King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math 102
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Second Major Exam
Semester I, 2001–2002 (011)

Name: KEY ID #: KEY

Section #: 7 (9:00 – 9:50), 11 (10:00 – 10:5) (please circle one)

Serial #: ________

Problem #		Points
1		16
2		16
3		16
4		16
5		16
6		16
7		25
8		25
9		25
10		29
Total:		200

FORM(A)

1. If $g(x) = \sqrt{x} f(x)$, $f(1) = 8$ and $f'(1) = 5$ then $g'(1) =$

(a) 5
(b) 4
(c) 9
(d) 13
(c) 40

$$g'(x) = \frac{1}{2\sqrt{x}} f(x) + \sqrt{x}\, f'(x)$$

$$g'(1) = \frac{1}{2} f(1) + f'(1)$$

Form (A) & (C) : $g'(1) = \frac{1}{2}(8) + 5 = 9 \Rightarrow$ c

Form (B) & (D) : $g'(1) = \frac{1}{2}(8) + 9 = 13 \Rightarrow$ d

2. Let $y = \exp(\sqrt{1 + 5x^3})$ then $y''(0) - y'(0) =$

(a) e
(b) 0
(c) $-e$
(d) $2e$
(e) $-2e$

$$y' = e^{\sqrt{1+5x^3}} \cdot \left(\sqrt{1+5x^3}\right)'$$

$$= e^{\sqrt{1+5x^3}} \cdot \frac{15x^2}{2\sqrt{1+5x^3}}$$

$$y'' = \left(e^{\sqrt{1+5x^3}}\right)' \cdot \frac{15x^2}{2\sqrt{1+5x^3}} + e^{\sqrt{1+5x^3}} \cdot \frac{60x\sqrt{1+5x^3} - \frac{15x^2}{\sqrt{1+5x^3}} \cdot (15x^2)}{4(1+5x^3)}$$

Now, $y'(0) = 0$, $y''(0) = 0 + 0 = 0$

Form (A) & (C): $y''(0) - y'(0) = 0 - 0 = 0 \Rightarrow$ b

Form (B) & (D) : $y''(0) + y'(0) = 0 + 0 = 0 \Rightarrow$ b

3. Let $f(x) = 5x - \sin 2x$ and $g(x) = f^{-1}(x)$ then $g'(\pi) =$

   (a) $\frac{1}{7}$

   (b) $\frac{1}{5}$

   (c) 3

   (d) $\frac{1}{3}$

   (e) 5

Let $y = 5x - \sin 2x$

$\frac{dy}{dx} = 5 - 2\cos 2x$

Now, $\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{5 - 2\cos 2x}$

So $g'(x) = \frac{d}{dx}\left[f^{-1}(x)\right] = \frac{dx}{dy} = \frac{1}{5 - 2\cos 2x}$

Form (A) & (B) : $g'(\pi) = \frac{1}{5 - 2\cos(2\pi)} = \frac{1}{5-2} = \frac{1}{3} \Rightarrow$ (d)

Form (C) & (D) : $g'(\frac{\pi}{2}) = \frac{1}{5 - 2\cos \pi} = \frac{1}{5+2} = \frac{1}{7} \Rightarrow$ (a)

4. When a spherical ball of metal is heated, the radius of the sphere increases by 0.1% per degree increase in temperature. Use differential to estimate the percentage increase in the volume of the ball per degree increase in temperature. ($V = \frac{4}{3}\pi r^3$) The estimate is:

   (a) 1.2%

   (b) 0.3%

   (c) 0.6%

   (d) 2.4%

   (e) 0.1%

$V = \frac{4}{3}\pi r^3 \Rightarrow dV = 4\pi r^2 dr$

$\frac{dV}{V} = \frac{4\pi r^2 \, dr}{\frac{4}{3}\pi r^3} \Rightarrow \frac{dV}{V} = 3\,\frac{dr}{r}$

$\Rightarrow \left(\frac{dV}{V} * 100\right)\% = 3\left(\frac{dr}{r} * 100\right)\%$

$\Rightarrow$ percentage in V = 3 (percentage in r)

$\boxed{\text{Form (A) \& (B)}}$: percentage in $V = 3\,(0.1\%) = 0.3\% \Rightarrow$ (b)

$\boxed{\text{Form (C) \& (D)}}$: percentage in $V = 3\,(0.2\%) = 0.6\% \Rightarrow$ (c)

5. The slope of the tangent line to the curve $y = \sin^{-1}(\tan x) + \tan^{-1}(1 + \ln(x+1))$ at the point $x = 0$ equals

