King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math 202 Final Exam
Semester II, 2006- (052)
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Name:	KEY	
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Sec:	2 (8:00 8:50am)	3 (9:00 9:50am)
Serial NO:		

FORM **A**

Q		Points
1		27
2		9
3		18
4		9
5		12
6		18
7		15
8		18
9		12
10		30
11		12
12		18
13		15
14		24
15		30
16		12
17		21
Total		300

(Q1) Solve $X' = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix} X$ (#29/page 352)

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{vmatrix} = (\lambda - 4)^2 = 0$$

$\Rightarrow \lambda = 4, 4$ are the eigenvalues 4

$[A - 4I \mid 0] \Longrightarrow K = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ 4

$[A - 4I \mid K] \Rightarrow P = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ 4

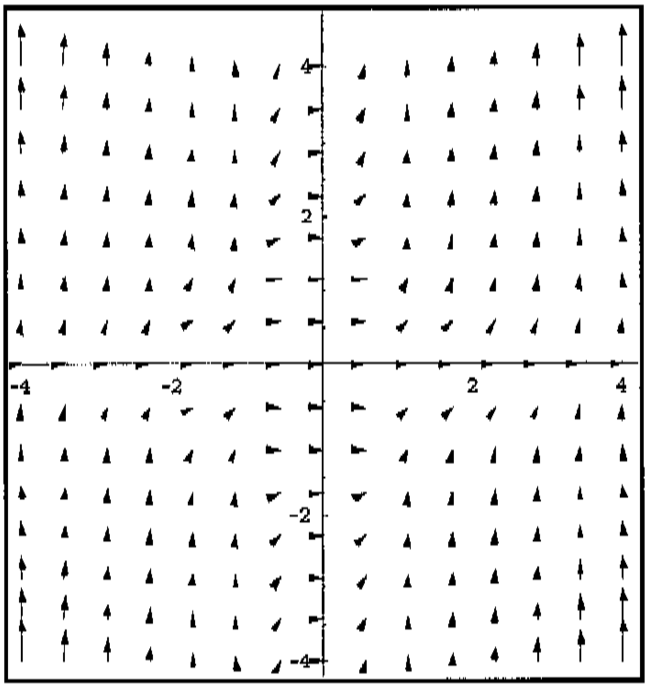
$X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}$ 4

$X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} t e^{4t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$ 4

