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 Department of Mathematics and Statistics
 Solution Exam 2 for Math 201 (081)

Q.1: (a) Find the point at which these lines intersect:

$$x = 1 + t, y = 1 - t, z = 2t \quad \text{and} \quad x = 2 - s, y = s, z = 2.$$

(b) Determine an equation of the plane that contains these lines.

Sol: (a) Solving the system

$$\begin{aligned} 1 + t &= 2 - s \\ 1 - t &= s \end{aligned}$$

we get $t = 1$ and $s = 2$. Putting $t = 1$ in the first equation or $s = 2$ in the second equation, we get the intersection point $(2, 0, 2)$.

(b) Direction vectors of these lines are $\vec{v}_1 = \langle 1, -1, 2 \rangle$ and $\vec{v}_2 = \langle -1, 1, 0 \rangle$. Since the required plane is passing through these lines, therefore these vectors are parallel to the plane. Hence the vector normal to

the plane is $\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = -2i - 2j + 0k$.

Equation of the plane is $-2(x - 2) - 2(y - 0) + 0(z - 2) = 0 \Rightarrow x + y - 2 = 0$.

Q.2: Consider the surface $\frac{z}{4} = \sqrt{x^2 + y^2}$.

(a) Describe the traces along the z -axis (parallel to xy -plane).

(b) Describe the traces along the x -axis (parallel to yz -plane).

(c) Identify and sketch the surface.

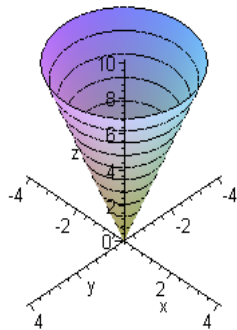
Sol: (a) For traces along z -axis, put $z = k$ with $k \geq 0$,

then $\sqrt{x^2 + y^2} = \frac{k}{4}$ or $x^2 + y^2 = \left(\frac{k}{4}\right)^2$ is a family of circles.

(b) For traces along x -axis, put $x = k$,

then $\sqrt{k^2 + y^2} = \frac{z}{4}$ or $4\sqrt{k^2 + y^2} = z$ is a family of straight lines. For $k = 0$, the line $z = 4|y|$ is passing through the origin $(0, 0, 0)$.

(c) The surface is a cone $\sqrt{x^2 + y^2} = \frac{z}{4}$



Q.3: Let $f(x, y) = \ln(36 - 4x^2 - 9y^2)$.

(a) Find and sketch the domain of f

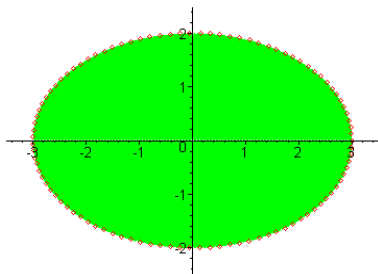
(b) Find the range of f

Sol: (a) $36 - 4x^2 - 9y^2 > 0$

$$4x^2 + 9y^2 < 36$$

$$\frac{x^2}{9} + \frac{y^2}{4} < 1$$

Therefore the domain is all points inside an open ellipse $\left\{ (x, y) \mid \frac{x^2}{9} + \frac{y^2}{4} < 1 \right\}$



(b) Range of f is $(-\infty, \ln 36]$

Q.4: If $f(x, y) = \frac{x^2 y}{x^4 + y^2}$, does the limit $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ exist? Justify your answer.

Sol: Let $(x, y) \rightarrow (0, 0)$ through y -axis, that is $x = 0$ and $y \rightarrow 0$, then

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2} = \lim_{y \rightarrow 0} \frac{0 \cdot y}{0 + y^2} = 0$$

Now let $(x, y) \rightarrow (0, 0)$ through x -axis, that is $y = 0$ and $x \rightarrow 0$, then

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2} = \lim_{x \rightarrow 0} \frac{x^2 \cdot 0}{x^4 + 0} = 0.$$

Now let $(x, y) \rightarrow (0, 0)$ through $y = x^2$, then

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2} = \lim_{x \rightarrow 0} \frac{x^2 \cdot x^2}{x^4 + x^4} = \lim_{x \rightarrow 0} \frac{x^4}{2x^4} = \lim_{x \rightarrow 0} \frac{1}{2} = \frac{1}{2}.$$

Since these limits are different, therefore limit $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist.

Q.5: Suppose over certain region of space the electrical potential V is given by $V(x, y, z) = 3xy^2 - y^2 + xyz$.

