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Perspective

Binomial coefficients and Nasir al-Din al-Tusi

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A historical note is given about the scientist Nasir al-Din al-Tusi legitimating the introduction of a new concept related to binomial coefficients. Al-Tusi binomial coefficients and binomial formulas are introduced and studied.

Key words: Binomial coefficients, binomial theorem, history of Mathematics.

INTRODUCTION

HISTORICAL NOTE AND NOTATIONS

Abu Jafar Muhammad Ibn Muhammad Ibn al-Hasan Nasir al-Din al-Tusi was born in Tus, Khurasan (now, Iran) in 17 February 1201 and died in Baghdad 25 June 1274. Al-Tusi was one of the greatest scientists, mathematicians, astronomers, philosophers, theologians and physicians of his time. Al-Tusi was an Arabic scholar whose writings became the standard texts in several disciplines for several centuries. They include editions of Euclid's Elements and Ptolemy's Almagest, as well as other books on mathematics and astronomy, and books on logic, ethics, and religion. Al-Tusi was known by a number of different names during his lifetime such as Muhaqqiq-i Tusi, Khwaja-yi Tusi and Khwaja Nasir. His proper name was Muhammad ibn Muhammad ibn al-Hasan al-Tusi.

In 1214, when al-Tusi was 13 years old, Genghis Khan, who was the leader of the Mongols, turned away from his conquests in China and began his rapid advance towards the west. Genghis Khan turned his attention again towards the east leaving his generals and sons in the west to continue his conquests. The Mongol invasion caused much destruction in both west and east. Fortunately, al-Tusi was able to study more advanced topics before seeing the effects of the Mongols conquests on his own regions. From Tus, al-Tusi went to Nishapur which is 75 km west of Tus; this was a good choice for al-Tusi to complete his education since it was an important centre of learning. There, al-Tusi studied Philosophy, Medicine and Mathematics. Particularly, he was taught Mathematics by Kamal al-Din ibn Yunus, who himself had been a pupil of Sharaf al-Din al-Tusi. In Nishapur, al-Tusi began to acquire a reputation as an outstanding scholar and became well known throughout the area.

Al-Tusi has wrote important works on Astronomy, logic, Mathematics and Philosophy. The first of these works, ``Akhlaq-i nasiri", was written in 1232; it was a work on ethics which al-Tusi dedicated to the Isma'ili ruler Nasir ad-Din Abd ar-Rahim. Al-Tusi was kidnaped by the Isma'ili Hasan Bin Sabah's agents and sent to Alamut where he remained until its capture by the Mongol Halagu Khan. Impressed by Al-Tusi's exceptional abilities and astrological competency, Ilkhanid Halagu Khan appointed him as one of his ministers. Later, has he been designated an administrator of Auqaf.

In 1262, al-Tusi built an observatory at Meragha (in the Azerbaijan region of north-western Iran) and directed its activity. It was equipped with the best instruments from Baghdad and other Islamic centers of learning. It contained a twelve-feet wall quadrant made from copper and an azimuth quadrant and 'turquet' invented by al-Tusi. Other instruments included Astrolabes, representtations of constellation, epicycles, and shapes of spheres. Al-Tusi designed several other instruments for the Observatory. After putting his Observatory to good use, making very accurate tables of planetary movements, al-Tusi published his work “Zij-i ilkhani (the Ilkhanic Tables)" (which was dedicated to Ilkhanid Halagu Khan), written first in Persian and later translated int Arabic. This work contains tables for computing the positions of the planets, and it also contains a star catalogue. The tables were developed from observations over a twelve-year period and were primarily based on original observations.

The major astronomical treatise of al-Tusi was “al-Tadhkira fi'ilm al-hay'a". This was written to give serious students a detailed acquaintance with astronomical and cosmological theory. In this treatise Nasir, gave a new model of lunar motion, essentially different from Ptolemy's. In his model Nasir, for the first time in the history of astronomy, employed a theorem invented by himself which, 250 years later, occurred again in Chapter IV of Book III of Copernicus' "De Revolutionibus" (On the revolutions of the heavenly spheres). This theorem runs as follows: If a point moves with uniform circular motion clockwise around the epicycle while the center of the epicycle moves counterclock-wise with half this speed along an equal deferent circle, the point will describe a straight-line segment.

