Done by Prof. Zaman

Math 202

2.5 (Homogeneous coeff.)

4. $x\,dx + (y-2x)\,dy = 0$

D.E. is homogeneous. Put

$y = ux \Rightarrow dy = u\,dx + x\,du$

so D E $\Rightarrow x\,dx + (ux - 2x)(u\,dx + x\,du) = 0$

$\Rightarrow x(1 - u^2 + 2u)\,dx = x^2(u+2)\,du$

or $\frac{x}{x^2}\,dx = \frac{u+2}{1-u^2+2u}\,du$

or $\frac{1}{x}\,dx = \frac{u-1+3}{(u-1)^2}\,du$

$= \left[\frac{1}{u-1} + \frac{3}{(u-1)^2}\right] du$

Integrate both sides

$\ln x = \ln(u-1) - \frac{3}{(u-1)} + C$

As $u = \frac{y}{x}$,

$\ln x = \ln\left(\frac{y}{x} - 1\right) - \frac{3}{\left(\frac{y}{x}-1\right)} + C$

$\Rightarrow \ln\left(\frac{x^2}{y-x}\right) = \frac{3x}{x-y} + C.$

6) $(y^2 + yx)\,dx - x^2\,dy = 0$

$y = ux \Rightarrow dy = u\,dx + x\,du$

D.E $\Rightarrow (u^2x^2 + u^2x)\,dx - x^2(u\,dx + x\,du) = 0$

$\Rightarrow u^2x^2\,dx - x^3\,du = 0$

or $\frac{dx}{x} = \frac{du}{u^2}$

Integration gives

$\ln x = -\frac{1}{u} + C$ or

$\ln x = -\frac{x}{y} + C$

$y = \frac{x}{C - \ln x}$.

10) $x\frac{dy}{dx} = y + \sqrt{x^2 - y^2}, \; x > 0$

homogeneous D E

$y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$

(This form is convenient here)

$\cancel{x}\left(u + x\frac{du}{dx}\right) = ux + \sqrt{x^2 - u^2x^2}$

$= \cancel{x}\left(u + \sqrt{1-u^2}\right)$

This gives $\qquad x \neq 0$

$x\frac{du}{dx} = \sqrt{1-u^2}$

or $\frac{du}{\sqrt{1-u^2}} = \frac{dx}{x}$

Integrate: $\sin^{-1}(u) = \ln x + C$

$u = \sin(\ln x + C)$

or $\frac{y}{x} = \sin(\ln x + C), \; x > 0.$

13) $(x + y e^{y/x})\,dx - x e^{y/x}\,dy = 0$

$y(1) = 0.$

Homogeneous D.E. Follow above procedure $y = ux$, $dy = u\,dx + x\,du$.

D.E $\Rightarrow (x + ux e^u)\,dx - x e^u(u\,dx + x\,du) = 0$

$\Rightarrow x\,dx - x^2 e^u\,du = 0$

$\Rightarrow \frac{dx}{x} = e^u\,du$

Integrate: $e^u = \ln x + C$

or $e^{y/x} = \ln x + C$

$y(1) = 0$ i.e. $x = 1, y = 0$

$\Rightarrow 1 = 0 + C \Rightarrow C = 1$

Finally, $e^{y/x} = \ln x + 1, \; x > 0$

Math 202

Bernoulli Eqn. (2.5)

Done by Prof. Zaman

18) $x\frac{dy}{dx} - (1+x)y = xy^2$

or $\frac{dy}{dx} - \frac{1+x}{x}y = y^2$ — $*$

$n=2$, Put $u = y^{-1}$

$$\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$$

Re-write $*$ as

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1+x}{x}\cdot y^{-1} = 1$$

Thus we get

$$-\frac{du}{dx} - \frac{1+x}{x}u = 1$$

or $\frac{du}{dx} + \frac{1+x}{x}u = -1$

(Linear D.E. in u)

$$I\cdot F = e^{\int\frac{1+x}{x}dx} = e^{\int(\frac{1}{x}+1)dx} = xe^{x}.$$

Thus solution is

$$u\,xe^{x} = -\int xe^{x}dx$$

$$= -\left[xe^{x} - \int e^{x}dx\right] = e^{x} - xe^{x} + C$$

Hence $ux = 1 - x + Ce^{-x}$

or $\frac{x}{y} = 1 - x + Ce^{-x}.$

$$y = \frac{x}{1-x+Ce^{-x}}$$

21) $x^2\frac{dy}{dx} - 2xy = 3y^4$, $y(1) = \frac{1}{2}$

Write in the form

$$\frac{dy}{dx} - \frac{2}{x}y = \frac{3}{x^2}y^4 \quad *$$

$n = 4$ so put $u = y^{-3}$

so that $-3y^{-4}\frac{dy}{dx} = \frac{du}{dx}$

Rewrite $*$ as

$$y^{-4}\frac{dy}{dx} - \frac{2}{x}y^{-3} = \frac{3}{x^2}$$

then $-\frac{1}{3}\frac{du}{dx} - \frac{2}{x}u = \frac{3}{x^2}$

or $\frac{du}{dx} + \frac{6}{x}u = -\frac{9}{x^2}$

Linear DE in u.

$$I\cdot F = e^{\int\frac{6}{x}dx} = x^6$$

Thus solution can be written as

$$ux^6 = -9\int x^4dx = -\frac{9}{5}x^5 + c$$

$$u = \frac{-9}{5x} + \frac{c}{x^6}$$

$$\Rightarrow y^{-3} = \frac{-9}{5x} + \frac{c}{x^6}$$

$y(1) = \frac{1}{2} \Rightarrow C = \frac{49}{5}.$

$$\Rightarrow y^{-3} = -\frac{9}{5x} + \frac{49}{5x^6}.$$

30) $\frac{dy}{dx} = \frac{3x+2y}{3x+2y+2}$, $y(-1) = -1$

Put $3x + 2y = u \Rightarrow 3 + 2\frac{dy}{dx} = \frac{du}{dx}$

so that $\frac{dy}{dx} = \frac{1}{2}\frac{du}{dx} - \frac{3}{2}$

D.E $\Rightarrow \frac{1}{2}\frac{du}{dx} - \frac{3}{2} = \frac{u}{u+2}$

$$\Rightarrow \frac{du}{dx} = 3 + \frac{2u}{u+2} = \frac{5u+6}{u+2}$$

or $\frac{u+2}{5u+6}du = dx$ But $\frac{u+2}{5u+6} = \frac{1}{5} + \frac{4}{5(5u+6)}$

$$\left[\frac{1}{5} + \frac{4}{5(5u+6)}\right]du = dx$$

$$\Rightarrow x = \frac{1}{5}u + \frac{4}{25}\ln|5u+6| + C$$

or $x = \frac{1}{5}(3x+2y) + \frac{4}{25}\ln|15x+10y+6|$

$y(-1) = -1 \Rightarrow C = -\frac{4}{25}\ln(19).$