King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math 202 Exam III
Semester II, 2008- (072)
Dr. Faisal Fairag

Name:	
ID:	
Serial NO:	

Q		Points
1		**16**
2		**20**
3		**20**
4		**10**
5		**10**
6		**10**
7		**10**
8		**10**
9		**24**
10		**15 (bonus)**
Total		**130**

Say a Prayer and Work Hard

(1) Solve

$$y''' + y'' = 8x^2$$

#1) Solve $y''' + y'' = 8x^2$ (41/page 167)

Step 1: Find y_c : $(D^3 + D^2)y = 0 \Rightarrow$ roots: $0, 0, -1$

Sols: $1, x, e^{-x}$

$y_c = c_1 + c_2 x + c_3 e^{-x}$ 2

Step 2: Find annihilator: $L = D^3$ 2 for $8x^2$

$D^5(D+1)y = 0 \Rightarrow$ roots: $0, 0, 0, 0, 0, -1$

Sols: $1, x, \underline{x^2}, \underline{x^3}, \underline{x^4}, e^{-x}$

2

Hence, $y_p = Ax^2 + Bx^3 + Cx^4 \Rightarrow y_p' = 2Ax + 3Bx^2 + 4Cx^3$

$y_p'' = 2A + 6Bx + 12Cx^2 \Rightarrow y_p''' = 6B + 24Cx$

$y_p''' + y_p'' = (6B + 24Cx) + (2A + 6Bx + 12Cx^2)$

$= (6B + 2A) + (24C + 6B)x + (12C)x^2 = 8x^2$

$\Rightarrow 6B + 2A = 0, \quad 24C + 6B = 0, \quad 12C = 8$ 6

$\Rightarrow C = \frac{2}{3}, \quad B = \frac{1}{6}(-24)(\frac{2}{3}) = -\frac{8}{3}, \quad A = -\frac{1}{3}B = \frac{8}{9}$

Hence, $y_p = \frac{8}{9}x^2 - \frac{8}{3}x^3 + \frac{2}{3}x^4$ 2

Therefore, The general solution is

$y = c_1 + c_2 x + c_3 e^{-x} + \frac{8}{9}x^2 - \frac{8}{3}x^3 + \frac{2}{3}x^4$ 2

(2) Solve $\boxed{x^2 y'' - 2y = x^3 \sin x} \Rightarrow y'' - \frac{2}{x^2} y = x \sin x$

it is Cauchy-Euler equation : $y = x^m$, $y'' = m(m-1)x^{m-2}$

Aux. eq : $m(m-1) - 2 = 0 \Rightarrow (m-2)(m+1) = 0$

$y_1 = x^2$, $y_2 = x^{-1}$ 4

$W = \begin{vmatrix} x^2 & x^{-1} \\ 2x & -x^{-2} \end{vmatrix} = -3$, $W_1 = \begin{vmatrix} 0 & x^{-1} \\ x\sin x & -x^{-2} \end{vmatrix} = -S$

$W_2 = \begin{vmatrix} x^2 & 0 \\ 2x & x\sin x \end{vmatrix} = x^3 \sin x$ 3

$u_1' = \frac{W_1}{W} = \frac{1}{3}\sin x \Rightarrow u_1 = -\frac{1}{3}\cos x$ 3

$u_2' = \frac{W_2}{W} = -\frac{1}{3}x^3 \sin x \Rightarrow u_2 = -\frac{1}{3}\int x^3 \sin x \, dx$

Now, $\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x$

x^3	+	$\sin x$
$3x^2$	−	$-\cos x$
$6x$	+	$-\sin x$
6	−	$\cos x$
0		$\sin x$

$u_2 = \frac{1}{3}x^3 \cos x - x^2 \sin x - 2x\cos x + 2\sin x$ 3

$y_p = u_1 y_1 + u_2 y_2$

$= -\frac{1}{3}x^2 \cos x + \frac{1}{3}x^2 \cos x - x\sin x - 2\cos x + \frac{2}{x}\sin x$

