King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math 202 Exam II
Semester II, 2006- (052)
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Name:	KEY		
ID:	KEY		
Sec:	2 (8:00 – 8:50am)	3 (9:00 – 9:50am)	
Serial NO:	KEY		

Q		Points
1		15
2		12
3		15
4		18
5		20
6		20
7		20
8		20
9		15
10		5
11		20
12		40
Total		220

(1) Find a linear differential operator that annihilates the given functions.

[a] $e^{-x} + 2xe^{x} - x^{2}e^{x}$ (#23/167)

$(D+1)(D-1)^3$

8

[b] $3 + e^{x}\cos 2x$ (#25/167)

$D(D^2-2D+5)$

7

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(2) Find the general solution of:

$y''' + 4y'' - 5y' = 0$ (#15/147)

$(D^3+4D^2-5D)y = 0$

auxi: $m^3+4m^2-5m = 0$

$m(m^2+4m-5) = 0$

$m(m+5)(m-1) = 0$

roots: $0, -5, 1$

indep. sol: $1, e^{-5x}, e^{x}$

$y = c_1 + c_2 e^{-5x} + c_3 e^{x}$ is the general sol.

4 4 4

(3) Solve $y''' - 3y'' = 8e^{3x} + 4\sin x$ ——— (1)

Given that $y_c = c_1 + c_2 x + c_3 e^{3x}$ is the general solution for the associated homogeneous equation.

$L_1 = (D^2+1)(D-3)$ is an anihilator for RHS,

(1) can be written as

$$(D^3 - 3D^2)y = 8e^{3x} + 4\sin x$$

$$D^2(D-3)y = 8e^{3x} + 4\sin x$$

$$\Rightarrow (D^2+1)D^2(D-3)^2 y = 0$$

roots : $i, -i, 0, 0, 3, 3$

3 $\tilde{y} = c_1\cos x + c_2 \sin x + \cancel{c_3} + \cancel{c_4 x} + \cancel{c_5 e^{3x}} + c_6 x e^{3x}$

Now, 2 $y_p = A\cos x + B\sin x + C x e^{3x}$

$$y_p' = -A\sin x + B\cos x + Ce^{3x} + 3Cxe^{3x}$$

$$y_p'' = -A\cos x - B\sin x + 3Ce^{3x} + 3Ce^{3x} + 9Cxe^{3x}$$

$$= -A\cos x - B\sin x + 6Ce^{3x} + 9Cxe^{3x}$$

$$y_p''' = A\sin x - B\cos x + 18Ce^{3x} + 9Ce^{3x} + 27Cxe^{3x}$$

$$= A\sin x - B\cos x + 27Ce^{3x} + 27Cxe^{3x}$$

$\boxed{\text{LHS of (1)}} = (A+3B)\sin x + (-B+3A)\cos x + (27C - 18C)e^{3x} + (27C - 27C)xe^{3x}$

$= (A+3B)\sin x + (3A-B)\cos x + 9Ce^{3x} = 8e^{3x} + 4\sin x$

$\Rightarrow 9C = 8$, $3A - B = 0$, $A + 3B = 4$

$\Rightarrow C = 8/9$, $A = 2/5$, $B = 6/5$

$$\boxed{y_p = \frac{2}{5}\cos x + \frac{6}{5}\sin x + \frac{8}{9} x e^{3x}}$$ 8

2 $y = y_c + y_p$ is the general solution.

(4) If we seek a power series solution

$$y = \sum_{n=0}^{\infty} c_n x^n \qquad \text{for the DE}$$

$$(x-1)y'' + y' = 0 \qquad (*)$$

we obtain the recurrence relation

$$c_{k+2} = \frac{(k+1)}{(k+2)} c_{k+1} \qquad k = 0,1,2,3,\ldots$$

Find two power series solutions of (*) about the ordinary point $x = 0$. (#23/248)

$\underline{k=0}$ $\quad c_2 = \frac{1}{2} c_1$

$\underline{k=1}$ $\quad c_3 = \frac{2}{3} c_2 = \frac{2}{3}\left(\frac{1}{2} c_1\right) = \frac{1\cdot 2}{2\cdot 3} c_1$

3 $\quad \underline{k=2}$ $\quad c_4 = \frac{3}{4} c_3 = \frac{3}{4}\left(\frac{1\cdot 2}{2\cdot 3} c_1\right) = \frac{1\cdot 2\cdot 3}{2\cdot 3\cdot 4} c_1$

