King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math 202 Exam II
Semester II, 2008- (072)
Dr. Faisal Fairag

Serial NO:	KEY
ID:	
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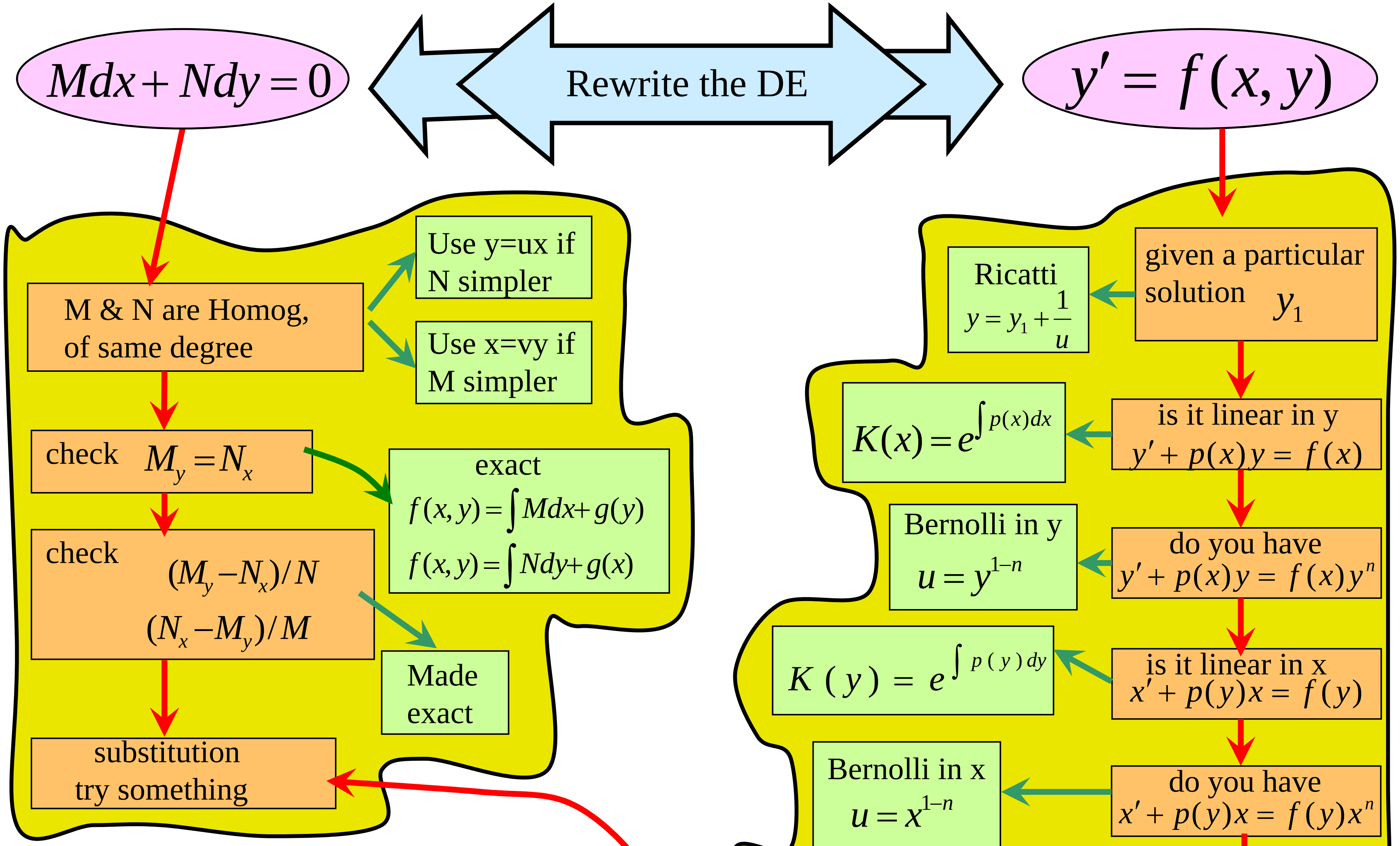
FORM B

Q		Points
1		20
2		20
3		30
4		15
5		15
6		30
Bonus		20
Total		130

(You are an **A+** student when you believe you are an **A+** student)

☺ Say a prayer & Good luck ☺

Math 202 - 072
First Order Differential Equations
Prepared by: Dr . Faisal Fairag



1) Consider the DE

$$y' = -4x^{-2} - x^{-1}y + y^2 \qquad (*)$$

Given that $y_1 = -2x^{-1}$ is a known solution of the equation (*). Transform the equation (*) into a linear DE. (DONOT SOLVE THE DE ONLY TRANSFORM IT INTO LINEAR)