$= -x\sin x - 2\cos x + \frac{2}{x}\sin x$ 4

The general solution for the DE :

$$\boxed{y = c_1 x^2 + c_2 x^{-1} - x\sin x - 2\cos x + \frac{2}{x}\sin x}$$

3

(3) Solve:

$$x^2y'' - 4xy' + 6y = 2x^4 + x^2$$

It is Cauchy-Euler eq. $\Rightarrow y'' - \frac{4}{x}y' + \frac{6}{x^2} = 2x$

Aux. eq: $m(m-1) - 4m + 6 = 0$

$m^2 - 5m + 6 = 0 \Rightarrow (m-3)(m-2) = 0$

$y_1 = x^2 \quad , y_2 = x^3$ 2

Step1: $W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4$ 2

$W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2+1 & 3x^2 \end{vmatrix} = -2x^5 - x^3$ 2 $\Rightarrow u_1' = \frac{W_1}{W} = -2x - \frac{1}{x}$ 2

$W_2 = \begin{vmatrix} x^2 & 0 \\ 2x & 2x^2+1 \end{vmatrix} = 2x^4 + x^2$ 2 $\Rightarrow u_2' = \frac{W_2}{W} = 2 + \frac{1}{x^2}$ 2

$u_1 = \int(-2x - \frac{1}{x})dx = -x^2 - \ln x$ 2

$u_2 = \int(2 + \frac{1}{x^2})dx = 2x - x^{-1}$ 2

$y_p = u_1y_1 + u_2y_2 = -x^4 - x^2\ln x + 2x^4 - x^2$

$= x^4 - x^2\ln x - x^2$ 2

Now, The general Sol is:

$y = \hat{c}_1x^2 + \hat{c}_2x^3 + x^4 - x^2\ln x - x^2$

or $\boxed{y = c_1x^2 + c_2x^3 + x^4 - x^2\ln x}$ 2

(4) **"DO NOT SOLVE THE DE"**

Use the substitution $x = e^t$ to transform the Cauchy-Euler equation

$$x^2 y'' - 4xy' + 6y = 0$$

to a DE with constant coefficients.

#4) $y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x} y_t$	2
$y'' = \frac{1}{x}\left(\frac{d^2y}{dt^2}\frac{1}{x}\right) + \frac{dy}{dt}\left(-\frac{1}{x^2}\right)$ $= \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) = \frac{1}{x^2}(y_{tt} - y_t)$	2
$x^2y'' - 4xy + 6y = 0$ $x^2\left[\frac{1}{x^2}(y_{tt} - y_t)\right] - 4x\left[\frac{1}{x} y_t\right] + 6y = 0$ $(y_{tt} - y_t) - 4y_t + 6y = 0$ $\boxed{y_{tt} - 5y_t + 6y = 0}$	6

(5) Given that $y_1 = x^{-1}$, $y_2 = x^4$ are solutions for the DE

$$x^2 y'' - 2xy' - 4y = 0 \qquad (*)$$

and $y_p = -1$ is a particular solution for

$$x^2 y'' - 2xy' - 4y = 4 \qquad (**)$$

The general solution for (**) is (circle the correct answer)

(a) $y = c_1 x^{-1} + c_2 x^4 - c_3$

(b) $y = c_1 + c_2 x^{-1} + x^4$

(c) $y = c_1 x^{-1} + c_2 x^4 - 1$ **(C)**

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(6) Given that $y_{p_1} = x^3$ and $y_{p_2} = 5x^3$ are, respectively, particular solutions of $y''' + 2y = 6 + 2x^3$ and $y''' + 2y = 30 + 10x^3$.

Find particular solutions of

$$y''' + 2y = 4x^3 + 12$$

Note that, $g_3 = 4x^3 + 12$ $= 7(6+2x^3) - (30+10x^3)$ $= 7g_1 - g_2$	2
Then $y_{p_3} = 7y_{p_1} - y_{p_2}$ (superposition Theorem 4.1) $y_{p_3} = 7(x^3) - 5x^3$	2
$\boxed{y_{p_3} = 2x^3}$	6

(7) Use Reduction of Order (4.2) method to find a second linearly independent solution for :

$$y'' - y = 0$$

Given that $y_1 = \cosh x$ is a solution.

