SOBER SPACES AND SOBER SETS

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Abstract. We introduce and study the notion of sober partially ordered sets. Some questions about sober spaces are also stated.

0. Introduction

The notion of sober spaces was introduced, firstly by the Grothendieck school. Several authors have been interested on sober spaces (see for example [1], [7], [9], [14]-[18], [1], [21], [23]-[25], [27]-[29])

In [11], H. Herrlich showed that sober spaces constitute the reflective hull of the Sierpinski space in the category TOP of topological spaces and continuous maps.

In [28], L. Skula has introduced sober spaces as b-closed subspaces of powers of the Sierpinski space.

Other investigations have described them in terms of certain open filters (see H. Herrlich [11] and S. S. Hong [19]).

R. Hoffman [14], has given a definition of "sober spaces" in terms of irreducible filters; this definition is "more natural" then any other topological approach to sober spaces.

Let L be a complete lattice. An element $x \in L$ is said to be coprime if for all finite subset F of L, if $x \leq \vee F$, then $F \cap (x \uparrow) \neq \emptyset$. We denote by $Spec_{\vee}(L)$ the family of all coprimes of L.

The hull-kernel topology on $Spec(L)$ is the topology whose closed sets are the $C = (\downarrow x) \cap Spec_{\vee}(L)$, where $x \in L$. A closed subset of a topological space X is irreducible if C is coprime in $\Gamma(X)$ (the lattice of closed sets of X). The space X is said to be *sober* if every irreducible closed subset C of X has a unique generic point (i.e., $C = \overline{\{x\}}$ for a unique $x \in X$).

We let SOB denote the full subcategory of TOP whose objects are sober spaces.

Recall that a continuous map $q : Y \longrightarrow Z$ is said to be a *quasihomeomorphism* if $U \longmapsto q^{-1}(U)$ defines a bijection $\mathcal{O}(Z) \longrightarrow \mathcal{O}(Y)$ [10], where $\mathcal{O}(Y)$ is the set of all open subset of the space Y. A subset S of a topological space X is said to be *strongly dense* in X, if S meets every nonempty locally closed subset of X [10]. Thus a subset S of X is strongly dense if and only if

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2000 *Mathematics Subject Classification.* 06B30; 06F30; 54A10; 54F05.

Key words and phrases. Alexandroff space; One-point compactification; Sober space; Spectral space.

The two authors are partially supported by the DGRST 03/UR/15-03.

the canonical injection $S \hookrightarrow X$ is a quasihomeomorphism. It is well known that a continuous map $q : X \longrightarrow Y$ is a quasihomeomorphism if and only if the topology of X is the inverse image by q of that of Y and the subset $q(X)$ is strongly dense in Y [10]. The notion of quasihomeomorphism is used in algebraic geometry and it has recently been shown that this notion arises naturally in the theory of some foliations associated to closed connected manifolds (see the papers [4], [5]).

The following properties are routine.

(a) $Spec_{\vee}(\Gamma(X))$ is a sober space.

(b) $\mu_X : X \longrightarrow Spec_{\vee}(\Gamma(X))$ is a quasihomeomorphism.

(c) The functor $Spec_{\vee} \circ \Gamma : TOP \longrightarrow SOB$ is left adjoint to the inclusion functor $I : SOB \hookrightarrow TOP$.

It is clear, that if X is a T_0-space, then X can be universally embedded into its soberification $Spec_{\vee}(\Gamma(X))$.

Given a poset $(X, \leq)$ and $x \in X$, the *generization* of x in X is $(\downarrow x) = \{y \in X \mid y \leq x\}$, the *specialization* of x in X is $(x \uparrow) = \{y \in X \mid y \geq x\}$. Let X have a topology $\mathcal{T}$ and a partial ordering $\leq$. We say that $\mathcal{T}$ is *compatible* with $\leq$ if $\overline{\{x\}} = (x \uparrow)$ for all $x \in X$.

We will say that a poset $(X, \leq)$ is a *sober set* if there is a topology $\mathcal{T}$ on X which is compatible with the ordering $\leq$.

The main purpose of this paper is the investigation of sober sets.

Let $(X, \mathcal{T})$ be a space which has a basis of compact open sets. We denote $X^{\star}$ its dual (equipped with the cocompact topology). We prove in Theorem 2.7, that if X and $X^{\star}$ are sober, then $(X, \leq)$ satisfies Kaplansky's two conditions K_1 and K_2, where $\leq$ is the ordering induced by the topology $\mathcal{T}$.