(*) is Ricatti equation (#35/page 79)

use $y = y_1 + u$

$$\boxed{y = -2x^{-1} + u} \quad \text{———} (1) \quad 4$$

$$y' = 2x^{-2} + u' \quad \text{———} (2) \quad 4$$

use (1), (2) in (*):

$$2x^{-2} + u' = -4x^{-2} - x^{-1}(-2x^{-1} + u) + (-2x^{-1} + u)^2$$

$$2x^{-2} + u' = -4x^{-2} + 2x^{-2} - x^{-1}u + 4x^{-2} - 4x^{-1}u + u^2$$

$$\Rightarrow u' = -5x^{-1}u + u^2 \quad \text{———} (**) \quad 4$$

(**) is Bernulli with $n = 2$

use $w = u^{1-n} \Rightarrow w = u^{-1}$ 4

$$u = w^{-1} \quad \text{———} (3) \quad \Rightarrow \quad u' = -w^{-2}w' \quad \text{———} (4)$$

use (3), (4) in (**):

$$-w^{-2} \cdot w' = -5x^{-1}w^{-1} + w^{-2} \quad \text{———} (***)$$

multiply (***) by $-w^2$

$$\boxed{w' = 5x^{-1}w - 1} \quad \text{is a linear DE} \quad 4$$

Another sol: use $y = y_1 + w^{-1} \Rightarrow y = -2x^{-1} + w^{-1}$ ——— (5) 5

$$\Rightarrow y' = 2x^{-2} - w^{-2}w' \quad \text{———} (6) \quad 5$$

use (5), (6) in (*): $2x^{-2} - w^{-2}w' = -4x^{-2} - x^{-1}(-2x^{-1} + w^{-1}) + (-2x^{-1} + w^{-1})^2$

$$\Rightarrow -w^{-2}w' = -5x^{-1}w^{-1} + w^{-2} \quad 5 \quad (\text{multiply by } -w^{2})$$

$$\Rightarrow \boxed{w' = 5x^{-1}w - 1} \quad \text{linear DE} \quad 5$$

2) Consider the DE

$$xdx + x^2 ydy = -4ydy \qquad (*)$$

Find an appropriate integrating factor K so that if we multiply the equation (*) by K then the new equation become an exact.

(DONOT SOLVE THE DE)

rewrite (*) as: $x\,dx + (x^2y + 4y)\,dy = 0$ —(1)

Now $M = x$, $N = x^2y + 4y$

$M_y = 0$ 5 , $N_x = 2xy$ 5

So, (*) is not exact DE.

$$\frac{N_x - M_y}{M} = \frac{2xy - 0}{x} = 2y$$ 5 function of y

integrating factor $= K(y) = e^{\int 2y\,dy} = e^{y^2}$ 5

multiply (1) by e^{y^2} give

$$\boxed{x e^{y^2} dx + (x^2 y e^{y^2} + 4y e^{y^2})\,dy = 0} \quad (**)$$

(**) is an exact DE

$\tilde{M} = x e^{y^2}$, $\tilde{N} = x^2 y e^{y^2} + 4y e^{y^2}$

$\tilde{M}_y = 2xy e^{y^2}$, $\tilde{N}_x = 2xy e^{y^2}$

So, $\tilde{M}_y = \tilde{N}_x$

$\frac{\#37}{\text{Sec } 2.4}$

3) Solve $(4x^3y-15x^2-y)dx+(x^4+3y^2-x)dy=0$ ——— (1)

(1) is an exact.

$M = 4x^3y-15x^2-y$, $N = x^4+3y^2-x$ ——— (2)

$M_y = 4x^3-1$ 3 , $N_x = 4x^3-1$ 3

$f = \int M\,dx + g(y)$

$= \int (4x^3y-15x^2-y)\,dx + g(y)$

$f = x^4y-5x^3-xy+g(y)$ ——— (*) 6

diff. w.r.t y

$\frac{\partial f}{\partial y} = x^4 - x + g'(y)$ ——— (3)

(2) & (3) & $N = \frac{\partial f}{\partial y} \Rightarrow x^4-x+g'(y) = x^4+3y^2-x$

$\Rightarrow g'(y) = 3y^2$ 3

$\Rightarrow g(y) = y^3$ 6 ——— (4)

(*) and (4) $\Rightarrow f(x,y) = x^4y-5x^3-xy+y^3$ 6

Hence, $x^4y-5x^3-xy+y^3 = c$ 3

is a family of solution for (1)

(#19 / Page 73)

4) Consider the DE $\quad x'(1+e^{y-x+5})=1 \quad$ (*)

We can solve (*) by transforming (*) into a ___Separable___ differential equation if we use the 7

following substitution ___$u=y-x+5$___. 8

(#28 / P. 78)

$$x' = \frac{1}{1+e^{y-\cancel{x}+5}}$$

$$y' = 1 + e^{y-x+5} \quad \text{———} (**)$$

let $u = y-x+5$

$u' = y'-1$

$$(**) \Rightarrow u'+1 = 1+e^u \Rightarrow u' = e^u$$

Separable

5) Consider the DE $\quad x(x-y)y'=1 \quad$ (*)

We can solve (*) by transforming (*) into a ___linear___ differential equation if we use the 7

following substitution ___$u=x^{-1}$___. 8

(#8 (d) / ch2 Rev)

$$y' = \frac{1}{x(x-y)} \Rightarrow y' = \frac{1}{x^2-xy}$$

$$x' = x^2 - \cancel{y}x \Rightarrow \boxed{x' + yx = x^2}$$

it is Bernoulli in x with $n=2$

use $u = x^{1-n} \Rightarrow u = x^{-1}$

(#10/ch2 Rev)