(a) $\frac{3}{2}$

(b) $\frac{1}{2}$

(c) 0

(d) $\frac{2}{3}$

(e) 1

Form (A) & (B) : $y' = \dfrac{\sec^2 x}{\sqrt{1-\tan^2 x}} + \dfrac{1/(x+1)}{1+(1+\ln(x+1))^2}$

slope $= y'(0) = \dfrac{\sec^2(0)}{\sqrt{1-0}} + \dfrac{1/(0+1)}{1+(1+0)^2} = 1 + \frac{1}{2}$

$y'(0) = \frac{3}{2} \Rightarrow$ (a)

Form (C) & (D) : $y' = \dfrac{\frac{1}{2}\sec^2 x}{\sqrt{1-\frac{1}{4}\tan^2 x}} + \dfrac{1/(x+1)}{1+(1+\ln(x+1))^2}$

slope $= y'(0) = \dfrac{\frac{1}{2}\sec^2(0)}{\sqrt{1-0}} + \dfrac{1/(0+1)}{1+(1+0)^2} = \frac{1}{2} + \frac{1}{2} = 1$

$\Rightarrow$ (e)

6. Use a differential to approximate $(1.986)^4$

(a) 15.4880

(b) 15.5567

(c) 15.5520

(d) 15.4880

(e) 15.3600

Let $f(x) = x^4$ and $x_0 = 2$

$f'(x) = 4x^3$

Now, $f(2) = 16$, $f'(2) = 32$

$f(x) \cong l(x) = f(2) + f'(2)(x-2)$

$f(x) \cong 16 + 32(x-2)$

Form (A) & (B) : $f(1.986) \cong 16 + 32(1.986-2) = 15.5520$

$\Rightarrow$ (c)

Form (C) & (D) : $f(1.984) \cong 16 + 32(1.984-2) = 15.4880$

$\Rightarrow$ (d)

7. Let $x\cos y = y - \frac{\pi}{2}$. Find $y''(0)$.

by implicit diff

$$\cos y - x y' \sin y = y' \Rightarrow y' + x y' \sin y = \cos y$$

$$y'(1 + x \sin y) = \cos y \Rightarrow \boxed{y' = \frac{\cos y}{1 + x \sin y}} \quad (*)$$

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Now, let us find $y(0)$:

substitute $x = 0$ in the original equation

$$(0) \cdot \cos[y(0)] = y(0) - \frac{\pi}{2}$$

$$0 = y(0) - \frac{\pi}{2} \Rightarrow \boxed{y(0) = \frac{\pi}{2}}$$

5

Now, let us find $y'(0)$:

$$y'(0) = \frac{\cos[y(0)]}{1 + (0) \cdot \sin[y(0)]} = \frac{\cos(\frac{\pi}{2})}{1 + 0} = 0$$

5

To find $y''(x)$, we diff $(*)$:

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$$y'' = \frac{-y' \sin y \,(1 + x \sin y) - (\sin y + x y' \cos y) \cos y}{(1 + x \sin y)^2}$$

$$y''(0) = \frac{0 - (\sin[y(0)] + 0 \cdot y'(0) \cdot \cos[y(0)]) \cos[y(0)]}{(1 + 0 \cdot \sin[y(0)])^2}$$

$$= \frac{-(\sin \frac{\pi}{2} + 0) \cos \frac{\pi}{2}}{(1 + 0)^2} = \frac{-(1+0)(0)}{1} = 0$$

Hence, $\boxed{y''(0) = 0}$

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8. Find $\lim\limits_{h->0}(1-3h)^{\frac{2}{h}}$ let $\omega = -3h \Rightarrow h = -\frac{1}{3}\omega$

$h \to 0$ as $\omega \to 0$ 5

Now, let $z = \lim\limits_{h \to 0} (1-3h)^{2/h}$

$$= \lim_{\omega \to 0} (1+\omega)^{\frac{2}{-\frac{1}{3}\omega}}$$

$$= \lim_{\omega \to 0} (1+\omega)^{-\frac{6}{\omega}}$$ 5

$$= \lim_{\omega \to 0} \left[(1+\omega)^{\frac{1}{\omega}}\right]^{-6}$$

$$= \left[\lim_{\omega \to 0} (1+\omega)^{\frac{1}{\omega}}\right]^{-6}$$ 5

$$= [e]^{-6} = e^{-6}$$

5 5

6

9. Let $h(x) = \dfrac{f(g(x))}{g(x)}$. Find $h''(2)$. Given that:

$$\boxed{** \ (fgh)' = f'gh + fg'h + fgh'}$$

$$f(2) = 2, \quad f'(2) = f''(2) = -1$$
$$g(2) = 2, \quad g'(2) = g''(2) = -1$$

$$h'(x) = \frac{[f(g(x))]' \cdot g(x) - g'(x) \cdot f(g(x))}{g^2(x)} \quad 5$$

$$= \frac{f'(g(x)) \cdot g'(x) \cdot g(x) - g'(x) \cdot f(g(x))}{g^2(x)} = \frac{(\star)}{g^2(x)}$$