Many historians claim that the Tusi-couple result was used by Copernicus after he discovered it in al-Tusi's work (see for example, Boyer (1947) and Dreyer (1953). However, Veselovsky (1973) shows that it is much more plausible to suppose that Copernicus took the argument he needed from Proclus' "Commentary on the first book of Euclid", and not from al- Tusi. There are two other astronomical treatises of al-Tusi. The first treatise, called the "Muiniya" and written in 1235, contains a standard account of Ptolemaic lunar and planetary theory. In the second treatise, the "Hall" (written between 1235 and 1256), al-Tusi uses the plane version of his "Tusi-couple" to explain the motion in longitude of the epicycle centre of the moon and the planets. The "Hall" does not yet include the spherical version of the "Tusi-couple", which is used in the "Tadhkira" to describe the prosneusis of the moon and the latitude theory of the planets. It is worth noting that the term "Tusi couple" is a modern one, coined by Edward Kennedy (1966).

It is also worth noting that al-Tusi has written revised Arabic versions of works by Autolycus, Aristarchus, Euclid, Apollonius, Archimedes, Hypsicles, Theodosius, Menelaus and Ptolemy. Ptolemy's Almagest was one of the works which Arabic/Muslim scientists studied intently. In 1247 al-Tusi wrote ``Tahrir al-Majisti (Commentary on the Almagest)" in which he introduced various trigonometrical techniques to calculate tables of sines; al-Tusi gave tables of sine with entries calculated to three sexagesimal places for each half degree of the argument. Ibadov (1968) has asserted that al-Tusi had found the value of the sine of one degree (with the precision up to the fifth decimal place). Ibadov considers some trigonometric propositions used for this purpose by Tusi and their relation with analogous results obtained by scientists of Central Asia and Western Europe.

An important mathematical contribution of al-Tusi was the creation of trigonometry as a mathematical discipline in its own right rather than as just a tool for astronomical applications. In "Treatise on the quadrilateral", al-Tusi gave the first extant exposition of the whole system of plane and spherical trigonometry. This work is really the first in history on trigonometry as an independent branch of pure Mathematics.

There is a method due to al-Tusi (dated 1265), from an apparently previously unexplored manuscript in Tashkent, for extracting roots of any order of a number and for determining the coefficients of the expansion of a binomial to any power (see for example Ahmedov, 1970).

This work is al-Tusi's version of methods developed by al-Karaji's school. In the manuscript, al-Tusi determined the coefficients of the expansion of a binomial to any power giving the binomial formula and the Pascal triangle relations between binomial coefficients.

Edwards (2002) has postulated that the work of al-Karaji in expanding the Binomial Triangle might have borrowed Brahmegupta's work, given that it was available and al-Karaji definitely had read other Hindu texts available in Baghdad which was the great cultural and scientific center of Muslims. The binomial coefficients have been studied in cultures around the world, both in the context of binomial expansions and in the question of how many ways to choose k items out of a collection of n things. Note also that, three names from China figure prominently in the story of the discovery of the binomial coefficients. These scientists are Chia Hsien, Yanghui (1261) and Chu Shih-chieh (1303). The idea of taking "six tastes one at a time, two at a time, three at a time, etc." was written down correctly in India 300 years before the birth of Christ in a book called the ``Bhagabati Sutra". Thus the Indian civilization is the earliest one that has an understanding of the binomial coefficients in their combinatorial form ``n choose k".

The interest in the binomial coefficients in India dealt with choosing and arranging things. However, mathematicians in the Middle East were interested almost entirely in expansions of polynomials; the work of the Indian Brahmagupta, which included the expansion of $(a+b)^3$, was available to the scholars of the Middle East. Some Historians consider that investigations on binomial coefficients by mathematicians in the Middle East may well have been inspired Brahmagupta's work. There is another scientist from the Middle East who worked with the binomial coefficients; namely, al-Samawal (a Jew born in Baghdad who died in 1180). Omar al-Khayyam is another famous Persian scientist who makes a claim to knowledge; he wrote a letter claiming to have been able to expand binomials to sixth power and higher, but the actual work does not survive; in the letter he mentions that he is aware not only of work done in India, but of Euclid's Elements. Al-Khayyam is best known in the West for his collection of poems "The Rubaiyat", which was translated into English in 1859 by Edward Fitzgerald.