6) Solve the IVP:

$$y(\ln x - \ln y)dx = (x\ln x - x\ln y - y)dy$$
$$y(e) = e$$

rewrite the DE as $y \ln \frac{x}{y}\, dx = \left[x \ln \frac{x}{y} - y\right] dy$ — (*)

3

This is a homog. DE, so let $\boxed{x = uy}$ — (1)

3

Then $\boxed{dx = u\,dy + y\,du}$ — (2)

3

subsitute (1) & (2) into (*) $\Rightarrow$

$$y \ln u\,(u\,dy + y\,du) = [uy \ln u - y]\,dy$$

$$\cancel{uy \ln u\, dy} + y^2 \ln u\, du = \cancel{uy \ln u\, dy} - y\,dy$$

$$\Rightarrow y^2 \ln u\, du = -y\,dy$$

divid by y^2.

$$\Rightarrow \boxed{\ln u\, du = -\frac{dy}{y}}$$ separable 12

integrate both sides

$$\int \ln u\, du = -\int \frac{dy}{y}$$

6

$$\Rightarrow u \ln|u| - u = -\ln|y| + C \quad \text{— (3)}$$

(1) and (3) give: $\boxed{\frac{x}{y} \ln\left|\frac{x}{y}\right| - \frac{x}{y} = -\ln|y| + C}$ 3

Another sol: rewrite as $y \ln\frac{x}{y}\,dx = \left[x\ln\frac{x}{y} - y\right]dy$

This is a homog DE, set let $\boxed{y = ux}$ — (1

$\Rightarrow dy = u\,dx + x\,du$ —— (2)

(1), (2) in (*) $\Rightarrow$

$$ux \ln\frac{1}{u}\,dx = \left[x\ln\frac{1}{u} - ux\right](u\,dx + x\,du)$$

$$\cancel{ux\ln\frac{1}{u}\,dx} = \cancel{xu\ln\frac{1}{u}\,dx} + x^2\ln\frac{1}{u}\,du - u^2x\,dx - ux^2\,du$$

$$\Rightarrow \quad u^2x\,dx = x^2\left(\ln\frac{1}{u} - u\right)du$$

divid by x^2u^2

$$\boxed{\frac{dx}{x} = \left(\frac{1}{u^2}\ln\frac{1}{u} - \frac{1}{u}\right)du}$$ separable

integrate both sides

$$\int\frac{dx}{x} = \int\left(\frac{1}{u^2}\ln\frac{1}{u} - \frac{1}{u}\right)du$$

$$\ln|x| = \frac{1}{u} - \frac{1}{u}\ln\left|\frac{1}{u}\right| - \ln|u| + C$$

$$\ln|x| = \frac{x}{y} - \frac{x}{y}\ln\left|\frac{x}{y}\right| - \ln\left|\frac{y}{x}\right| + c$$

$$\frac{x}{y}\ln\left|\frac{x}{y}\right| - \frac{x}{y} = -\ln\left|\frac{y}{x}\right| - \ln|x| + c$$

$$\frac{x}{y}\ln\left|\frac{x}{y}\right| - \frac{x}{y} = -\ln|y| + c$$

Aside: let $w = \frac{1}{u} \Rightarrow dw = \frac{-du}{u^2}$

$$\int\frac{1}{u^2}\ln\frac{1}{u}\,du = -\int \ln w\,dw$$

$$= -\left[w\ln w - w\right]$$

$$= w - w\ln w$$

$$= \frac{1}{u} - \frac{1}{u}\ln\frac{1}{u}$$

Bonus) Consider the DE

$$\overbrace{(4x^5+4x^3y^2-2x^5-2xy^4)}^{\tilde{M}}dx+\overbrace{(4y^3x^2+4y^5-2yx^4-2y^5)}^{\tilde{N}}dy=0 \qquad (*)$$

Find an appropriate integrating factor K so that if we multiply the equation (*) by K then the new equation become an exact.

$$K(x,y) = (x^2+y^2)^{-2} \quad \text{——} (1)$$

multiply (*) by $K(x,y)$

$$K\tilde{M}\,dx + K\tilde{N}\,dy = 0 \quad \text{——} (2)$$

Now, $M = (x^2+y^2)^{-2}\tilde{M}$, $N = (x^2+y^2)^{-2}\tilde{N}$

$$M_y = -4y(x^2+y^2)^{-3}\tilde{M} + (x^2+y^2)^{-2}(8x^3y-8xy^3)$$

$$= -16x^3y^3(x^2+y^2)^{-3} \quad \text{——} (3)$$

$$N_x = -4x(x^2+y^2)^{-3}\tilde{N} + (x^2+y^2)^{-2}(8y^3x-8yx^3)$$

$$= -16x^3y^3(x^2+y^2)^{-3} \quad \text{——} (4)$$

from (3) and (4) $\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Hence, (2) is an exact.