When $(X, \leq)$ is a totally ordered set, we prove, in Theorem 2.9, that $(X, \leq)$ is a sober set if and only if each nonempty subset of X has an infimum.

It is also proved that a space X is sober if and only if its one-point compactification is sober. Some questions concerning T_0-compactifications and Wallman compactification are also stated.

1. Sober Spaces and Sober Sets

Let X be a topological space.

(1) The open sets of the hull-kernel topology on $Spec_{\vee}(\Gamma(X))$ are of the form $U^s = \{C \in Spec_{\vee}(\Gamma(X)) \mid C \cap U \neq \emptyset\}$, where U is open in X.

(2) If $C \in Spec_{\vee}(\Gamma(X))$, then the closure $\overline{\{C\}}$ of $\{C\}$ for the hull-kernel topology is $(\downarrow C)$. Thus the hull-kernel topology on $Spec_{\vee}(\Gamma(X))$ is compatible with the order $\supseteq$.

Let $\leq$ be a quasiorder on a set X. A subset Y of X is said to be *left-directed* in $(X, \leq)$ if for each $x, y \in Y$, there is some $z \in Y$ such that $z \leq x$ and $z \leq y$.

Let $(X, \mathcal{T})$ be a topological space. By the *quasiorder induced by the topology* $\mathcal{T}$, we mean the binary relation $\leq$ defined on X by; $x \leq y$ if and only if $y \in \overline{\{x\}}$.

Remarks 1.1. (1) Let $(X, \mathcal{T})$ be a topological space and $\leq$be the quasiorder induced by the topology $\mathcal{T}$. If Y is a subset of X and Y is left-directed, then Y is irreducible in $(X, \mathcal{T})$.

(2) Suppose that $(X, \mathcal{T})$ is an Alexandroff space (any intersection of open sets of $(X, \mathcal{T})$ is open). Then a subset $Y \subseteq X$ is irreducible if and only if Y is left-directed (see R. Hoffman [17, Lemma 1.1]).

The following definitions are natural.

Definitions 1.2. (1) *Let $(X, \leq)$ be a quasiordered set and Y a nonempty subset of X. By a quasiinfimum (resp. quasisupremum) of Y, we mean an element $x \in X$ such that $x \leq y$ (resp. $y \leq x$) for each $y \in Y$; and if $z \in X$ is such that $z \leq y$ (resp. $y \leq z$) for each $y \in Y$, we have $z \leq x$ (resp. $x \leq z$).*

(2) *A sequence $(x_n)_{n \in \mathbb{N}}$ in a quasiordered set is said to be quasistationnary if there is $p \in \mathbb{N}$ such that $x_n \leq x_p$ and $x_p \leq x_n$ for each $n \geq p$.*

A space X is called *quasisober* [19] if and only if every nonempty irreducible closed subset has at least a generic point. Clearly, a space is sober if and only if it is a quasisober T_0-space.

Proposition 1.3. *Let $(X, \mathcal{T})$ be a quasisober space and $\leq$ the quasiorder induced by the topology $\mathcal{T}$. Then each left-directed subset of $(X, \leq)$ has a quasiinfimum in its closure.*

Proof. Let Y be a left-directed subset of X. Then Y is an irreducible subset of X, by Remark 1.1 $\overline{Y}$ has a generic point a; $\overline{\{a\}} = \overline{Y}$. Thus $a \leq y$, for each $y \in Y$. On the other hand, if $z \in X$ is such that $z \leq y$, for each $y \in Y$, then $Y \subseteq \overline{\{z\}}$. Therefore, $\overline{Y} \subseteq \overline{\{z\}}$; and consequently, $z \leq a$, proving that a is an infimum of Y. ☐

Recall that an Alexandroff space is a topological space in which any intersection of open sets is open. A T_0-Alexandroff space is sometimes called discrete Alexandroff. It is worth noting that Alexandroff spaces have several applications especially in digital topology and theoretical computer science. Also, Alexandroff T_0-spaces have been studied, by Ivashchenko, as discrete topological models of continuous spaces in theoretical physics [20].

The following result gives a complete characterization of Alexandroff quasisober spaces.

Theorem 1.4. *Let X be an Alexandroff space. Then the following statements are equivalent:*

(*i*) *X is quasisober;*

(*ii*) *Each decreasing sequence of $(X, \leq)$ is quasistationary, where $\leq$ is the quasiorder induced by the topology.*

Proof. $(i) \Longrightarrow (ii)$ Let $(x_n)_{n\in\mathbb{N}}$ be a decreasing sequence of $(X, \leq)$. Since $Y = \{x_n \mid n \in \mathbb{N}\}$ is left-directed, then Y has a quasiinfimum a in $\overline{Y}$, by Proposition 1.3; so that $\overline{\{a\}} = \overline{Y}$.