In Europe, there are many authors who can fairly lay claim to having made a serious study of the binomial coefficients, several of them long before Blaise Pascal was even born. Blaise Pascal was not the first man in Europe to study the binomial coefficients, and never claimed to be such; indeed, both Blaise Pascal and his father Etienne had been in correspondence with Father Marin Mersenne, who published a book with a table of binomial coefficients in 1636.

It is very reasonable to claim that the manuscripts written by Arabic/Muslims scientists have deeply influenced the philosophical and scientific thoughts during Renaissance Europe; libraries inherited by Europeans in

Spain (after the exit of Muslims from Andalusia) is a gigantic scientific treasure that contributed to the development of Europe. We think also that, there is no need to deny brilliant contributions of medieval Arabic/Muslims scientists. Undoubtedly Arabic/Muslim civilization has contributed meaningfully to the human civilization; contrary to what some Western politicians, who ignore the history of civilizations, think.

To close this historical note, let us postulate that it is not easy to give evidence for the claim of precedence for al-Tusi in finding binomial coefficients and Pascal's triangle. Nevertheless, the aim of this paper is the introduction of a new type of binomial coefficients that will be called al-Tusi binomial coefficients (in honour to Nasir al-Din al-Tusi).

In order to make this paper as self contained as possible (and also relatively accessible to any scientist), here are most of notions and concepts used.

Let $n, p \in N$. Set $A_n^p := \frac{n!}{(n-p)!}$, for $n \geq p$ and $A_n^p = 0$, otherwise; where $n! := n(n-1)...2$.

Recall, also, the classical binomial coefficient: $\binom{n}{p} := \frac{n!}{p!(n-p)!}$, for $n \geq p$ and $\binom{n}{p} =$ 0, other wise. Then we have the well known recurrence relations: $A_n^p = A_{n-1}^p + pA_{n-1}^{p-1}$ for, $p \geq 1$; and $\binom{n}{p} = \binom{n-1}{p} + \binom{n-1}{p-1}$ ``Pascal's formula" for $n, p \geq 1$.

These recurrence relations are well known to all school children. Looking at the above recurrence relations in two variables and in honour to Nasir al-Din al-Tusi, we will introduce a new concept. Firstly, as promised, let us recall some elementary definitions.

- By a binary operation on a set S, we mean a map $* : S \times S \longrightarrow S$.
- A group is a mathematical system consisting of a nonempty set G, a binary operation denoted by + (and considered as abstract addition) and the axioms:

(1) $(a+b)+c = a+(b+c)$, for any elements of G (associativity).

(2) G contains an element e such that $a+e = e+a = a$ for each a in G (the element e is called an identity element of the group).

(3) For each a in G, there is an element $-a$ in G such that $a+(-a)=(-a)+a=e$. The element $-a$ is called an (inverse of a).

If the following additional axiom is assumed, the group is said to be a commutative or Abelian group:

(4) $a+b=b+a$, for any elements a and b in G (commutativity).

- A system $(R,+,\times)$ consisting of a nonempty set R and two binary operations, called addition and multiplication, on R is called a ring if the following conditions are satisfied:

Addition: R is an Abelian group relative to addition.

Multiplication: R is a semigroup relative to multiplication (i.e., multiplication satisfies the associativity property).

Distribution: For any elements a, b, and c of R, we have the following equalities:

$a(b+c) = ab+ac$ and $(b+c)a = ba+ca.$

The identity element for addition is denoted by 0 and the additive inverse of a is written as $(-a)$. If in addition, multiplication is commutative we say that R is a commutative ring. A unitary ring is a ring having a multiplicative identity element (denoted by 1) distinct from 0.

Now, we are in a position to introduce a new concept.

