Q.1 i) Find the angle between the planes $x-2y+z=1$ and $2x+y+z=1$.

ii) Find parametric equations for the line of intersection of these two planes.

Solution

i) Normal vectors of the two given planes are

$$\vec{n}_1 = \langle 1, -2, 1\rangle \quad \text{and} \quad \vec{n}_2 = \langle 2, 1, 1\rangle$$

The required angle is

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right)\\ &= \cos^{-1}\left(\frac{2-2+1}{\sqrt{1+4+1}\sqrt{4+1+1}}\right) = \cos^{-1}\left(\frac{1}{6}\right)\end{aligned}$$

ii) We first need to find one point on the line. Setting $z=0$ gives

$$\begin{aligned}x - 2y &= 1\\ 2x + y &= 1.\end{aligned}$$

We get $x = 3/5$ and $y = -1/5$. So, the point $(3/5, -1/5, 0)$ lies on the line of intersection of the two planes. Since the line is lying on both planes, it is perpendicular to both normal vectors. Thus a vector $\vec{v}$ parallel to the line is

$$\begin{aligned}\vec{v} = \vec{n}_1 \times \vec{n}_2 &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix}\\ &= -3\vec{i} + \vec{j} + 5\vec{k}\end{aligned}$$

So, the parametric equations of the required line are

$$x = \frac{3}{5} - 3t, \quad y = -\frac{1}{5} + t, \quad z = 5t.$$

Q.2 a) Consider the quadric surface $4x^2 - 2y^2 + z^2 + 8 = 0$.

i) Find the traces of the surface in the vertical planes $y = k$. (k is a constant)

ii) Identify and sketch the surface.

Solution

i) Trace in the vertical plane $y = k$ is

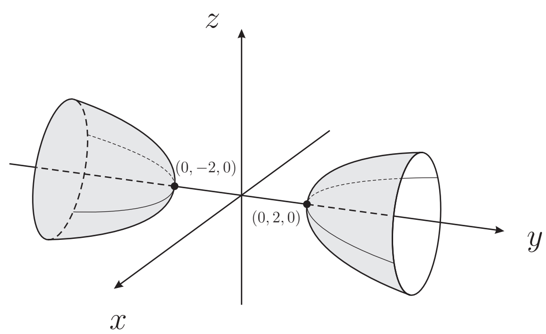
$$4x^2 + z^2 = 2k^2 - 8, \quad y = k \qquad \text{or} \qquad x^2 + \frac{z^2}{4} = \frac{k^2}{2} - 2, \quad y = k.$$

An ellipse for $|k| > 2$, points for $|k| = 2$ and no traces for $|k| < 2$.

ii) Dividing both sides of the equation of the surface by -8 gives

$$-\frac{x^2}{2} + \frac{y^2}{4} - \frac{z^2}{8} = 1$$

which is a hyperboloid of two sheets, the axis is y-axis.



b) The cylindrical coordinates of a point are $(\sqrt{6}, \pi/4, \sqrt{2})$. Find the rectangular and spherical coordinates of the point.

Solution

Rectangular coordinates:

$$x = r\cos\theta = \sqrt{6}\cos(\pi/4) = \sqrt{3}$$
$$y = r\sin\theta = \sqrt{6}\sin(\pi/4) = \sqrt{3}$$
$$z = \sqrt{2} \qquad \Longrightarrow \qquad (x, y, z) = (\sqrt{3}, \sqrt{3}, \sqrt{2}).$$

Spherical coordinates:

$$\rho^2 = r^2 + z^2 = 6 + 2 = 8 \qquad \Longrightarrow \qquad \rho = 2\sqrt{2}$$
$$\theta = \pi/4$$
$$\phi = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{\sqrt{2}}{2\sqrt{2}}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \pi/3$$
$$\Longrightarrow \quad (\rho, \theta, \phi) = (2\sqrt{2}, \pi/4, \pi/3)$$

Q.3 a) Let $f(x,y,z) = \sqrt{16 - x^2 - y^2 - z^2}$

i) Find and describe the domain of f

ii) Find the range of f

Solution

i) The given function f is defined if $16 - x^2 - y^2 - z^2 \geq 0$. So, the domain of f is

$$D = \{(x,y,z) |\ x^2 + y^2 + z^2 \leq 16\},$$

which consists of all points on or within the sphere $x^2+y^2+z^2 = 16$. That is, the domain of f is the solid sphere $x^2 + y^2 + z^2 \leq 16$.

ii) Observe that

$$0 \leq \sqrt{16 - (x^2 + y^2 + z^2)} \leq \sqrt{16}.$$

So the range of f is

$$R = \{w |\ 0 \leq w \leq 4\} = [0, 4].$$

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b) Let

$$f(x,y) = \begin{cases} \dfrac{3xy}{x^2 + xy + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Check whether or not f is continuous at $(0,0)$.