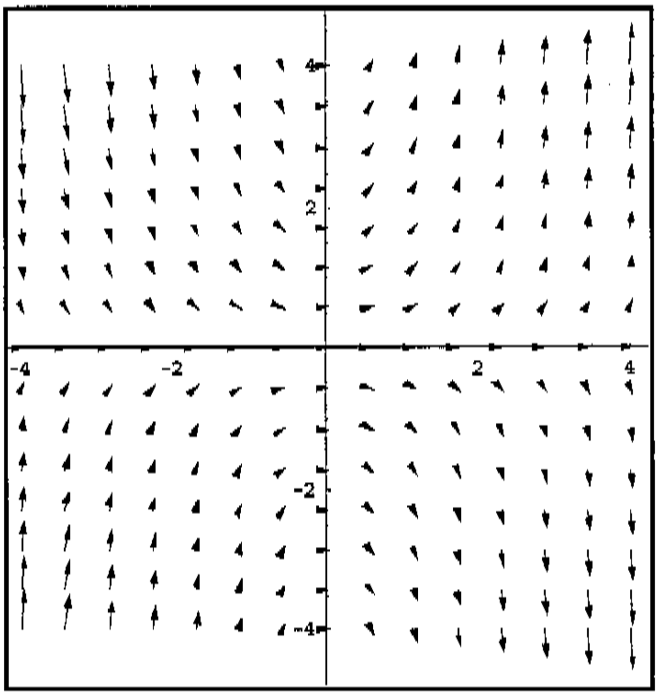
$X = c_1 X_1 + c_2 X_2$ 4

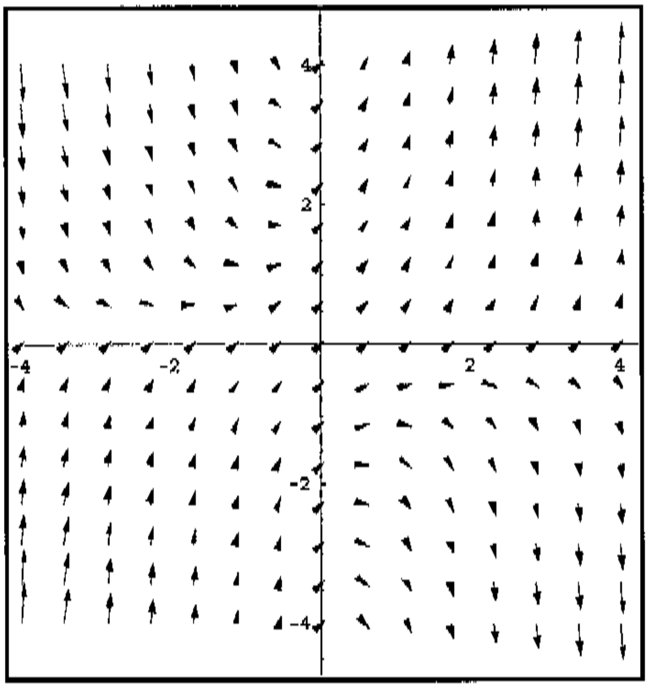
(Q2) The computer-generated directional field of $y' = x^2 y^2$ is :

(a) a

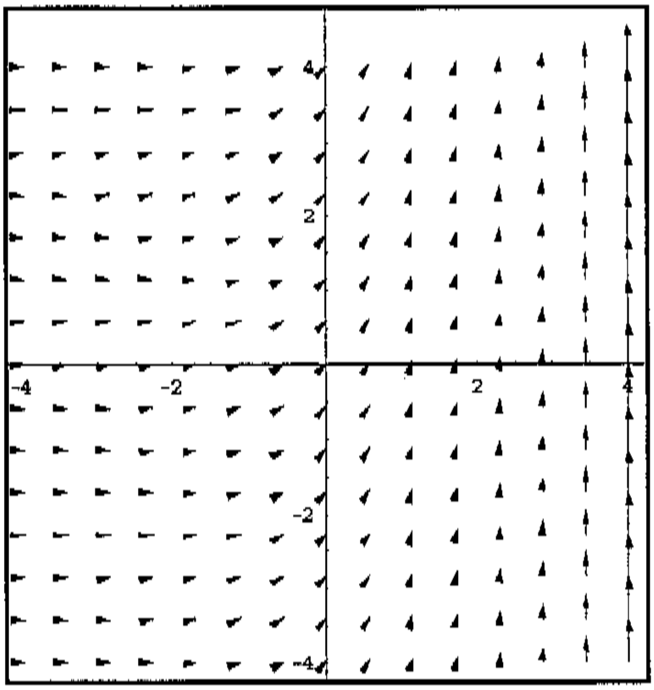


(b)





(c)



(d)

(Q3) Find a particular solution X_P for the system

$$X' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} X + \begin{pmatrix} e^t \cos t \\ e^t \sin t \end{pmatrix}$$

Hint:

$\Phi(t) = \begin{pmatrix} -e^t \sin t & e^t \cos t \\ e^t \cos t & e^t \sin t \end{pmatrix}$ is a fundamental matrix of the associated homogeneous system.

$$\Phi^{-1}(t) = \begin{bmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{bmatrix} e^{-t}$$

6

$$U = \int \Phi^{-1} F \, dt = \int \begin{bmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{bmatrix} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^{-t} e^{t} \, dt$$

$$= \int \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

6

$$X_p = \Phi U = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} t e^t$$

6

$X_p = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} t e^t$

(Q4) Transform the system

$$3y_1'''-5y_2'+t^2=0$$
$$7y_2''+5y_1''=e^t$$

Into a first-order system

$$X'(t)=AX(t)+F(t).$$

$x_1 = y_1 \Rightarrow x_1' = y_1' = x_2$

$x_2 = y_1' \Rightarrow x_2' = y_1'' = x_3$

$x_3 = y_1'' \Rightarrow x_3' = y_1''' = \frac{5}{3}y_2' - \frac{1}{3}t^2 = \frac{5}{3}x_5 - \frac{1}{3}t^2$