But $(\downarrow a)$ is an open subset of X containing a, this yields $(\downarrow a) \cap Y \neq \emptyset$. Hence there is $p \in \mathbb{N}$ such that $x_p \in (\downarrow a)$. Therefore, $x_n \leq x_p \leq a \leq x_n \leq x_p$, for each $n \geq p$, proving that the sequence $(x_n)_{n\in\mathbb{N}}$ is quasistationary.

$(ii) \Longrightarrow (i)$ Let C be an irreducible closed subset of X. Suppose that C has no generic point. Pick $x_0 \in C$. Then $\overline{\{x_0\}} \subseteq C$; so that there is $y_0 \in C$ such that $y_0 \notin \overline{\{x_0\}}$; consequently, $\overline{\{x_0\}} \cup \overline{\{y_0\}} \subseteq C$. Since C is left-directed (see Remarks 1.1), there is $x_1 \in C$ such that $x_1 \leq x_0$ and $x_1 \leq y_0$.

Necessarily, $x_0 \not\leq x_1$; if not $\overline{\{x_0\}} = \overline{\{x_1\}}$ and thus $y \in \overline{\{x_1\}} = \overline{\{x_0\}}$, a contradiction. One may do the same thing for $x_1 \in C$ in order to get $x_2 \in C$ such that $x_2 \leq x_1$ and $x_1 \not\leq x_2$ etc ... This procedure provides a decreasing sequence $(x_n)_{n\in\mathbb{N}}$ of elements of C which is not quasistationary, a contradiction. It follows that C has a generic point. □

Links between sober and quasisober spaces is given by S. S. Hong in [19, Proposition 2.2], by proving that a space is quasisober if and only if its T_0-reflection is sober.

The following provides a more general result concerning sober and quasisober spaces.

Theorem 1.5. *Let $q : X \longrightarrow Y$ be a quasihomeomorphism. Then the following properties hold:*
(1) *If q is onto, then the following statements are equivalent:*
  (i) *X is quasisober;*
  (ii) *Y is quasisober.*
(2) *Suppose that Y is sober; then the following statements are equivalent*
  (i) *q is onto.*
  (ii) *X is quasisober.*

Proof. (1) $(i) \Longrightarrow (ii)$ Remark that, we need not suppose q onto for this implication.

Suppose that X is quasisober. Let C be an irreducible closed subset of Y. Then $q^{-1}(C)$ is an irreducible closed subset of X(see [2, Lemma 3.1]). Hence there exists $x \in q^{-1}(C)$ such that $q^{-1}(C) = \overline{\{x\}}$. Thus $q^{-1}(C) = \overline{\{x\}} \subseteq q^{-1}(\overline{\{q(x)\}}) \subseteq q^{-1}(C) = \overline{\{x\}}$.

It follows that $q^{-1}(C) = q^{-1}(\overline{\{q(x)\}})$. Therefore, $C = \overline{\{q(x)\}}$ so that C has a generic point.

We conclude that Y is quasisober.

$(ii) \Longrightarrow (i)$ We begin by remarking that if q is an onto quasihomeomorphism, then q is a closed map and $q^{-1}(q(C)) = C$, for each closed subset C of X (see [8, Lemma 1.1]).

Let C be an irreducible closed subset of X.

According to [2, Lemma 3.1] and the fact that q is a quasihomeomorphism, there exists an irreducible closed subset D of Y such that $C = q^{-1}(D)$.

Hence D has a generic point y. Let $x \in X$ such that $y = q(x)$. Thus $C = q^{-1}(\overline{\{q(x)\}})$. But q is a closed map; this yields

$$C = q^{-1}(\overline{\{q(x)\}}) = q^{-1}(q(\overline{\{x\}})) = \overline{\{x\}}.$$

Therefore, X is quasisober.

(2) It remains to prove that, if X is quasisober, then q is onto.

Assume that X is quasisober. For each $y \in Y$, $q^{-1}(\overline{\{y\}})$ is an irreducible closed subset of X, by [2, Lemma 3.1]. Hence $q^{-1}(\overline{\{y\}})$ has a generic point x. clearly, we have

$$\overline{\{x\}} \subseteq q^{-1}(\overline{\{q(x)\}}) \subseteq q^{-1}(\overline{\{y\}}) = \overline{\{x\}}.$$

Thus $q^{-1}(\overline{\{q(x)\}}) = q^{-1}(\overline{\{y\}})$; so that $\overline{\{q(x)\}} = \overline{\{y\}}$; but since Y is a T_0-space, we get $q(x) = y$.