$\underline{k=3}$ $\quad c_5 = \frac{4}{5} c_4 = \frac{4}{5}\left(\frac{1\cdot 2\cdot 3}{2\cdot 3\cdot 4} c_1\right) = \frac{1\cdot 2\cdot 3\cdot 4}{2\cdot 3\cdot 4\cdot 5} c_1$

$\vdots$

$k = m-2$ $\quad c_m = \frac{m-1}{m} c_{m-1} = \frac{1\cdot 2\cdot 3\cdots(m-1)}{2\cdot 3\cdot 4\cdots(m)} c_1$

$$= \frac{(m-1)!}{m!} c_1 = \frac{1}{m} c_1$$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

$$= c_0 + c_1 x + \frac{1}{2} c_1 x^2 + \frac{1}{3} c_1 x^3 + \frac{1}{4} c_1 x^4 + \cdots$$

$$= c_0 (1) + c_1 \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots\right)$$

$y_1(x) = 1$ and $y_2(x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n}$$

6 $\qquad$ 6

with $|x| < 1$ $\quad$ ($x = 1$ is a singular point)

3

(5) Find a particular solution y_p for

$$y'' + 3y' + 2y = \sin(e^x)$$

given that $y_1 = e^{-x}$, $y_2 = e^{-2x}$ are two linearly independent solutions for the associated homogeneous DE. (#13/172)

2 $W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x}$

2 $W_1 = \begin{vmatrix} 0 & e^{-2x} \\ \sin e^x & -2e^{-2x} \end{vmatrix} = -e^{-2x}\sin(e^x) \Rightarrow u_1' = \frac{W_1}{W} = e^{x}\sin e^{x}$

2 $W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \sin e^x \end{vmatrix} = e^{-x}\sin(e^x) \Rightarrow u_2' = \frac{W_2}{W} = -e^{2x}\sin(e^x)$

2 $u_1 = \int e^x \sin(e^x)\,dx = -\cos(e^x)$

2 $u_2 = -\int e^{2x}\sin(e^x)\,dx = e^x\cos(e^x) - \sin(e^x)$

$y_p = u_1 y_1 + u_2 y_2$

$= -e^{-x}\cos(e^x) - e^{-2x}\sin(e^x) + e^{-x}\cos(e^x)$

$= -e^{-2x}\sin(e^x)$

10 $\boxed{y_p = -e^{-2x} \cdot \sin(e^x)}$

(6) Find two powe series solutions of the DE

$$y'' + (\sin x)y = o$$

about the ordinary point $x = 0$.

[Hint: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ..... = x - \frac{x^3}{6} + \frac{x^5}{120} - .....$]

(#33/248)

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^n = 2c_2 + 6c_3 x + 12 c_4 x^2 + 20 c_5 x^3 + \cdots$$

$$y'' + (\sin x) y =$$

$$= \left[2c_2 + 6c_3 x + 12 c_4 x^2 + 20 c_5 x^3 + \cdots\right] + \left[x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 - \cdots\right]\left[c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots\right]$$

$$= (2c_2) + (6c_3 + c_0)x + (12c_4 + c_1)x^2 + (20c_5 + c_2 - \tfrac{1}{6}c_0)x^3 + (30c_6 + c_3 - \tfrac{1}{6}c_1) + \cdots$$

$$\Rightarrow c_2 = 0, \quad c_3 = -\tfrac{1}{6}c_0, \quad c_4 = -\tfrac{1}{12}c_1, \quad c_5 = \tfrac{1}{120}c_0 - \tfrac{1}{20}c_2 = \tfrac{1}{120}c_0$$

5

$$c_6 = \tfrac{1}{180}c_1 - \tfrac{1}{30}c_3 = \tfrac{1}{180}c_1 + \tfrac{1}{180}c_0$$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$

$$= c_0 + c_1 x + 0 - \tfrac{1}{6}c_0 x^3 - \tfrac{1}{12}c_1 x^4 + \tfrac{1}{120}c_0 x^5 + \left(\tfrac{1}{180}c_0 + \tfrac{1}{180}c_1\right)x^6 + \cdots$$

$$= c_0\left[1 - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 + \tfrac{1}{180}x^6 + \cdots\right] + c_1\left[x - \tfrac{1}{12}x^4 + \tfrac{1}{180}x^6 + \cdots\right]$$

6 $y_1 = 1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{180}x^6 + \cdots$

6 $y_2 = x - \frac{1}{12}x^4 + \frac{1}{180}x^6 + \cdots$

$\Big\}\ |x| < \infty$

3

(7) Solve: $x^2 y'' - 3xy' + 3y = 2x^4 e^x$ ——— (1)

Given that $y_1 = x$ is a solution to the associated homogeneous equation.