$$h''(x) = \frac{(\star)' \cdot g^2(x) - 2 \cdot g(x) \cdot g'(x) \cdot (\star)}{g^4(x)} \quad 10$$

use (**) to find $(\star)'$

$$(\star)' = f''(g(x)) \, [g'(x)]^2 \cdot g(x) + f'(g(x)) \cdot g''(x) \, g(x) + f'(g(x)) [g'(x)]^2 - g''(x) \cdot f(g(x)) - [g'(x)]^2 f'(g(x))$$

Now, $(\star)(2) = f'(g(2)) \cdot g'(2) \cdot g(2) - g'(2) \cdot f(g(2))$
$= f'(2) \cdot (-1) \cdot (2) - (-1) \cdot f(2) = 2 + 2 = 4$

$$(\star)'(2) = f''(2) \cdot [g'(2)]^2 \cdot g(2) + f'(2) \cdot g''(2) \cdot g(2) + f'(2) [g'(2)]^2 - g''(2) \cdot f(2) - [g'(2)]^2 \cdot f'(2)$$

$= (-1)(1) \cdot (2) + (-1)(-1)(2) + (-1)(1) - (-1)(2) - (1) \cdot (-1)$

$= -2 + 2 - 1 + 2 + 1 = 2 \quad 10$

$$h''(2) = \frac{2 \cdot g^2(2) - 2 \, g(2) \, g'(2) \cdot 4}{g^4(2)} = \frac{2 \cdot (2)^2 - (2)(2)(-1)(4)}{(2)^4} = \frac{3}{2}$$

10. Let $F(x) = f(2g(x))$ where $f(x) = x^4 + x^3 + 1$ for $0 \le x \le 2$ and $g(x) = f^{-1}(x)$. Find $F''(3)$.

$f'(x) = 4x^3 + 3x^2$, $f''(x) = 12x^2 + 6x$

$f'(1) = 7$, $f'(2) = 44$ $f''(1) = 18$, $f''(2) = 60$

Find $g(3)$: $z = g(3) \iff f(z) = 3 \Rightarrow z^4 + z^3 + 1 = 3$

$\Rightarrow z^4 + z^3 - 2 = 0 \Rightarrow z = 1 \Rightarrow \boxed{g(3) = 1}$

Find $g'(3)$: $g'(x) = \dfrac{1}{f'(g(x))} \Rightarrow g'(3) = \dfrac{1}{f'(g(3))} = \dfrac{1}{f'(1)}$

$\Rightarrow g'(3) = \dfrac{1}{7} \Rightarrow \boxed{g'(3) = \frac{1}{7}}$

Find $g''(3)$: $g''(x) = -\dfrac{[f'(g(x))]'}{[f'(g(x))]^2} = -\dfrac{f''(g(x)) \cdot g'(x)}{[f'(g(x))]^2}$

$g''(3) = -\dfrac{f''(g(3)) \cdot g'(3)}{[f'(g(3))]^2} = -\dfrac{f''(1) \cdot \frac{1}{7}}{[f'(1)]^2} = -\dfrac{18 \cdot \frac{1}{7}}{7^2} = -\dfrac{18}{7^3}$

$\boxed{g''(3) = \frac{18}{7^3}}$

Find $F''(x)$: $F'(x) = f'(2g(x)) \cdot 2 \cdot g'(x)$

$F''(x) = [f'(2g(x))]' \cdot 2 \cdot g'(x) + f'(2g(x)) \cdot 2 \cdot g''(x)$

$= f''(2g(x)) \cdot 2 \cdot g'(x) \cdot 2 \cdot g'(x) + f'(2g(x)) \cdot 2 \cdot g''(x)$

$F''(3) = f''(2g(3)) \cdot 2 \cdot g'(3) \cdot 2 \cdot g'(3) + f'(2g(3)) \cdot 2 \cdot g''(3)$

$= f''(2) \cdot 2 \cdot \frac{1}{7} \cdot 2 \cdot \frac{1}{7} + f'(2) \cdot 2 \cdot \left(-\frac{18}{7^3}\right)$

$= \dfrac{240}{7^2} - \dfrac{1584}{7^3} = \dfrac{1680 - 1584}{343} = \dfrac{96}{343} = 0.2799$