This proves that q is an onto map. □

Let us recall the T_0-reflection of a topological space. It is well known that to each topological X space there is a universal T_0-space: Let $\sim$ be the equivalence relation defined on X by; $x \sim y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$. We denote by $\mathbf{T}_0(X)$ the quotient space $X/\sim$, it is called the T_0-reflection of X.

Corollary 1.6. [19, Proposition 2.2] *Let X be a topological space. Then X is quasisober if and only if its T_0-reflection is sober.*

Proof. Let $\mathbf{T}_0(X)$ be the T_0-reflection of X. Then the canonical map $\mu_X : X \longrightarrow \mathbf{T}_0(X)$ is an onto quasihomeomorphism, so that Theorem 1.5 completes the proof. □

2. Quasisober sets

Definition 2.1. A quasiordered set $(X, \leq)$ is said to be *quasisober* if there exists a quasisober topology X which is compatible with $\leq$ (i.e., $\overline{\{x\}} = (x \uparrow)$, for each $x \in X$). When $\leq$ is an ordering and $(X, \leq)$ is quasisober, we say that $(X, \leq)$ is a *sober set.*

Remarks 2.2. (1) Let $(X, \leq)$ be a quasiordered set. The order-reflection of $(X, \leq)$ is the poset $(Y, \preceq)$, where Y is the quotient set of X by the equivalence relation $\equiv$ defined by $x \equiv y$ if and only if $x \leq y$ and $y \leq x$, the order $\preceq$ on Y is defined by $\overline{x} \preceq \overline{y}$ if and only if $x \leq y$ (where $\overline{x}$ is the equivalence class of x).

(2) If $\mathcal{T}$ is a topology which is compatible with the quasisober $\leq$, then the quotient space $X/\mathcal{T}$ is homeomorphic to the T_0-reflection of X and the order induced by the quotient topology is isomorphic to the order $\preceq$ defined on Y.

Combining Remarks 2.2, and Corollary 1.6, we get easily the following:

Proposition 2.3. *A quasiordered set $(X, \leq)$ is quasisober if and only if its order-reflection is a sober set.*

Sober sets may be interpreted in term of the coprime spectrum of a lattice $\Gamma(Y)$.

Proposition 2.4. *Let* $(X, \leq)$ *be a poset. Then the following statements are equivalent:*

(i) $(X, \leq)$ *is a sober set;*

(ii) *There exists a topological space* Y *such that* $(X, \leq)$ *is isomorphic to* $(Spec_{\vee}(\Gamma(Y)), \supseteq)$.

Proof. (i) Suppose that $(X, \leq)$ is sober. Let $\mathcal{T}$ be an order compatible sober topology on X. Then X is homeomorphic to its soberification $Spec_{\vee}(\Gamma(Y))$ (equipped with the hull-kernel topology). The map $\mu_X : X \longrightarrow Spec_{\vee}(\Gamma(Y))$ defined by $\mu_X(x) = \overline{\{x\}}$ is a homeomorphism. Thus μ_X induces an isomorphism from $(X, \leq)$ onto $(Spec_{\vee}(\Gamma(Y)), \supseteq)$.

$(ii) \Longrightarrow (i)$ Let $\varphi : (X, \leq) \longrightarrow Spec_{\vee}(\Gamma(Y)), \supseteq)$ be an isomorphism of posets. Let $\mathcal{T}_Y$ be the hull-kernel topology on $(Spec_{\vee}(\Gamma(Y)), \supseteq)$. The inverse image of $\mathcal{T}_Y$ by φ defined by $\mathcal{T} = \{\varphi^{-1}(U) : U \in O(Spec_{\vee}(\Gamma(Y)))\}$ is a topology on X and $\varphi(X, \mathcal{T}) \longrightarrow (Spec_{\vee}(\Gamma(Y)), \mathcal{T}_Y)$ is a homeomorphism. Therefore, $\mathcal{T}$ is a sober topology on X which is compatible with $\leq$. $\square$

Problem 2.5. Characterize sober posets in purely order-theoretical conditions.

Proposition 2.6. *There is at most one Noetherian quasisober topology inducing a given quasiorder on a space* X.

Proof. Note first that the proof is similar to that of [13, Proposition 14].

