$$= (0) + (0)x + (1)x^2 + (-\tfrac{1}{2} - \tfrac{1}{2})x^3 + (\tfrac{1}{12} + \tfrac{1}{4} + \tfrac{1}{12})x^4$$

$$+ (-\tfrac{1}{144} - \tfrac{1}{24} - \tfrac{1}{24} - \tfrac{1}{144})x^5 + \cdots$$

$$y_1^2 = x^2 - x^3 + \frac{5}{12}x^4 - \frac{7}{72}x^5 + \cdots$$

Step2: Find $1/y_1^2$ (long division)

$$\frac{1}{x^2} + \frac{1}{x} + \frac{7}{12} + \frac{19}{12}x + \cdots$$

$$x^2 - x^3 + \frac{5}{12}x^4 - \frac{7}{72}x^5 + \cdots \;\big)\; 1$$

$$\mp 1 \pm x \mp \frac{5}{12}x^2 \pm \frac{7}{72}x^3 \mp \cdots$$

$$x - \frac{5}{12}x^2 + \frac{7}{72}x^3 + \cdots$$

$$\mp x \pm x^2 \mp \frac{5}{12}x^3 + \cdots$$

$$\frac{7}{12}x^2 - \frac{23}{72}x^3 + \cdots$$

$$\mp \frac{7}{12}x^2 \pm \frac{7}{12}x^3 + \cdots$$

$$\frac{19}{12}x^3 + \cdots$$

$$\mp \frac{19}{12}x^3 + \cdots$$

$$0 + \cdots$$

So $\frac{1}{[y_1(x)]^2} = x^{-2} + x^{-1} + \frac{7}{12} + \frac{19}{12}x + \cdots$

Step 3:

$$\int \frac{dx}{y_1^2} = \int \left(x^{-2} + x^{-1} + \frac{7}{12} + \frac{19}{12}x + \cdots\right) dx$$

$$= -x^{-1} + \ln x + \frac{7}{12}x + \frac{19}{24}x^2 + \cdots$$

Step 4:

$$y_2 = y_1 \int \frac{dx}{y_1^2} = y_1 \left(-x^{-1} + \ln x + \frac{7}{12}x + \frac{19}{24}x^2 + \cdots\right)$$

$$= y_1 \ln x + y_1\left(-x^{-1} + \frac{7}{12}x + \frac{19}{24}x^2 + \cdots\right)$$

$$= y_1 \ln x + \left(x - \frac{1}{2}x^2 + \frac{1}{12}x^3 + \frac{1}{144}x^4 + \cdots\right)\left(-x^{-1} + \frac{7}{12}x + \frac{19}{24}x^2 + \cdots\right)$$

$$= y_1 \ln x + (-1) + \left(-\frac{1}{2}\right)x + \left(\frac{7}{12} - \frac{1}{12}\right)x^2 + \left(\frac{19}{24} - \frac{7}{24} - \frac{1}{144}\right)x^3 + \cdots$$

$$= y_1 \ln x + \left(-1 - \frac{1}{2}x + \frac{1}{2}x^2 - \frac{71}{144}x^3 + \cdots\right)$$

$$y_2(x) = C y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^n$$

Note: $r_2 = 0$

Hence, $C = 1$, $b_0 = -1$, $b_1 = -\frac{1}{2}$, $b_2 = \frac{1}{2}$, $b_3 = -\frac{71}{144}$