$x_4 = y_2 \Rightarrow x_4' = y_2' = x_5$

$x_5 = y_2' \Rightarrow x_5' = y_2'' = -\frac{5}{7}y_1'' + \frac{1}{7}e^t = -\frac{5}{7}x_3 + \frac{1}{7}e^t$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5/3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -5/7 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{3}t^2 \\ 0 \\ \frac{1}{7}e^t \end{bmatrix}$$

6 3

$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5/3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -5/7 & 0 & 0 \end{bmatrix}$	$F(t) = \begin{bmatrix} 0 \\ 0 \\ -1/3\, t^2 \\ 0 \\ 1/7\, e^t \end{bmatrix}$

(Q5) $x_0 = 0$ is a regular singular point of the DE

$$x^2 y'' + xy' + (x^2 - \frac{1}{9})y = 0$$

Find the indicial roots of the singularity.

(a) $r_1 = 1$, $r_2 = -1$

(b) $r_1 = \frac{1}{9}$, $r_2 = \frac{-1}{9}$

(c) $r_1 = \frac{1}{3}$, $r_2 = \frac{-1}{3}$

(d) $r_1 = 0$, $r_2 = \frac{-1}{2}$

12

$$y'' + \frac{1}{x} y' + (1 - \frac{1}{9} x^{-2}) y = 0$$

$$P = \frac{1}{x} \Rightarrow p = 1 \Rightarrow p(0) = 1$$

$$Q = 1 - \frac{1}{9} x^{-2} \Rightarrow q = x^2 - \frac{1}{9} \Rightarrow q(0) = -\frac{1}{9}$$

$$r(r-1) + r - \frac{1}{9} = 0$$

$$r^2 - \frac{1}{9} = 0$$

$$(r - \frac{1}{3})(r + \frac{1}{3}) = 0$$

$$r_1 = \frac{1}{3} \ , \ r_2 = -\frac{1}{3}$$

(Q6) A tank contains 200 liters of fluid in which 30 grams of salt is dissolved. Brine containing 1 gram of salt per liter is then pumped into the tank at a rate of 4 L/min; the well-mixed solution is pumped out at the same rate. What is the concentration of the salt in the tank after 50 min? (19/p94)

(a) 0.4328 gram of salt per liter

(b) 0.3127 gram of salt per liter

(c) 0.6873 gram of salt per liter

(d) 0.5672 gram of salt per liter

18

$$\frac{dA}{dt} = R_{in} - R_{out}$$

$$R_{in} = (4\ L/min)(1\ gram/L) = 4\ gram/min$$

$$R_{out} = (4\ L/min)\cdot\left(\frac{A}{200}\ gram/L\right) = \frac{A}{50}\ gm/min$$

$$\frac{dA}{dt} = 4 - \frac{A}{50} \Rightarrow \frac{dA}{dt} = \frac{200-A}{50} \qquad A(0) = 30$$

$$\frac{dA}{200-A} = \frac{1}{50}\,dt \Rightarrow -\ln|200-A| = \frac{1}{50}t + c$$

$$|200-A| = e^{-\frac{1}{50}t}\cdot e^{-c} \Rightarrow 200 - A = \pm \bar{e}^{c}\, e^{-\frac{1}{50}t}$$

$$\Rightarrow \boxed{A = 200 + c\,e^{-t/50}} \qquad 30 = A(0) = 200 + c \Rightarrow c = -170$$

$$\boxed{A(t) = 200 - 170\,e^{-t/50}}$$

$$\text{Concentration after 50 min} = \frac{A(50)}{200} = \frac{200 - 170\bar{e}^{1}}{200} = 1 - \frac{170}{200e} \cong 0.6873$$