It is easy to see that a quasisober Noetherian topology on a space X is determined by the quasiorder (induced by the topology).

Indeed, if C is a closed subset of X, then there exist finitely many irreducible closed subsets C_1, ..., C_n of X such that $C = C_1 \cup ... \cup C_n$ (since X is Noetherian). On the other hand, X is quasisober; so that there is $x_i \in X$ such that $C_i = \overline{\{x_i\}}$, for each $i \in \{1, ..., n\}$.

But $\overline{\{x_i\}} = (x_i \uparrow)$. This proves that closed sets are completely determined by the quasiorder. $\square$

Recall that a poset $(X, \leq)$ is said to be *spectral* if there is a commutative ring R with unit such that $(X, \leq)$ is isomorphic with $(Spec(R), \subseteq)$[26].

In order that an order set $(X, \leq)$ be spectral it is necessary (but not sufficient [26]) that it satisfies two conditions:

K_1: Each nonempty totally ordered subset of X has a supremum and an infimum (that is, X is up-complete and down-complete)

K_2: For each $a < b$ in X, there exist two adjacent elements $a_1 < b_1$ such that $a \leq a_1 < b_1 \leq b$ (that is, X is weakly atomic).

These properties were noted, for a ring spectrum by I. Kaplansky (see [22, Theorems 9 and 11]), and then are called respectively the *first condition* and the *second condition* of Kaplansky.

Let X be a space which has a basis of compact open sets. We denote $X^\star$ its dual (equipped with the cocompact topology). Remark that if $\leq$ is the quasiorder induced by the topology of X, then the topology of $X^\star$ induces the reverse order $\geq$ of $\leq$ [13].

Theorem 2.7. *Let $(X, \mathcal{T})$ be a space which has a basis of compact open sets and $\leq$ the order induced by the topology $\mathcal{T}$. If X and $X^\star$ are sober, then $(X, \leq)$ satisfies K_1 and K_2.*

Proof. (K_1) Let C be a nonempty totally ordered subset of $(X, \leq)$. By Proposition 1.3 $(C, \leq)$ has an infimum in $\overline{C}^{\mathcal{T}}$ and $(C, \geq)$ has an infimum in $\overline{C}^{\star}$. It follows that, in $(X, \leq)$, C has an infimum and a supremum.

(K_2) Let $x < y$ in X. Consider a maximal chain C between x and y. Since $x \notin \overline{\{y\}}$, there is a compact open subset U of X such that $x \in U$ and $y \notin U$.

Let us denote by $x_1 = Sup(C_1)$, where $C_1 = \{t \in C \mid t \in U\}$ and $y_1 = Inf(C_2)$, where $C_2 = \{t \in C \mid t \notin U\}$.

Since $C_1 \subseteq U$ and U is closed in $X^\star$, we get $x_1 \in U$.

On the other hand, $C_2 \subseteq (X - U)$ and $(X - U)$ is closed in $(X, \mathcal{T})$; so that $y_1 \notin U$.

For each $t \in C_1$ and $s \in C_2$, we have $t < s$ [if not $t \in \overline{\{s\}}$, contradicting the fact that $t \in U$ and $s \notin U$].

We conclude that $x_1 \leq y_1$; and as $x_1 \in U$ and $y_1 \notin U$. This forces $x_1 < y_1$.

If we suppose that $x_1 < y_1$ are not adjacent, then there is $z \in X$ such that $x_1 < z < y_1$; and necessarily $z \notin C$. Therefore, $C \cup \{z\}$ is a chain between x and y, against the maximality of C. □

The following question is suggested by the anonymous referee.

Question 2.8. Under which hypothesis, the converse of Theorem 2.7 is true?

The following gives a complete characterization of totally ordered sober sets.

Theorem 2.9. *Let $(X, \leq)$ be a totally ordered set. Then the following statements are equivalent:*

(i) $(X, \leq)$ is a sober set;

(ii) Each nonempty subset of X has an infimum.

Proof. $(i) \Longrightarrow (ii)$ Follows immediately from Proposition 1.3.

$(ii) \Longrightarrow (i)$ Recall that the cop-topology on an order set (closure of points) $(X, \leq)$ is the topology on X which has $\{(x \uparrow) : x \in X\}$ as a sub-basis of closed sets.

Clearly, the closed sets of the cop-topology on a totally ordered set $(X, \leq)$ in which every nonempty subset has an infimum are $\emptyset$, X and $(x \uparrow)$ (for each $x \in X$).

