ON $\mathbf{T_D}$-SPACES

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Abstract. This paper deals with some new properties of T_D-spaces. These properties are used in order to give an intrinsic topological characterization of the Goldman spectrum of a commutative ring.

1. Introduction. There are three famous separation axioms in topology namely, T_0, T_1, and T_2.

We denote by **TOP** the category of topological spaces with continuous maps as morphisms, and by $\mathbf{TOP}_i$ the full subcategory of **TOP** whose object are T_i-spaces. It is well-known that $\mathbf{TOP}_{i+1}$ is a reflective subcategory of $\mathbf{TOP}_i$, for $i = -1, 0, 1$, with $\mathbf{TOP}_{-1} = \mathbf{TOP}$. Thus, $\mathbf{TOP}_i$ is reflective in **TOP**, for each $i = 0, 1, 2$ [10]. In other words, there is a universal T_i-space for every topological space X; we denote it by $\mathbf{T}_i(X)$. The assignment $X \mapsto \mathbf{T}_i(X)$ defines a functor $\mathbf{T}_i$ from **TOP** onto $\mathbf{TOP}_i$, which is a left adjoint functor of the inclusion functor $\mathbf{TOP}_i \hookrightarrow \mathbf{TOP}$.

The T_D separation axiom was introduced by Aull and Thron [1]. Recall that a topological space X is said to be a T_D-space if for each $x \in X$, $\{x\}$ is locally closed.

For the separation axioms T_0, T_1, T_2, T_D, we have classically the following implications:

$$T_2 \implies T_1 \implies T_D \implies T_0.$$

We denote by $\mathbf{TOP}_D$ the full subcategory of **TOP** whose objects are T_D-spaces.

Unfortunately, the T_D property is not reflective in **TOP**. Indeed, in [2], Brümer has proved that the countable product of the Sierpinski space is not a T_D-space. On the other hand, according to Herrlich and Strecker [7], if a subcategory **A** is reflective in a category **B**, then for each category **I**, **A** is closed under the formation of **I**-limits in **B** [7]. [Taking **I**, a discrete category, one can see that in particular **A** is closed under products in **B**.] Therefore the full subcategory $\mathbf{TOP}_D$ of **TOP** whose objects are T_D-spaces is not reflective in **TOP**.

This paper deals with some new categorical properties of T_D-spaces (see Theorem 1.8).

Let R be a commutative ring with unit. We denote by $Spec(R)$ the set of all prime ideals of R.

A topology $\mathcal{T}$ on a set X is defined to be *spectral* [8] (and $(X, \mathcal{T})$ is called a *spectral space*) if the following conditions hold:

(i) $\mathcal{T}$ is sober;
(ii) the compact open subsets of X form a basis of $\mathcal{T}$;

(iii) the family of compact open subsets of X is closed under finite intersections.

In a remarkable paper, M. Hochster proved that a topological space is homeomorphic to the prime spectrum of some ring if and only if it is a spectral space [8]. In the same paper, Hochster characterized the space of maximal ideals of a ring. When a particular subset of the spectrum of a ring is given, a classical question, of whether we can give a topological characterization of that subspace, is asked.

A prime ideal $\mathfrak{p}$ of R is said to be a Goldman ideal (G-ideal, for short) if there exists a maximal ideal $\mathfrak{M}$ of the polynomial ring $R[X]$ such that $\mathfrak{p} = \mathfrak{M} \cap R$. Goldman ideals are important objects of investigation in algebra and algebraic geometry. Note, in particular, that G-ideals have been used by Goldman [5] and Krull [11] for a short inductive proof of the Nullstellensatz. It is a part of the folklore of algebra that $\mathfrak{p}$ is a G-ideal of R if and only if $\{\mathfrak{p}\}$ is locally closed in $Spec(R)$ (endowed with the hull-kernel topology).

The subspace of $Spec(R)$, whose elements are G-ideals is called the *Goldman spectrum* of R and it is denoted by $Gold(R)$.

As in [4], by a *goldspectral space* we mean a topological space X which is homeomorphic to $Gold(R)$ for some ring R.

A natural question is "give an intrinsic topological characterization of goldspectral spaces".

The goal of this paper is to re-prove our characterization of goldspectral spaces [4] in a short elegant manner. More precisely, using our main result Theorem 1.5, we give an intrinsic topological characterization of the Goldman prime spectrum of a commutative ring (see Theorem 2.2). We prove that a topological space X is goldspectral if and only if X satisfies the following conditions:

(a) X is compact and has a basis of compact open subsets which is closed under finite intersections.
(b) X is a T_D-space.

2. $\mathbf{T_D}$-spaces and Quasihomeomorphisms.

