

(8.2)
Math 202

45) $A = \begin{pmatrix} 1 & -12 & -14 \\ 1 & 2 & -3 \\ 1 & 1 & -2 \end{pmatrix}$

I $\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -12 & -14 \\ 1 & 2-\lambda & -3 \\ 1 & 1 & -2-\lambda \end{vmatrix} = 0$

After simplification

$\Rightarrow (1-\lambda)(\lambda^2 + 25) = 0$

$\Rightarrow \lambda = 1, \lambda = \pm 5i$

II $\lambda = 1, (A - \lambda I)|_{\lambda=1}$ gives

$\begin{bmatrix} 0 & -12 & -14 \\ 1 & 1 & -3 \\ 1 & 1 & -3 \end{bmatrix}$ ① $R_1 \leftrightarrow R_2$
 ② $R_3 - R_1$
 ③ $-\frac{1}{12} R_2$
 ④ $R_1 - R_2$

$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{25}{6} \\ 0 & 1 & \frac{7}{6} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{pmatrix} \frac{25}{6}\alpha \\ -\frac{7}{6}\alpha \\ \alpha \end{pmatrix}$

For $\alpha = 6, K_1 = \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix}$
 $X_1 = \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^t$

Next, $\lambda = 5i, (A - \lambda I)|_{\lambda=5i}$

$= \begin{pmatrix} 1-5i & -12 & -14 \\ 1 & 2-5i & -3 \\ 1 & 1 & -2-5i \end{pmatrix}$

Then $R_1 \leftrightarrow R_3, R_2 - R_1$

$\frac{1}{1-5i} R_2, R_3 + (13-5i)R_2$
 and $R_1 - R_2$ leads to

$\begin{pmatrix} 1 & 0 & -1-5i \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

This gives $x_3 = t, x_2 = t$

$x_1 = 1 + 5i$

So $K = \begin{pmatrix} 1+5i \\ 1 \\ 1 \end{pmatrix} \pm \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$

$B_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, B_2 = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$

$X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \sin t$

$X_3 = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \sin t$

$X = C_1 X_1 + C_2 X_2 + C_3 X_3 =$

$\left(C_1 \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cos t - C_2 \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \sin t + C_3 \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \cos t + C_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \sin t \right)$

Using $X(0) = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}$

We get

$\begin{cases} 25C_1 + C_2 + 5C_3 = 4 & \text{---(1)} \\ -7C_1 + C_2 = 6 & \text{---(2)} \\ 6C_1 + C_2 = -7 & \text{---(3)} \end{cases}$

Solving (2), (3) $C_1 = -1, C_2 = -1$

From (1) $-25 - 1 + 5C_3 = 4$

$\Rightarrow C_3 = 6$

Hence the solution can be written as

$-\begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^t - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cos t + \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \sin t$
 $+ 6 \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \cos t + 6 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \sin t$