[Hint: use the method of reduction of order (sec 4.2) to find a second solution y_2]

(1) written as $y'' - \frac{3}{x} y' + \frac{3}{x^2} y = 2x^2 e^x$

$p(x) = -\frac{3}{x}$, $k(x) = e^{\int \frac{3}{x} dx} = e^{3 \ln x} = x^3$

$y_2 = y_1 \int \frac{k}{y_1^2} dx = x \int \frac{x^3}{x^2} dx = x \int x \, dx = \frac{1}{2} x^3$ 6

$W = \begin{vmatrix} x & \frac{1}{2}x^3 \\ 1 & \frac{3}{2}x^2 \end{vmatrix} = \frac{3}{2}x^3 - \frac{1}{2}x^3 = x^3$

$W_1 = \begin{vmatrix} 0 & \frac{1}{2}x^3 \\ 2x^2 e^x & \frac{3}{2}x^2 \end{vmatrix} = -x^5 e^x \Rightarrow u_1' = \frac{W_1}{W} = -x^2 e^x$

$W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^2 e^x \end{vmatrix} = 2x^3 e^x \Rightarrow u_2' = \frac{W_2}{W} = 2e^x$

$u_1 = -\int x^2 e^x dx = -x^2 e^x + 2x e^x - 2e^x$

$u_2 = 2\int e^x dx = 2e^x$

$y_p = u_1 y_1 + u_2 y_2 = \cancel{-x^3 e^x} + 2x^2 e^x - 2x e^x + \cancel{x^3 e^x}$

6 $y_p = 2x e^x (x - 1)$

8 $y = c_1 x + \frac{1}{2} c_2 x^3 + 2x e^x (x - 1)$

(8) Find a power series solution of the nonhomogeneous equation $y'' - xy = e^x$ about the ordinary point $x = 0$. (#36/248)

[Hint: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\ldots$] $= \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + \cdots$$

$$y'' = 2c_2 + 6c_3 x + 12 c_4 x^2 + 20 c_5 x^3 + 30 c_6 x^4 + 42 c_7 x^5 + \cdots$$

$$LH = y'' - xy = (2c_2 + 6c_3 x + 12 c_4 x^2 + 20 c_5 x^3 + 30 c_6 x^4 + 42 c_7 x^5 + \cdots)$$
$$- (c_0 x + c_1 x^2 + c_2 x^3 + c_3 x^4 + c_4 x^5 + c_5 x^6 + \cdots)$$

$$= (2c_2) + (6c_3 - c_0)x + (12 c_4 - c_1) x^2 + (20 c_5 - c_2) x^3$$
$$+ (30 c_6 - c_3) x^4 + (42 c_7 - c_4) x^5 + \cdots$$

$$RH = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \frac{1}{120} x^5 + \cdots$$

match coeff from LH and RH

$$\boxed{\begin{array}{l} 2c_2 = 1 \;;\; 6c_3 - c_0 = 1 \;;\; 12c_4 - c_1 = \frac{1}{2} \\ 20c_5 - c_2 = \frac{1}{6} \;;\; 30c_6 - c_3 = \frac{1}{24} \;;\; 42c_7 - c_4 = \frac{1}{120} \end{array}}$$ 3

$\Downarrow$ 3

$$\boxed{\begin{array}{l} c_2 = \frac{1}{2} \;;\; c_3 = \frac{1}{6} + \frac{1}{6} c_0 \;;\; c_4 = \frac{1}{24} + \frac{1}{12} c_1 \\ c_5 = \frac{1}{120} + \frac{1}{20} c_2 = \frac{1}{30} \;;\; c_6 = \frac{1}{30}\left(\frac{1}{24} + c_3\right) = \frac{1}{144} + \frac{1}{180} c_0 \;;\; c_7 = \frac{1}{840} + \frac{1}{504} c_1 \end{array}}$$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + \cdots$$
$$= c_0 + c_1 x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{6} c_0 x^3 + \frac{1}{24} x^4 + \frac{1}{12} c_1 x^4 + \frac{1}{30} x^5 + \frac{1}{144} x^6 + \frac{1}{180} c_0 x^6 + \cdots$$
$$= c_0 \underbrace{\left[1 + \frac{1}{6} x^3 + \frac{1}{180} x^6 + \cdots\right]}_{y_1} + c_1 \underbrace{\left[x + \frac{1}{12} x^4 + \frac{1}{504} x^7 + \cdots\right]}_{y_2} + \underbrace{\frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \frac{1}{30} x^5 + \frac{1}{144} x^6 + \cdots}_{y_p}$$