(Q7) The following IVP

$$X' = \begin{bmatrix} \frac{1}{\sqrt{10-t}} & 5 \\ t^2 & \sin t \end{bmatrix} X + \begin{bmatrix} \frac{\cos t}{t} \\ \ln(t+50) \end{bmatrix}$$

$$X(-1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

has a unique solution on the interval

(a) $(0,10)$

(b) $(0,+\infty)$

(c) $(-\infty,0)$

(d) $(-10,10)$

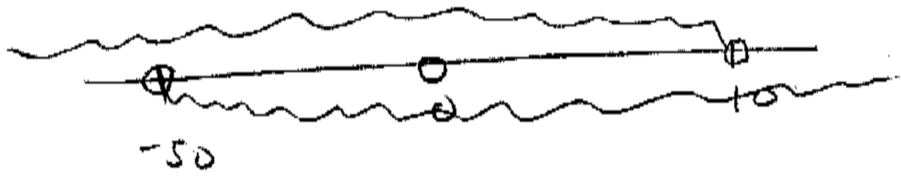
(e) $(-10,0)$

15

$10 - t > 0 \Rightarrow 10 > t$

$t + 50 > 0 \Rightarrow t > -50$

$\frac{\cos t}{t} \Rightarrow t \neq 0$



Also, $-1 \in (-50, 0)$

$\Rightarrow (-10, 0)$ is the interval

(Q8) If X(t) is the solution of the IVP (39/p352)

$X' = AX$

$X(0) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ where

$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

Hint: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector for the matrix A associated with the eigenvalue $\lambda = 0$

$\begin{bmatrix} 1 \\ -1 \\ i \end{bmatrix}$ is an eigenvector for the matrix A associated with the eigenvalue $\lambda = i$

Then $X(\frac{\pi}{2}) =$

(a) $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{0 \cdot t} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \\ i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{B_1} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B_2} i$$

$$X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \sin t = \begin{bmatrix} \cos t \\ -\cos t \\ -\sin t \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cos t + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \sin t = \begin{bmatrix} \sin t \\ -\sin t \\ \cos t \end{bmatrix}$$

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3$$

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = X(0) = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow c_3 = 1, \quad -c_2 = 1, \quad c_1 + c_2 = 0$$

$$\Rightarrow c_3 = 1, \quad c_2 = -1, \quad c_1 = 1$$

$$X(t) = \begin{bmatrix} 1 - \cos t + \sin t \\ \cos t - \sin t \\ \sin t + \cos t \end{bmatrix}$$

18

$$X(\tfrac{\pi}{2}) = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

(Q9)The form of two linearly independent solutions of the DE

$$xy'' - xy' + y = 0$$

(r_1, r_2 are the indicial roots) (27/p258)

(a) $y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}, c_0 \neq 0, y_2 = \sum_{n=0}^{\infty} b_n x^{n+r_2}, b_0 \neq 0$

(b) $y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}, c_0 \neq 0, y_2 = Cy_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}, b_0 \neq 0$

(c) $y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}, c_0 \neq 0, y_2 = y_1(x)\ln x + \sum_{n=1}^{\infty} b_n x^{n+r_1}$

(d) none of the above

$$y'' - y' + \frac{1}{x}y = 0$$

$$P = -1 \Rightarrow p = -x \Rightarrow p(0) = 0$$

$$Q = \frac{1}{x} \Rightarrow q = x \Rightarrow q(0) = 0$$

The indicial equation

$$r(r-1) = 0 \Rightarrow r_1 = 1 \ , \ r_2 = 0$$

$$\Rightarrow r_1 - r_2 = 1 = \text{positive integer}$$

$$\boxed{b}$$

12

(Q10) x=0 is a regular singular point of the DE (19/p287)

$$3xy'' + (2-x)y' - y = 0$$

Use the method of Frobenius to obtain two linearly independent series solutions about x=0.