Solution

Along line $y = x$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 1$$

Along x-axis $(y = 0)$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

Since limits along two different paths are different, the limit does not exist. So,

$$\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0) = 0.$$

Hence $f(x,y)$ is not continuous at $(0,0)$.

Note: (**alternative**) Since the limit along the line $y = x$ is equal to 1 and $f(0,0) = 0$, we can conclude that f is not continuous at $(0,0)$.

Q.4 a) The equation $xy + xz^3 - 2yz = 5$ defines z as an implicit function of x and y. Find $\dfrac{\partial z}{\partial x}$ at the point $(3, 2, 1)$.

Solution

Put $F(x, y, z) = xy + xz^3 - 2yz - 5$, then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} = -\frac{y + z^3}{3xz^2 - 2y},$$

So at the point $(3, 2, 1)$,

$$\frac{\partial z}{\partial x} = -\frac{2 + 1}{9 - 4} = -\frac{3}{5}.$$

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b) Find the linearization of $f(x, y) = \sqrt{x^2 + y^2}$ at the point $(4, 3)$

Solution

At $(4, 3)$, the linearization of f is

$$L(x, y) = f(4, 3) + f_x(4, 3)(x - 4) + f_y(4, 3)(y - 3).$$

Since

$$f_x = \frac{x}{\sqrt{x^2 + y^2}}, \quad f_y = \frac{y}{\sqrt{x^2 + y^2}},$$

we get

$$f_x(4, 3) = \frac{4}{5}, \quad f_y(4, 3) = \frac{3}{5}.$$

Hence

$$L(x, y) = 5 + \frac{4}{5}(x - 4) + \frac{3}{5}(y - 3).$$

or

$$L(x, y) = \frac{4}{5}x + \frac{3}{5}y.$$

Q.5 i) Find the directional derivative of the function $f(x,y) = \ln(x^2 + y^2)$ at the point $(1,2)$ in the direction of $\vec{v} = \langle -1, 2 \rangle$

ii) Find the maximum rate of change of f at the point $(1,2)$.

Solution

i)

$$\nabla f(x,y) = \Big\langle f_x(x,y), f_y(x,y) \Big\rangle = \left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle$$

Therefore,

$$\nabla f(1,2) = \left\langle \frac{2}{5}, \frac{4}{5} \right\rangle.$$

On the other hand,

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle.$$

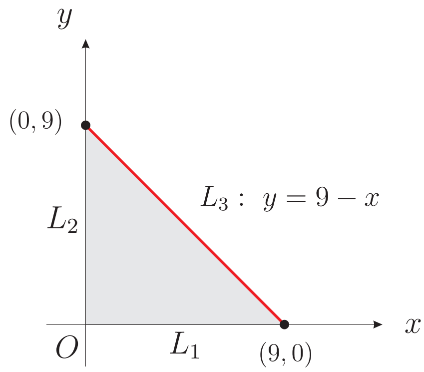
The directional derivative of f at the point $(1,2)$ in the direction of $\vec{v}$ is

$$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u} = \left(\frac{2}{5}\right)\left(\frac{1}{\sqrt{5}}\right) + \left(\frac{4}{5}\right)\left(\frac{2}{\sqrt{5}}\right) = \frac{6\sqrt{5}}{25}.$$

ii) The maximum rate of change of f at the point $(1,2)$ is

$$|\nabla f(1,2)| = \sqrt{\frac{4}{25} + \frac{16}{25}} = \frac{2\sqrt{5}}{5}.$$

Q.6 Find the absolute maximum and minimum values of $f(x,y) = 2 + 2x + 2y - x^2 - y^2$ on the closed triangular region in the first quadrant bounded by the lines $x = 0,\ y = 0,\ y = 9 - x$.



Solution

Step 1) Solving the following system

$$f_x = 2 - 2x = 0$$
$$f_y = 2 - 2y = 0$$

gives $x = y = 1$. The point $(1,1)$ is a critical point in the triangle.

Step 2) (Boundary Test)

- On $L_1:\ y = 0 \implies f(x,0) = 2 + 2x - x^2,\quad 0 \le x \le 9.$
  $(1,0)$ is the critical point
- On $L_2:\ x = 0 \implies f(0,y) = 2 + 2y - y^2,\quad 0 \le y \le 9.$
  $(0,1)$ is the critical point
- On $L_3:\ y = 9 - x \implies f(x, 9-x) = -2x^2 + 18x - 61,\ 0 \le x \le 9.$
  $(9/2, 9/2)$ is the critical point

Step 3) (To decide absolute extreme values)

Values of f at the critical points:

$$f(1,1) = 4$$
$$f(1,0) = 3$$
$$f(0,1) = 3$$
$$f(9/2, 9/2) = -41/2$$

Values of f at the end points:

$$f(0,0) = 2$$
$$f(9,0) = -61$$
$$f(0,9) = -61$$

Thus, the absolute minimum value $= -61$ and the absolute maximum value $= 4$