Thus the cop-topology on $(X, \leq)$ is an order-compatible sober topology on X. Therefore, $(X, \leq)$ is a sober set. □

Example 2.10. A sober set $(X, \leq)$ that does not satisfy Kaplansky's two conditions (K_1) and (K_2).

Let $X = [0, 1[$ equipped with the natural order induced by that of $\mathbb{R}$. According to Theorem 2.9, $(X, \leq)$ is a sober set. However, it is clear that $(X, \leq)$ does not satisfy Kaplansky's two conditions (K_1) and (K_2).

3. Compactifications

Let X be a topological space, set $\widetilde{X} = X \cup \{\infty\}$ with the topology whose members are the open subsets of X and all subsets U of $\widetilde{X}$ such that $\widetilde{X} \setminus U$ is a closed compact subset of X. The space $\widetilde{X}$ is called the *Alexandroff extension* of X (or the *one-point compactification of* X). It is well known that the following properties hold :

(i) $\widetilde{X}$ *is compact and* X *is open in* $\widetilde{X}$.

(ii) X *is dense in* $\widetilde{X}$.

(iii) $\widetilde{X}$ *is Hausdorff if and only if* X *is locally compact and Hausdorff.*

Remark 3.1. Let X be a topological space and $\widetilde{X}$ the Alexandroff extension of X. Then the following properties hold:

(1) $\widetilde{X}$ is a T_0-space if and only if X is a T_0-space.

(2) Let C be a closed subset of $\widetilde{X}$.

- If $\infty \notin C$, then C is closed in X.

- If $\infty \in C$, then $C \setminus \{\infty\}$ is closed in X.

(3) If C is an irreducible closed subset of $\widetilde{X}$ such that $\infty \notin C$, then C is an irreducible closed subset of X.

(4) Let C be a closed subset of $\widetilde{X}$ such that $\infty \notin C$. If $C = \overline{\{a\}}^X$, then $C = \overline{\{a\}}^{\widetilde{X}}$.

Theorem 3.2. *Let* X *be a topological space. Then the following statements are equivalent:*

(i) X *is a sober space;*

(ii) The one-point compactification $\widetilde{X}$ *is a sober space.*

Proof.

$(ii) \Longrightarrow (i)$ Let C be a nonempty irreducible closed subset of X. Then $\overline{C}^{\widetilde{X}}$ is an irreducible closed subset of $\widetilde{X}$. Since $\widetilde{X}$ is sober, there exists $x \in C$ such that $\overline{C}^{\widetilde{X}} = \overline{\{x\}}^{\widetilde{X}}$. Hence, by Remark 3.1 (4), $C = \overline{C}^{\widetilde{X}} \cap X = \overline{\{x\}}^{\widetilde{X}} \cap X = \overline{\{x\}}^X$. Thus C has a generic point in X, proving that X is sober.

$(i) \Longrightarrow (ii)$ Let C be a nonempty irreducible closed subset of $\widetilde{X}$. Then three cases arise:

Case 1: $\infty \notin C$. In this case C is an irreducible closed subset of X. Thus $C = \overline{\{x\}}^X$, since X is sober. Therefore, $C = \overline{\{x\}}^{\widetilde{X}}$.

Case 2: $C = \{\infty\}$. In this case we have $C = \overline{\{\infty\}}^{\widetilde{X}}$.

Case 3: $C = D \cup \{\infty\}$, where D is a nonempty closed subset of X. In this case D is irreducible in X, so that D has a generic point in X: there exists $x \in D$ such that $D = \overline{\{x\}}^X$. Hence $C = \overline{\{x\}}^X \cup \{\infty\}$, this leads to the fact that $\overline{\{x\}}^X$ is not closed in $\widetilde{X}$, by irreducibility of C. Consequently, $\overline{\{x\}}^{\widetilde{X}} = \overline{\{x\}}^X \cup \{\infty\} = C$, proving that $\widetilde{X}$ is sober. □

The previous result incites us to ask the following question.

Question 3.3. (1) Let X be a space (resp. a T_1-space). We denote by $\beta_\omega X$ (resp. WX) the T_0-compactification of X introduced by Herrlich in [12] (resp. The Wallman compactification of X introduced in [30]). Are the following equivalences true:

$$[X \text{ is sober}] \Longleftrightarrow [\beta_\omega X \text{ is sober}];$$

$$[X \text{ is sober}] \Longleftrightarrow [WX \text{ is sober}]?$$

(2) More precisely, we ask when is $\beta_\omega X$ (resp. WX) sober?

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