(a) Compute the rate of change of the potential at $A(1, 1, -1)$ in the direction of $\vec{u} = 2\hat{i} + \hat{j} - 3\hat{k}$.

(b) In which direction does V changes most rapidly?

(c) What is the maximum rate of change at A ?

Sol: Gradient vector is $\nabla V(x, y, z) = \langle 3y^2 + yz, 6xy - 2y + xz, xy \rangle$

$$\text{At } A(1, 1, -1) \nabla V(1, 1, -1) = \langle 2, 3, 1 \rangle$$

$$\text{The unit vector in the direction of } \vec{u} \text{ is } \hat{u} = \frac{\vec{u}}{\|\vec{u}\|} = \left\langle \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-3}{\sqrt{14}} \right\rangle.$$

Rate of change of V in the direction of \vec{u} is

$$\begin{aligned} D_{\vec{u}} V(1, 1, -1) &= \nabla V(1, 1, -1) \cdot \frac{\vec{u}}{\|\vec{u}\|} = \langle 2, 3, 1 \rangle \cdot \left\langle \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-3}{\sqrt{14}} \right\rangle \\ &= \frac{4 + 3 - 3}{\sqrt{14}} = \frac{4}{\sqrt{14}}. \end{aligned}$$

(b) V changes most rapidly at $A(1, 1, -1)$ in the direction of the gradient vector $\nabla V(1, 1, -1) = \langle 2, 3, 1 \rangle$.

(c) The maximum rate of change at $A(1, 1, -1)$ is $\|\nabla V(1, 1, -1)\| = \sqrt{4 + 9 + 1} = \sqrt{14}$.

Q.6: Find the maximum and minimum values of $f(x, y) = xy - x^3y^2$ over the region $R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

Sol: Find critical points inside the region R .

$$f_x(x, y) = y - 3x^2y = y(1 - 3x^2y), \quad f_y(x, y) = x - 2x^3y = x(1 - 2x^2y).$$

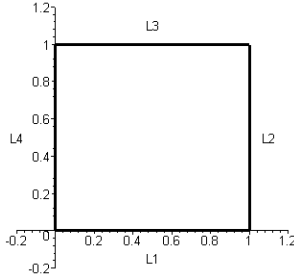
$$f_x(x, y) = 0, \quad f_y(x, y) = 0 \Rightarrow x = 0, y = 0, \text{ OR } 3x^2y = 1 \text{ and } 2x^2y = 1.$$

But the system $3x^2y = 1$ and $2x^2y = 1$ has no solution inside the region R .

Thus the function $f(x, y) = xy - x^3y^2$ has no critical point inside R .

The boundary of R is a square consists of line segments $L_1 = \{(x, y) | y = 0, 0 \leq x \leq 1\}$,

$L_2 = \{(x, y) | x = 1, 0 \leq y \leq 1\}$, $L_3 = \{(x, y) | y = 1, 0 \leq x \leq 1\}$, $L_4 = \{(x, y) | x = 0, 0 \leq y \leq 1\}$



On L_1 , $g(x) = f(x, 0) = 0$ for all $0 \leq x \leq 1$.

On L_2 , $g(y) = f(1, y) = y - y^2$, $0 \leq y \leq 1$ and $g(0) = f(1, 0) = 0$, $g(1) = f(1, 1) = 0$.

For the critical points on L_2 , $g'(y) = 1 - 2y = 0 \Rightarrow y = \frac{1}{2}$ and $g(\frac{1}{2}) = f(1, \frac{1}{2}) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

On L_3 , $g(x) = f(x, 1) = x - x^3$, $0 \leq x \leq 1$ and $g(0) = f(0, 1) = 0$, $g(1) = f(1, 1) = 0$.

For the critical points on L_3 , $g'(x) = 1 - 3x^2 = 0 \Rightarrow x = \pm \frac{1}{\sqrt{3}}$. But $-\frac{1}{\sqrt{3}} \notin R$.

and $g(\frac{1}{\sqrt{3}}) = f(\frac{1}{\sqrt{3}}, 1) = \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{2}{3\sqrt{3}}$.

On L_4 , $g(y) = f(0, y) = 0$ for all $0 \leq y \leq 1$.

Thus maximum value of f is $\frac{2}{3\sqrt{3}}$ and minimum value of f is 0.