Definition 1

Let $(R,+,\times)$ be a commutative unitary ring and $f : IN \times IN \longrightarrow R$ be a map. We say that $(f(n,p))$ are al-Tusi Binomial coefficients (TBCs, for short)}, if the following properties hold:

$(i) f(n,0) = 1$, for each $n \in IN$;

$(ii) f(n,p) = 0$, for each $p > n$;

(iii) There exist $\alpha, \beta \in R$ such that

$f(n,p) = f(n-1,p) + (\alpha + p\beta) f(n-1,p-1),$ foreach $n, p \geq 1.$

If there is no confusion, $f(n,p)$ will be denoted by T_n^p.

It is easily seen that the double sequences (A_n^p) and $\binom{n}{p}$ are TBCs. The name ``Binomial Coefficients" allotted to al-Tusi will be justified in Section 2.

In mathematics, the binomial theorem is an important formula giving the expansion of powers of sums. The simplest version of that theorem is the following: let x, y be two elements of a ring $(R,+,\times)$ such that $xy = yx$.

Then we have a well known ``Newton's Binomial Formula":

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.$$

The proof is a straightforward induction and can be found in all good high school advanced algebra books. In the following, we generalize the previous binomial formula in the noncommutative setting.

Definition 2

Let R be a ring and $(A,+,\bullet,\times)$ be an R-algebra with unit element 1. Let $(F_n, n \in IN)$ be a sequence of elements of A. We say that $(F_n, n \in IN)$ is a Newton's formula if there following properties hold:

(1) $(F_n, n \in IN)$ is iterated under * (i.e.; $F_{n+1} = F_1 * F_n$, for each $n \in IN$).
(2) $x * 1 = x$, for each $x \in A$.
(3) $x * (y + z) = x * y + x * z$, for each $x, y, z \in A$.
(4) $x * (\lambda y) = (\lambda x) * y = \lambda(x * y)$, for each $\lambda \in R$.

[Such an operation * will be called a compatible binary operation with the structure of R-algebra on A.]
We will, simply, write $F_n = (F_1)^{*n}$.

In connection with al-Tusi Binomial coefficients and Newton's binomial formula, we introduce the following concept.

Definition 3

Let R be a commutative ring and x, y be two indeterminates over R. We call al-Tusi Binomial formula (TBF, for short) the sequence $(F_n(x, y), n \in IN)$ in $R[x, y]$ defined by $F_n(x, y) = \sum_{i=0}^{n} T_n^i x^i y^{n-i}$, where (T_n^i) are TBCs.

The main goal of this paper is to compute TBCs and show that each TBF is a Newton's formula in the sense of Definition 2.

Note that, more details and applications of the concepts introduced, in this paper, will be considered for publication in a suitable Mathematical Journal.

BINOMIAL COEFFICIENTS

The following result computes TBCs.

Theorem 1: Let $(R, +, \times)$ be a commutative unitary ring and $(T_n^p, n, p \in IN)$ be a double sequence of elements of R such that $T_n^0 = 1$, for each $n \in IN$ and $T_n^p = 0$, for each $p > n$. Then the following statements are equivalent:

(i) (T_n^p) are TBCs;

(ii) There exist α, β in R such that $T_n^p = \binom{n}{i}(\prod_{i=1}^{p}(\alpha + i\beta))$, for each $n \in IN$ and $p \in IN \setminus \{0\}$.

Proof:

$(ii) \Longrightarrow (i)$. Straightforward.

$(i) \Longrightarrow (ii)$. We use induction on $k \in IN$.

-- If $k = 1$, then T_n^1 satisfies the recurrence relation:

$$T_n^1 = T_{n-1}^1 + (\alpha + \beta)T_{n-1}^0$$
$$= T_{n-1}^1 + (\alpha + \beta).$$

Thus, clearly,

$T_n^1 = n(\alpha + \beta) = \binom{n}{1}(\alpha + \beta)$. -- Suppose that $T_n^l = \binom{n}{l}(\prod_{i=1}^{l}(\alpha + i\beta))$, for $1 \le l \le k$; and let us compute T_n^{k+1}.