#7) $y'' - y = 0 \Rightarrow P(x) = 0$ $K(x) = e^{\int 0\,dx} = 1$	2
$y_2 = y_1 \int \frac{K}{y_1^2}\,dx$ $= \cosh x \int \frac{dx}{\cosh^2 x} = \cosh x \int \operatorname{sech}^2 x\,dx$	3
$= \cosh x \left[\tanh x\right] = \sinh x$	5

(8) Find a linear differential operator that annihilates the function

$$x^3 e^{4x} \cos 3x + x^2 e^{-3x} \sin 4x$$

#8 $L = \left[D^2 - 2\alpha D + (\alpha^2+\beta^2)\right]^n$	
$L_1 = \left[D^2 - 8D + 25\right]^4$ annihilate the first term	3
$L_2 = \left[D^2 + 6D + 25\right]^3$ annihilate the second term	3
$L_1 L_2 = \left[D^2 - 8D + 25\right]^4 \cdot \left[D^2 + 6D + 25\right]^3$ annihilate the function.	4

(9) True or False:

(2 each)

1) The functions $y_1(x) = 2e^{3x}$ and $y_2(x) = e^{-3x}$ form a fundamental set of solutions of the DE $y'' - 9y = 0$. **T**

2) If $y_1 = 1, y_2 = x, y_3 = e^{3x}$ then $W(y_1, y_2, y_3) = 9e^{3x}$. **T**

3) $L = (D+2)^2(D-1)^3$ is an annihilator for the function $f(x) = xe^x - e^{-2x} + x^2e^x$ **T**

4) $y = x^2$ is the only solution on any interval containing $x = -1$ for the IVP: $(x^2+1)y'' - xy' = 1$, $y(-1) = 1$, $y'(-1) = -2$. **T**

5) $x^2y'' - 2y' + 1 = 0$ is the associated homogeneous equation of $x^2y'' - 2y' + 1 = \sin x$. **F**

6) A constant multiple of a solution of a linear differential equation is also a solution. **F**

7) A set of functions is linearly independent if at least one function can be expressed as a linear combination of the remaining functions. **F**

8) The IVP :

$$3y'' + (x-2)y' = \frac{1}{x+3},$$

$y(0) = 0, y'(0) = 1$ has a unique solution on the interval [0,4] . **T**

9) $y'' + y' = 0$, $y(0) = 1, y(1) = 0$ is a boundary value problem. **T**

10) $L_1L_2 f = L_2L_1 f$ (where $L_1 = xD - 1, L_2 = D + 1, f(x) = x$). **F**

11) $(e^{\frac{\pi}{4}i} + e^{-\frac{\pi}{4}i})^2 = 0$ **F**

12) $L = D^2 + 1$ is an annihilator for $f(x) = \sin x \cos x$ **F**

(10) BOUNUS

Solve :

$$\frac{xy''}{y'} + 1 = x^2 y' \qquad \text{———} (1)$$

Let $u = y' \Rightarrow u' = y'' \Rightarrow$ DE(1) :

$$\boxed{\frac{x u'}{u} + 1 = x^2 u}$$ is a first order DE

multiply by u : $x u' + u = x^2 u^2$ ——— (2)

(2) is Bernoulli DE in u with $n=2$

(See Example 2 pagee 77 to solve (2))

$$u = \frac{1}{-x^2 + c_1 x} \Rightarrow y' = \frac{1}{-x^2 + c_1 x}$$

$$y = \int \frac{dx}{-x^2 + c_1 x} = -\int \frac{dx}{x^2 - c_1 x}$$

$$\boxed{y = \frac{1}{c_1} \ln \left| \frac{x}{x - c_1} \right| + c_2}$$

Aside : partial fraction

$$\frac{1}{x^2 - c_1 x} = \frac{1}{x(x - c_1)} = \frac{-1/c_1}{x} + \frac{1/c_1}{x - c_1}$$

$$\int \frac{dx}{-x^2 + c_1 x} = -\int \frac{dx}{x^2 - c_1 x}$$

$$= -\left[-\frac{1}{c_1} \int \frac{dx}{x} + \frac{1}{c_1} \int \frac{dx}{x - c_1} \right]$$

$$= \frac{1}{c_1} \ln |x| - \frac{1}{c_1} \ln |x - c_1|$$

$$= \frac{1}{c_1} \ln \left| \frac{x}{x - c_1} \right| + c_2$$

☺ Wish you a FULL MARK