subsitute $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, we obtain

$$(3r^2 - r)c_0 x^{r-1} + \sum_{k=1}^{\infty}\left[3(k+r-1)(k+r)c_k + 2(k+r)c_k - (k+r)c_{k-1}\right]x^{k+r-1} = 0$$

indicial equation: $r(3r-1) = 0$

indicial roots: $r_1 = \frac{1}{3}$, $r_2 = 0$ 4

Case $r=0$: $c_k = \frac{c_{k-1}}{3k-1}$ $k = 1, 2, 3, \ldots$

$c_1 = \frac{1}{2}c_0$, $c_2 = \frac{1}{10}c_0$, $c_3 = \frac{1}{80}c_0$ 6

case $r = \frac{1}{3}$. $c_k = \frac{c_{k-1}}{3k}$ $k = 1, 2, 3, \ldots$

$c_1 = \frac{1}{3}c_0$, $c_2 = \frac{1}{18}c_0$, $c_3 = \frac{1}{162}c_0$ 6

$y_1 = 1 + \frac{1}{2}x + \frac{1}{10}x^2 + \frac{1}{80}x^3 + \cdots$ 7

$y_2 = x^{1/3}\left(1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 + \cdots\right)$ 7

(Q11) Without solving the DE, use the substitution $x = e^t$ to transform the given Cauchy-Euler equation (33/p178)

$$x^2 y'' + 10xy' + 8y = x^2$$

to a DE with constant coefficients

$$x = e^t$$

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}$$ 3

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$ 3

$$LHS = \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 10\frac{dy}{dt} + 8y$$

$$= \frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 8y$$

$$\boxed{\begin{gathered}\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 8y = e^{2t} \\ y'' + 9y' + 8y = e^{2t}\end{gathered}}$$ 6

(Q12) Solve (21/p178)

$$x^2y'' - xy' + y = 2x \Rightarrow y'' - \frac{1}{x}y' + \frac{1}{x^2} = \frac{2}{x}$$

it is cauchy- Euler

$$y - x^m, \quad y' = m x^{m-1}, \quad y'' = m(m-1)x^{m-2}$$

$$LHS = m(m-1)x^m - mx^m + x^m = 0$$

$$[m(m-1) - m + 1]\, x^m = 0$$

$$\Rightarrow m^2 - 2m + 1 = 0$$ auxilling

$$(m-1) = 0$$

Solutions $y_1 = x, \quad y_2 = x\ln x$ 2

$$W = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x$$ 1

$$f(x) = \frac{2}{x}$$

$$u_1' = \frac{-2\ln x}{x} \qquad u_2' = \frac{2}{x}$$ 2

$$u_1 = -(\ln x)^2 \qquad u_2 = 2\ln x$$ 2

$$y_p = -x(\ln x)^2 + 2x(\ln x)^2$$
$$= x(\ln x)^2$$ 4

The general sol is

$$\boxed{\begin{aligned} y &= c_1 y_1 + c_2 y_2 + y_p \\ &= c_1 x + c_2 x\ln x + x(\ln x)^2 \end{aligned}}$$ 7

(Q13) If y(x) is the solution of the IVP

$$xy' + y = x^2y^2, \qquad y(1) = 2$$

then y(2)=

(a) -1

(b) 1

(c) 0

(d) 2

(e) -2

$$y' + \frac{1}{x}y = xy^2$$ Bernoulli

$$u = y^{-1} \Rightarrow y = u^{-1} \Rightarrow y' = -u^{-2}u'$$

DE become

$$-u^{-2}u' + \frac{1}{x}u^{-1} = xu^{-2}$$

$$\Rightarrow u' - \frac{1}{x}u = -x$$ linear

$$p(x) = e^{-\int \frac{1}{x}dx} = e^{\ln x^{-1}} = \frac{1}{x}$$

$$\frac{1}{x}u' - \frac{1}{x^2}u = -1$$

$$\frac{d}{dx}\left[\frac{1}{x}u\right] = -1$$

$$\frac{1}{x}u = -x + c \Rightarrow u = -x^2 + cx$$

$$\Rightarrow y^{-1} = -x^2 + cx \Rightarrow \boxed{y = \frac{1}{cx - x^2}}$$

$$2 = y(1) = \frac{1}{c-1} \Rightarrow 2c - 2 = 1 \Rightarrow c = 3/2$$

$$\boxed{y = \frac{1}{\frac{3}{2}x - x^2}}$$

$$y(2) = \frac{1}{3-4} = -1$$

(Q14) Without solving classify each of the following equations as to:

