EXTENSION OF THE BRAMBLE PASCIAK PRECONDITIONER

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Abstract. Murphy, Golub and Wathen [1] propose the block diagonal and block triangular Schur complement preconditioners for systems of saddle-point (or KKT) form. These preconditioners are extended by inserting a nonzero parameter α in (2,2) block.

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AMS subject classifications. 65F10, 15A23, 65N99

1. Introduction.

Consider the symmetric saddle point problem

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \tag{1.1}$$

with symmetric positive definit $A \in \mathbb{R}^{n\times m}$, symmetric positive semidefinite $C \in \mathbb{R}^{m\times m}, n \geq m$, and $B \in \mathbb{R}^{m\times n}$ of row full rank m, if preconditioned on the left by

$$P = \begin{bmatrix} A_0 & 0 \\ B & -I \end{bmatrix} \qquad \text{with} \qquad P^{-1} = \begin{bmatrix} A_0^{-1} & 0 \\ BA_0^{-1} & -I \end{bmatrix}, \tag{1.2}$$

results in the nonsymmetric matrix

$$T = P^{-1}\mathcal{A} \begin{bmatrix} A_0^{-1}A & A_0^{-1}B^T \\ BA_0^{-1}A - B & BA_0^{-1}B^T \end{bmatrix} \tag{1.3}$$

which turns out to be self-adjoint (symmetric) in the inner product $< \cdot, \cdot >_H$ defined by $< u, v >_H := u^T H v$ where

$$H = \begin{bmatrix} A - A_0 & 0 \\ 0 & I \end{bmatrix}. \tag{1.4}$$

2. The generalized Bramble-Pasciak preconditioner.

The Bramble-Pasciak CG method requires that the matrix

$$H = \begin{bmatrix} A - A_0 & 0 \\ 0 & I \end{bmatrix}. \tag{2.1}$$

is positive definite so it is necessarly to scale the matrix A_0 such that $A - A_0$ is positive definite. We introduce a nonzero parameter γ_1 and γ_2 in this matrix H. So we have now two parameters and the matrix A_0 to work on in order to meet this positive requirement giving us more freedom of choices. We introduce the preconditioner

$$P = \begin{bmatrix} A_0 & 0 \\ \frac{1}{\gamma_1}B & \gamma_2 I \end{bmatrix} \qquad \text{and} \qquad P^{-1} = \begin{bmatrix} A_0^{-1} & 0 \\ -\frac{1}{\gamma_1\gamma_2}BA_0^{-1} & \frac{1}{\gamma_2}I \end{bmatrix} \tag{2.2}$$

and by left preconditioning with P, we obtain

$$T = P^{-1}\mathcal{A} \begin{bmatrix} A_0^{-1}A & A_0^{-1}B^T \\ -\frac{1}{\gamma_1\gamma_2}BA_0^{-1}A + \frac{1}{\gamma_2}B & -\frac{1}{\gamma_1\gamma_2}BA_0^{-1}B^T \end{bmatrix}. \tag{2.3}$$

Simple algebra shows that T is self-adjoint in the inner product induced by

$$H = \begin{bmatrix} A - \gamma_1 A_0 & 0 \\ 0 & -\gamma_1\gamma_2 I \end{bmatrix}. \tag{2.4}$$

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