Let us first recall some notions which were introduced by Grothendieck school, such as quasihomeomorphisms, strongly dense subsets and sober spaces.

If X is a topological space, we denote by $\mathfrak{O}(X)$ the set of all open subsets of X. Recall that a continuous map $q\colon Y \longrightarrow Z$ is called a *quasihomeomorphism* if $U \longmapsto q^{-1}(U)$ defines a bijection $\mathfrak{O}(Z) \longrightarrow \mathfrak{O}(Y)$. A subset S of a topological space X is said to be *strongly dense* in X, if S meets every nonempty locally closed subset of X. Thus, a subset S of X is strongly dense if and only if the canonical embedding $S \hookrightarrow X$ is a quasihomeomorphism. It is well-known that a continuous map $q\colon X \longrightarrow Y$ is a quasihomeomorphism if and only if the topology of X is the inverse image by q of that of Y and the subset $q(X)$ is strongly dense in Y [6].

A subspace Y of X is called *irreducible*, if each nonempty open subset of Y is dense in Y (equivalently, if C_1 and C_2 are two closed subsets of X such that $Y \subseteq C_1 \cup C_2$, then $Y \subseteq C_1$ or $Y \subseteq C_2$). Let C be a closed subset of a space X; we say that C *has a generic point* if there exists $x \in C$ such that $C = \overline{\{x\}}$. Recall that a topological space X is said to be *sober* if any nonempty irreducible closed subset of X has a unique generic point.

The main result of this section is Theorem 1.5. Before stating it, we need a sequence of lemmas.

Lemma 1.1 ([3]). Let X be a topological space. Then the following properties hold:

(1) if X is a T_0-space which has a basis of compact open subsets, then $Gold(X)$ is strongly dense in X;
(2) if $Gold(X)$ is strongly dense in X, then it is the smallest strongly dense subset of X.

Lemma 1.2 (Never two without three). Let $p: X \longrightarrow Y$ and $q: Y \longrightarrow Z$ be two continuous maps. If two among the three maps $(p, q, q \circ p)$ are quasihomeomorphisms, then so is the third one.

Lemma 1.3. Let X be a topological space and A a strongly dense subset of X. Then A is strongly dense in each subspace of X containing A.

Proof. The proof is straightforward; but I would like to check it in terms of quasihomeomorphisms. Clearly, B is strongly dense in X. Hence, the canonical embeddings $i: A \hookrightarrow X$, $j: B \hookrightarrow X$ are quasihomeomorphisms. If we let $t: A \hookrightarrow B$ be the canonical embedding, then $i = j \circ t$. By Lemma 1.2, $t: A \hookrightarrow B$ is a quasihomeomorphism; this means that A is strongly dense in B.

Lemma 1.4 ([3]). Let $q: X \longrightarrow Y$ be a quasihomeomorphism. Then the following properties hold:

(1) if X is a T_0-space, then q is injective;
(2) if X is sober and Y is a T_0-space, then q is a homeomorphism.

Now, we are in a position to state our main result.

Theorem 1.5. Let X be a T_0-space and Y be a topological space such that $Gold(X)$ is strongly dense in X and $Gold(Y)$ is strongly dense in Y. Let $q: X \longrightarrow Y$ be a quasihomeomorphism. Then the following properties hold:

(a) $q(Gold(X)) = Gold(q(X)) = Gold(Y)$;
(b) the induced map $q_G: Gold(X) \longrightarrow Gold(Y)$ which carries x to $q(x)$ is a homeomorphism.

Proof.

(a) Let us consider the map $q_1: X \longrightarrow q(X)$ induced by q. Let $j: q(X) \longrightarrow Y$ be the canonical embedding; then j is a quasihomeomorphism. Since $q = j \circ q_1$, we get q_1 a quasihomeomorphism (by "Never two without three"). Now, since X is a T_0-space, q_1 is injective, by Lemma 1.4. Thus, q_1 is a bijective quasihomeomorphism; so that it is a homeomorphism.

It follows that $Gold(q(X)) = q_1(Gold(X)) = q(Gold(X))$. Since X is homeomorphic to $q(X)$, $Gold(q(X))$ is strongly dense in $q(X)$. But $q(X)$ is strongly dense in Y; this forces $Gold(q(X))$ to be strongly dense in Y.

On the other hand, $Gold(Y)$ is the smallest strongly dense subset of Y (see Lemma 1.1); this yields

$$Gold(Y) \subseteq Gold(q(X)) = q(Gold(X)) \subseteq q(X).$$

By Lemma 1.3, $Gold(Y)$ is strongly dense in $q(X)$; but Lemma 1.1 says that $Gold(q(X))$ is the smallest strongly dense subset of $q(X)$; consequently, $Gold(q(X)) \subseteq Gold(Y)$. We conclude that $q(Gold(X)) = Gold(q(X)) = Gold(Y)$.