4 4 4

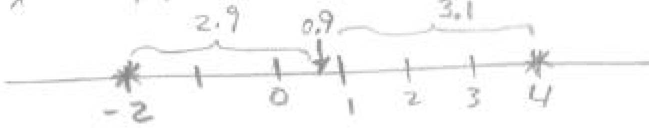
2 $|x| < \infty$

(9) Without solving the DE

$$(x+2)(x-4)y'' + y' + 5y = 0.$$

A power series solution about the ordinary point $x = 0.9$ converges at least on some interval defined by

$x = -2, 4$ are singular points



a) $|x-0.9| < 2.9$

b) $|x-0.9| < 3.1$

c) $|x| < 2$

d) $|x-0.9| < 2.1$

e) $|x-0.9| < 3.9$

$|x - 0.9| < 2.9$

5

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(10) If $y_1 = 1,\ y_2 = x,\ y_3 = e^{2x}$ then

$W(y_1, y_2, y_3) =$

$$W = \begin{vmatrix} 1 & x & e^{2x} \\ 0 & 1 & 2e^{2x} \\ 0 & 0 & 4e^{2x} \end{vmatrix} = 4e^{2x}$$

a) e^{2x}

b) $8e^{2x}$

c) 1

d) $4e^{2x}$

e) $8e^{6x}$

5

(11) Given that

y_{p_1} is a particular solution of $y'' - xy' = 4 - 4x^2 = g_1$

y_{p_2} is a particular solution of $y'' - xy' = 2 - 2x^2 - x = g_2$

y_{p_3} is a particular solution of $y'' - xy' = 3x = g_3$

then $y_{p_3} =$

a) $-y_{p_2} + \frac{1}{2} y_{p_1}$

b) $y_{p_1} + y_{p_2}$

c) $3y_{p_2} - \frac{3}{2} y_{p_1}$

d) $\frac{1}{2} y_{p_2} - y_{p_1}$

e) $-3y_{p_2} + \frac{3}{2} y_{p_1}$

$$3x = g_3 = c_1 g_1 + c_2 g_2$$

$$= c_1(4-4x^2) + c_2(2-2x^2-x)$$

$$= (-4c_1 - 2c_2)x^2 + (-c_2)x + (4c_1 + 2c_2)$$

$$\Rightarrow \left.\begin{array}{l} -4c_1 - 2c_2 = 0 \\ -c_2 = 3 \\ 4c_1 + 2c_2 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} c_2 = -3 \\ c_1 = 3/2 \end{array}$$

20

$$3x = g_3 = \frac{3}{2} g_1 - 3 g_2$$

$$\Rightarrow y_{p_3} = \frac{3}{2} y_{p_1} - 3 y_{p_2}$$

5 each

(12) True or False:

a) $L = D^2 - xD$ is a linear operator. (T)

b) If y_1, y_2 are two solutions of an n-th order nonhomogeneous differential equation on an interval I then the linear combination $y = c_1 y_1 + c_2 y_2$ where c_1, c_2 are arbitrary constants, is also a solution. (F)

c) If a set of two functions is linearly dependent, then one function is simply a constant multiple of the other. (T)

d) The set of functions $f_1(x) = x$, $f_2(x) = |x|$ is linearly independent on $(-\infty, -10)$. (F)

e) $y_1 = \frac{1}{5} e^{3x}$ and $y_2 = -e^{-3x}$ form a fundamental set of solutions of the DE $y'' - 9y = 0$. (T)

f) $y = 3$ is the complementary function for the equation $y'' + 9y = 27$. (F)

g) The point $x = 0$ is an ordinary point of the DE $y'' + (e^x) y' + (\sin x) y = 0$. (T)

h) The point $x = 0$ is a singular point of the DE $y'' + (e^x) y' + (\ln x) y = 0$. (T)

☺ Wish you a FULL MARK