Q.7: Let $f(x, y) = \sin\left(\frac{x}{y}\right) + e^{\frac{x}{y}} + 3x$. Then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$ is equal to:

(a) 0

(b) $\sin\left(\frac{x}{y}\right) - e^{\frac{x}{y}} + 3$

(c) $\cos\left(\frac{x}{y}\right) + e^{\frac{x}{y}}$

(d) $\cos\left(\frac{x}{y}\right) - e^{\frac{x}{y}}$

(e) $3x$ ----- \rightarrow Correct Answer

Sol: $\frac{\partial f}{\partial x} = \cos\left(\frac{x}{y}\right) \cdot \frac{1}{y} + e^{\frac{x}{y}} \cdot \frac{1}{y} + 3$ and $x\frac{\partial f}{\partial x} = \frac{x}{y} \cos\left(\frac{x}{y}\right) + \frac{x}{y} e^{\frac{x}{y}} + 3x$.

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{y}\right) \cdot \frac{-x}{y^2} + e^{\frac{x}{y}} \cdot \frac{-x}{y^2}$$

$$y\frac{\partial f}{\partial y} = \frac{-x}{y} \cos\left(\frac{x}{y}\right) - \frac{x}{y} e^{\frac{x}{y}}$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3x.$$

Q.8: The distance between the point $(2, -3, 4)$ and the plane $x + 2y + 2z - 13 = 0$ is equal to:

(a) 7

(b) 3 ----- \rightarrow Correct Answer

(c) 1

(d) 4

(e) 2

Sol: $d = \frac{|2 + 2(-3) + 2(4) - 13|}{\sqrt{1 + 4 + 4}} = 3$.

Q.9 : Find $\frac{\partial z}{\partial x} \Big|_{(\frac{\pi}{4}, 1, 1)}$ when $x - z + 1 = \arctan(yz)$.

- (a) 0
- (b) 1
- (c) $\frac{1}{2}$
- (d) $\frac{2}{3}$ ----- → Correct Answer
- (e) $-\frac{2}{3}$

Sol: Here $F(x, y, z) = x - z + 1 - \arctan(yz) = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{1}{-1 - \frac{y}{1+(yz)^2}} = \frac{1}{\frac{1+(yz)^2+y}{1+(yz)^2}} = \frac{1+(yz)^2}{1+(yz)^2+y}$$

$$\frac{\partial z}{\partial x} \Big|_{(\frac{\pi}{4}, 1, 1)} = \frac{1+(1)^2}{1+(1)^2+1} = \frac{2}{3}.$$

Q.10: If $(0, -2\sqrt{3}, -2)$ is a point in rectangular coordinates and ρ, θ, ϕ are its spherical coordinates, then $\rho + \tan \phi + \csc \theta$ is equal to:

- (a) $3 - \sqrt{3}$ ----- → Correct Answer
- (b) $3 + \sqrt{3}$
- (c) $5 + \sqrt{3}$
- (d) $5 - \sqrt{3}$
- (e) None of these

Sol: $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0 + 12 + 4} = 4$
 $\cos \phi = \frac{z}{\rho} = \frac{-2}{4} = \frac{-1}{2} \Rightarrow \phi = \frac{2\pi}{3}$
 $\cos \theta = \frac{x}{\rho \sin \phi} = \frac{0}{4 \sin \frac{2\pi}{3}} = 0 \Rightarrow \theta = \frac{3\pi}{2}, (\theta \neq \frac{\pi}{2} \text{ since } y = -2\sqrt{3} < 0)$
 $\rho + \tan \phi + \csc \theta = 4 + \tan \frac{2\pi}{3} + \csc \frac{3\pi}{2} = 4 + (-\sqrt{3}) + (-1) = 3 - \sqrt{3}.$

Q.11: If (x, y) changes from $(2, -1)$ to $(1.96, -0.95)$ in the function $z = x^2 - xy + 3y^2$, then the value of the differential dz is:

- (a) 1.6
- (b) 0.45
- (c) -0.60 ----- → Correct Answer
- (d) 0.03
- (e) -1.5

Sol: Here $x = 2, y = -1, \Delta x = dx = -0.04,$ and $\Delta y = dy = 0.05.$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = (2x - y) dx + (-x + 6y) dy$$

$$= (4 + 1)(-0.04) + (-2 - 6)(0.05) = -0.2 - 0.4 = -0.6.$$

Q.12: Suppose that $w = r^2 \cos(2\theta)$, where r and θ are the polar coordinates, then $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}$ is equal to:

- (a) $3(x - y)$
- (b) $2(x - y)$ ----- → Correct Answer
- (c) $3(x + y)$
- (d) $2x - 3y$
- (e) $3x + 2y$

Sol: $w = r^2 (\cos^2 \theta - \sin^2 \theta) = x^2 - y^2$ and $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = 2x - 2y = 2(x - y).$