

$$4. \quad x dx + (y - 2x) dy = 0$$

D.E. is homogeneous. Put

$$y = ux \Rightarrow dy = u dx + x du$$

$$\text{so D.E.} \Rightarrow x dx + (ux - 2x)(u dx + x du) = 0$$

$$\Rightarrow x(1 - u^2 + 2u) dx = x^2(u+2) du$$

$$\text{or } \frac{x}{x^2} dx = \frac{u+2}{1-u^2+2u} du$$

$$\text{or } \frac{1}{x} dx = \frac{u-1+3}{(u-1)^2} du$$

$$= \left[ \frac{1}{u-1} + \frac{3}{(u-1)^2} \right] du$$

Integrate both sides

$$\ln x = \ln(u-1) - \frac{3}{(u-1)} + C$$

$$\text{As } u = \frac{y}{x},$$

$$\ln x = \ln\left(\frac{y}{x} - 1\right) - \frac{3}{\left(\frac{y}{x} - 1\right)} + C$$

$$\Rightarrow \ln\left(\frac{x^2}{y-x}\right) = \frac{3x}{x-y} + C.$$

$$b) \quad (y^2 + yx) dx - x^2 dy = 0$$

$$y = ux \Rightarrow dy = u dx + x du$$

$$\text{D.E.} \Rightarrow (u^2 x^2 + u^2 x) dx - x^2(u dx + x du) = 0$$

$$\Rightarrow u^2 x^2 dx - x^3 du = 0$$

$$\text{or } \frac{dx}{x} = \frac{du}{u^2}$$

Integration gives

$$\ln x = -\frac{1}{u} + C \quad \text{or}$$

$$\ln x = -\frac{x}{y} + C$$

$$y = \frac{x}{C - \ln x}$$

$$10) \quad x \frac{dy}{dx} = y + \sqrt{x^2 - y^2}, \quad x > 0$$

homogeneous D.E.

$$y = ux \Rightarrow \frac{dy}{dx} = u + x \frac{du}{dx}$$

(This form is convenient here)

$$x(u + x \frac{du}{dx}) = ux + \sqrt{x^2 - u^2} x$$

$$= x(u + \sqrt{1 - u^2})$$

This gives  $x \neq 0$

$$x \frac{du}{dx} = \sqrt{1 - u^2}$$

$$\text{or } \frac{du}{\sqrt{1 - u^2}} = \frac{dx}{x}$$

$$\text{Integrate: } \sin^{-1}(u) = \ln x + C$$

$$u = \sin(\ln x + C)$$

$$\text{or } \frac{y}{x} = \sin(\ln x + C), \quad x > 0.$$

$$13) \quad (x + y e^{y/x}) dx - x e^{y/x} dy = 0$$

$$y(1) = 0.$$

Homogeneous D.E. Follow above procedure  $y = ux, dy = u dx + x du.$

$$\text{D.E.} \Rightarrow (x + ux e^u) dx - x e^u (u dx + x du) = 0$$

$$\Rightarrow x dx - x^2 e^u du = 0$$

$$\Rightarrow \frac{dx}{x} = e^u du$$

$$\text{Integrate: } e^u = \ln x + C$$

$$\text{or } e^{y/x} = \ln x + C$$

$$y(1) = 0 \text{ i.e. } x=1, y=0$$

$$\Rightarrow 1 = 0 + C \Rightarrow C = 1$$

$$\text{Finally, } e^{y/x} = \ln x + 1, \quad x > 0$$

## Bernoulli Eqn

Done by Prof. Zaman

$$18) x \frac{dy}{dx} - (1+x)y = x y^2$$

$$\text{or } \frac{dy}{dx} - \frac{1+x}{x} y = y^2 \quad *$$

$$n=2, \text{ Put } u = y^{-1}$$

$$\frac{du}{dx} = -\frac{1}{y^2} \frac{dy}{dx}$$

Re-write \* as

$$\frac{1}{y^2} \frac{dy}{dx} - \frac{1+x}{x} \cdot y^{-1} = 1$$

Thus we get

$$-\frac{du}{dx} - \frac{1+x}{x} u = 1$$

$$\text{or } \frac{du}{dx} + \frac{1+x}{x} u = -1$$

(Linear D.E. in u)

$$I \cdot F = e^{\int \frac{1+x}{x} dx} = e^{\int (\frac{1}{x} + 1) dx} = x e^x$$

Thus solution is

$$u x e^x = -\int x e^x dx$$

$$= -\left[ x e^x - \int e^x dx \right] = e^x - x e^x + C$$

$$\text{Hence } u x = 1 - x + C e^{-x}$$

$$\text{or } \frac{x}{y} = 1 - x + C e^{-x}$$

$$y = \frac{x}{1 - x + C e^{-x}}$$

$$21) x^2 \frac{dy}{dx} - 2xy = 3y^4, \quad y(1) = \frac{1}{2}$$

Write in the form

$$\frac{dy}{dx} - \frac{2}{x} y = \frac{3}{x^2} y^4 \quad *$$

$$n=4 \text{ so put } u = y^{-3}$$

$$\text{so that } 3 y^{-4} \frac{dy}{dx} = \frac{du}{dx}$$

Rewrite \* as

$$y^{-4} \frac{dy}{dx} - \frac{2}{x} y^{-3} = \frac{3}{x^2}$$

$$\text{then } -\frac{1}{3} \frac{du}{dx} - \frac{2}{x} u = \frac{3}{x^2}$$

$$\text{or } \frac{du}{dx} + \frac{6}{x} u = -\frac{9}{x^2}$$

Linear D.E. in u

$$I \cdot F = e^{\int \frac{6}{x} dx} = x^6$$

Thus solution can be written as

$$u x^6 = -9 \int x^{-4} dx = -\frac{9}{5} x^5 + C$$

$$u = -\frac{9}{5x} + \frac{C}{x^6}$$

$$\Rightarrow y^{-3} = -\frac{9}{5x} + \frac{C}{x^6}$$

$$y(1) = \frac{1}{2} \Rightarrow C = \frac{49}{5}$$

$$\Rightarrow y^{-3} = -\frac{9}{5x} + \frac{49}{5x^6}$$

$$30) \frac{dy}{dx} = \frac{3x+2y}{3x+2y+2}, \quad y(-1) = -1$$

$$\text{Put } 3x+2y = u \Rightarrow 3+2 \frac{dy}{dx} = \frac{du}{dx}$$

$$\text{so that } \frac{dy}{dx} = \frac{1}{2} \frac{du}{dx} - \frac{3}{2}$$

$$D.E \Rightarrow \frac{1}{2} \frac{du}{dx} - \frac{3}{2} = \frac{u}{u+2}$$

$$\Rightarrow \frac{du}{dx} = 3 + \frac{2u}{u+2} = \frac{5u+6}{u+2}$$

$$\text{or } \frac{u+2}{5u+6} du = dx$$

$$\frac{1}{5u+6} \text{ But } \frac{u+2}{5u+6} = \frac{1}{5} + \frac{4}{5(5u+6)}$$

$$\left[ \frac{1}{5} + \frac{4}{5(5u+6)} \right] du = dx$$

$$\Rightarrow x = \frac{1}{5} u + \frac{4}{25} \ln|5u+6| + C$$

$$\text{or } x = \frac{1}{5} (3x+2y) + \frac{4}{25} \ln|5(3x+2y)+6|$$

$$y(-1) = -1 \Rightarrow C = -\frac{4}{25} \ln(19)$$