By induction hypothesis, we have

$$T_n^{k+1} = T_{n-1}^{k+1} + \binom{n-1}{k}(\prod_{i=1}^{k+1}(\alpha + i\beta)),$$

which gives immediately the following:

$$T_n^{k+1} = (\sum_{i=k}^{n-1}\binom{i}{k})(\prod_{i=1}^{k+1}(\alpha + i\beta))$$
$$= \binom{n}{k}(\prod_{i=1}^{k+1}(\alpha + i\beta)),$$

finishing the induction.

THE BINOMIAL FORMULA

Examples

(1). Let $(E, +, \bullet, \times)$ be a unitary algebra over a ring R. Then $\times$ is a compatible binary operation with the structure of R-algebra on E.

(2) Let $(R, +, \times)$ be a commutative ring with unit. Then $\times$ is a compatible binary operation with the structure of R-algebra on $(R, +, \times, \times)$. If $x, y \in R$ are such that $xy = xy$, we let $B_n(x, y)$ be the sum $\sum_{i=0}^{n}\binom{n}{i}x^i y^{n-i}$, then $B_n(x, y)$ is iterated by the operation $\times$ (according to the Newton's binomial formula).

Naturally, it will be very nice if every TBF is a Newton formula. This will be carried out in the following result.

Theorem 2

Let R be a commutative ring and x, y be two indeterminates over R. Then each TBF is a Newton's Formula in the R-algebra $(R[x, y], +, \bullet, \times)$.

Proof

Let $(F_n(x, y), n \in IN)$ be a TBF in $R[x, y]$. We are aiming to construct a binary operation * on $R[x, y]$ such that

$F_n = (F_1)^{*n}$ and is compatible with the structure of R -algebra on $R[x,y],+,.,\times)$.

Let us write $F_n(x,y) = \sum_{i=0}^{n} T_n^i x^i y^{n-i}$, where ($T_n^i$) are TBCs. There exist $\alpha, \beta \in R$ such that

$T_n^k = T_{n-1}^k + (\alpha + k\beta) T_{n-1}^{k-1}$,

for each $n, k \geq 1$. Then, for $n \in IN \setminus \{0\}$, we have

$$F_n(x,y) = \sum_{k=0}^{n} T_n^k x^k y^{n-k}$$

$$= y^n + \sum_{k=1}^{n} T_{n-1}^k x^k y^{n-k} + \sum_{k=1}^{n} (\alpha + k\beta) T_{n-1}^{k-1} x^k y^{n-k}$$

$$= y^n + y(\sum_{k=1}^{n-1} T_{n-1}^k x^k y^{(n-1)-k}) + \alpha \sum_{k=1}^{n} T_{n-1}^k x^k y^{n-k}$$
$$+ \beta \sum_{k=1}^{n} k T_{n-1}^{k-1} x^k y^{n-k}$$

$$= y^n + y(F_{n-1}(x,y) - y^{n-1}) + \alpha \sum_{k=0}^{n-1} T_{n-1}^k x^{k+1} y^{(n-1)-k}$$
$$+ \beta \sum_{k=0}^{n-1} (k+1) T_{n-1}^k x^{k+1} y^{(n-1)-k}$$

$$= yF_{n-1}(x,y) + \alpha x F_{n-1}(x,y) + \beta x F_{n-1}(x,y)$$

$$= (y + (\alpha + \beta)x) F_{n-1}(x,y) + \beta x^2 \frac{\partial F_{n-1}}{\partial x}(x,y)$$
$$+ \beta x^2 \frac{\partial F_{n-1}}{\partial x}(x,y)$$

$$= F_1(x,y).F_{n-1}(x,y) + \beta x^2 \frac{\partial F_{n-1}}{\partial x}(x,y).$$

We define $* : R[x,y] \times R[x,y] \longrightarrow R[x,y]$, by $f * g = fg + \beta x^2 \frac{\partial g}{\partial x}$. Then clearly, * is a binary operation which is compatible with the structure of R -algebra on $(R[x,y],+,\bullet,\times)$; and we have $F_n = (F_1)^{*n}$ for each $n \in IN \setminus \{0\}$. Therefore, $(F_n, n \in IN)$ is a BF.

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