(b) Since the induced map $q_1: X \longrightarrow q(X)$ is a homeomorphism and $q_1(Gold(X)) = Gold(Y)$, the mapping $q_G: Gold(X) \longrightarrow Gold(Y)$ defined by $x \longmapsto q(x)$ is also a homeomorphism.

Proposition 1.6. Every quasihomeomorphism between two T_D-spaces is a homeomorphism.

Proof. It follows immediately from Theorem 1.5 (b).

Note also that one may give an easy direct proof. Indeed, let $q: X \longrightarrow Y$ be a quasihomeomorphism between two T_D-spaces. Hence, q is injective by Lemma 1.4. On the other hand, $q(X)$ is strongly dense in Y and every point set of Y is locally closed; so that $q(X) = Y$. Thus, q is a bijective quasihomeomorphism. Therefore, q is a homeomorphism.

The following concept, motivated by Proposition 1.6, proves to be useful.

Definition 1.7. Let $\mathbf{C}$ be a category. By a *categoroid* of $\mathbf{C}$ we mean a full subcategory of $\mathbf{C}$ closed under isomorphisms in which all arrows are isomorphisms.

Theorem 1.4 and Definition 1.6 immediately give the following categorical properties of T_D-spaces.

Theorem 1.8. Let $\mathbf{C}$ be the category where objects are topological spaces X such that $Gold(X)$ is strongly dense in X and arrows are quasihomeomorphisms. Let $\mathbf{C}_1$ be the full subcategory of $\mathbf{C}$ whose objects are T_D-spaces. Then $\mathbf{C}_1$ is a coreflective categoroide of $\mathbf{C}$. The coreflector is $\mu_X: Gold(X) \hookrightarrow X$.

Remark 1.9. $\mathbf{C}_1$ is strictly contained in $\mathbf{C}$. Let Y be an infinite set equipped with the cofinite topology and $\omega \notin Y$. Set $X = Y \cup \{\omega\}$ and equip it with the topology whose closed sets are X and the closed sets of Y. Clearly, $Gold(X) = Y$ is strongly dense in X. However, X is not a T_D-space, since $\{\omega\}$ is not locally closed.

3. The Goldman Spectrum of a Ring. Our next investigation of Theorem 1.5 is a new proof of our main result of [4], which gives an intrinsic topological characterization of the Goldman spectrum of a commutative ring.

We need to recall the the notion of soberification of a topological space which proved to play an important part in the next theorem.

Let X be a topological space and $\mathbb{S}(X)$ the set of all nonempty irreducible closed subsets of X [6]. Let U be an open subset of X and set

$$\widetilde{U} = \{C \in \mathbb{S}(X) \mid U \cap C \neq \emptyset\}.$$

Then the collection $(\widetilde{U}$, U is an open subset of $X)$ provides a topology on $\mathbb{S}(X)$ and the following properties hold [6]:

(i) the map $\mu_X : X \longrightarrow \mathbb{S}(X)$ which carries x to $\overline{\{x\}}$ is a quasihomeomorphism;
(ii) $\mathbb{S}(X)$ is a sober space.

The topological space $\mathbb{S}(X)$ is called the *soberification* of X.

Before stating the result which characterizes goldspectral spaces, let us give a straightforward remark.

Remark 2.1. If $q: X \longrightarrow Y$ is a quasihomeomorphism, then the following properties hold:

(a) Let U be an open subset of Y, then U is compact if and only if $q^{-1}(U)$ is compact.
(b) X has a basis of compact open subsets closed under finite intersections if and only if so is Y.

Theorem 2.2. Let X be a topological space. Then X is goldspectral if and only if X satisfies the following properties:

(a) X is compact;
(b) X has a basis of compact open subsets;
(c) the intersection of two compact open subsets is compact;
(d) X is a T_D-space.

Proof.

• For each ring R, $Gold(R) = Gold(Spec(R))$ is a T_D-space. Since the canonical embedding $Gold(R) \hookrightarrow Spec(R)$ is a quasihomeomorphism and $Spec(R)$ satisfies properties (a), (b), and (c), then so is $Gold(R)$, by Remark 2.1.

• Conversely, let X be a space satisfying properties (a), (b), (c), and (d). Let $\mathbb{S}(X)$ be the soberification of X and $\mu_X : X \longrightarrow \mathbb{S}(X)$ defined by $\mu_X(x) = \overline{\{x\}}$ the canonical embedding of X into $\mathbb{S}(X)$.

According to Theorem 1.5, X is homeomorphic to $Gold(\mathbb{S}(X))$. Thus, Remark 2.1 implies immediately that $\mathbb{S}(X)$ is a spectral space, completing the proof.

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