(a) separable (b) homogeneous (c) exact
(d) linear in x (e) linear in y (f) Bernolli

(I)	$y' = (x+1)^2$	a	e	c	
(II)	$y' = \dfrac{y+1}{x}$	a	d	e	
(III)	$y' + \dfrac{y}{x} = xy^2$	f			
(IV)	$(x+y)^2 dx + (2xy + x^2 - 1) dy = 0$	C			

3 each

-1 for wrong

(Q15) State the order of the given differential equation. Determine weather the equation is ODE or PDE. Determine weather the equation is linear or nonlinear.

	Equation	Type		Linearity		Order
1	A simple model for the shape of a tsunami, or tidal wave, is given by $$\frac{dW}{dx} = W\sqrt{4-2W}$$ (page 111)	ODE	PDE	linear	nonlinear	1
2	The model of the free pendulum given by $$\frac{d^2\theta}{dt^2} + 2\lambda\frac{d\theta}{dt} + \omega^2 \sin\theta = 0$$ Where λ, ω are constants.	ODE	PDE	linear	nonlinear	2
3	The steady state heat conduction equation $$-\frac{\partial}{\partial x}(k_1\frac{\partial u}{\partial x}) - \frac{\partial}{\partial y}(k_2\frac{\partial u}{\partial y}) = -\frac{\partial \rho}{\partial x}$$ Where k_1 and k_2 are constants.	ODE	PDE	linear	nonlinear	2
4	The equation of a vibrating membrane $$\frac{\partial^2 u}{\partial t^2} - a^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) = 0$$ Where a is a constant.	ODE	PDE	linear	nonlinear	2
5	The current $i(t)$ in an LRC series circuit satisfies $$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E'(t)$$ Where L, R, C are constants.	ODE	PDE	linear	nonlinear	2

2 each

(Q16) The eigenvalues of the matrix

$$A = \begin{bmatrix} 9 & 1 & 1 \\ 1 & 9 & 1 \\ 1 & 1 & 9 \end{bmatrix} \text{ are:}$$

(a) 8, 11, 11.

(b) 0, 3, -4.

(c) 3, 3, 4.

(d) 8, 8, 11.

(e) 9, 9, 9.

$$|A - \lambda I| = \begin{vmatrix} 9-\lambda & 1 & 1 \\ 1 & 9-\lambda & 1 \\ 1 & 1 & 9-\lambda \end{vmatrix}$$

$$= -(\lambda - 8)^2 (\lambda - 11)$$

12

(Q17) True or False

(a)	The point x=-2 is a regular singular point of the DE $(x^2-4)y''+3(x-2)y'+5y=0$	T
(b)	X=2 is an ordinary point of the DE $3x(x+2)y''+(x-2)y'-y=0$	T
(c)	We can not find two linearly independent solutions of the DE $xy''+y=0$ in the form of a power series centered at x=2.	F
(d)	The differential operator $[D^2-2D+2]^5$ annihilates the function $x^2e^x\cos x$	T
(e)	$(\frac{1}{e^{i\beta}})^2=\cos(2\beta)-i\sin(2\beta)$	T
(f)	If A is a 4x4 constant matrix and $X_1(t), X_2(t), X_3(t)$ are linearly independent solutions of the homogeneous system $X'=AX$, then $X(t)=c_1X_1(t)+c_2X_2(t)+c_3X_3(t)$ is the general solution of the system.	F
(g)	If $\Phi(t)$ is a fundamental matrix of the system $X'=AX$, then $\Phi'(t)=A\Phi(t)$.	T

3 each