Q.1 a) Find a Cartesian equation of the curve whose polar equation is given by $r = \dfrac{\sin 2\theta - \cos 2\theta}{\sin\theta\cos\theta}$

Solution

$$r = \frac{\sin 2\theta - \cos 2\theta}{\sin\theta\cos\theta} = \frac{2\sin\theta\cos\theta - \cos^{\theta} + \sin^2\theta}{\sin\theta\cos\theta}$$

$$= 2 - \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = 2 - \frac{1}{\tan\theta} + \tan\theta$$

We know that $r = \sqrt{x^2+y^2}$ and $\tan\theta = y/x$. Thus, a Cartesian equation of the given polar equation is

$$\sqrt{x^2+y^2} = 2 - \frac{x}{y} + \frac{y}{x}.$$

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b) Find the area of the surface obtained by rotating the curve with parametric equations $x = a\cos^3\theta,\ y = a\sin^3\theta,\ 0 \le \theta \le \pi/2$ about the x-axis.

Solution

$$S = 2\pi\int_\alpha^\beta y\sqrt{\left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2}\, d\theta$$

$$= 2\pi\int_0^{\pi/2} a\sin^3\theta\sqrt{9a^2\sin^4\theta\cos^2\theta + 9a^2\cos^4\theta\sin^2\theta}\ d\theta$$

$$= 2\pi\int_0^{\pi/2} a\sin^3\theta\sqrt{9a^2\sin^2\theta\cos^2\theta(\sin^2\theta+\cos^2\theta)}\ d\theta$$

$$= 2\pi\int_0^{\pi/2} a\sin^3\theta\sqrt{9a^2\sin^2\theta\cos^2\theta}\ d\theta$$

$$= 2\pi\int_0^{\pi/2} a\sin^3\theta\ (3a\sin\theta\cos\theta)\ d\theta, \qquad (\text{since } \ 0 \le \theta \le \pi/2)$$

$$= 2\pi\int_0^{\pi/2} 3a^2\sin^4\theta\cos\theta)\ d\theta = 6\pi a^2\int_0^{\pi/2}\sin^4\theta\cos\theta\ d\theta$$

$$= 6\pi a^2\left[\frac{\sin^5\theta}{5}\right]_0^{\pi/2} = \frac{6\pi a^2}{5}$$

Q.2 Consider the polar equation $r = 1 - \sin\theta$.

i) Sketch the curve of the given polar equation

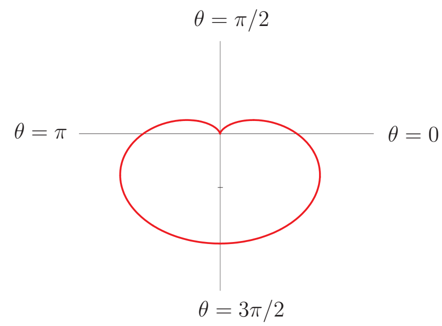
ii) Find the equation of the tangent line to the polar curve at $\theta = \pi/3$

Solution

i)

θ	$0 \to \pi/2$	$\pi/2 \to \pi$	$\pi \to 3\pi/2$	$3\pi/2 \to 2\pi$
$r = 1 - \sin\theta$	$1 \to 0$	$0 \to 1$	$1 \to 2$	$2 \to 1$

The graph of the polar equation is given by the following picture.



ii)

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{-\cos\theta\sin\theta + (1-\sin\theta)\cos\theta}{-\cos^2\theta - (1-\sin\theta)\sin\theta} = \frac{\cos\theta - \sin 2\theta}{-\sin\theta - \cos 2\theta}$$

At $\theta = \pi/3$, we have

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{\cos\pi/3 - \sin 2\pi/3}{-\sin\pi/3 - \cos 2\pi/3} = \frac{1/2 - \sqrt{3}/2}{-\sqrt{3}/2 + 1/2} = 1,$$

$$x = \left(1 - \sin\frac{\pi}{3}\right)\cos\frac{\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{4}, \quad y = \left(1 - \sin\frac{\pi}{3}\right)\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} - \frac{3}{4}$$

The equation of the tangent line to the polar curve at $\theta = \pi/3$ is

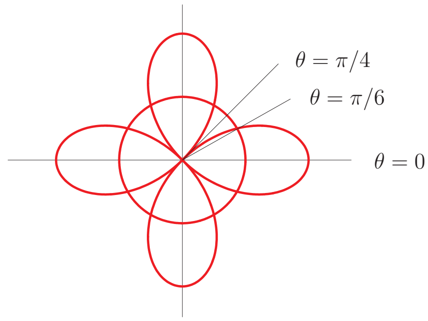
$$y - \left(\frac{\sqrt{3}}{2} - \frac{3}{4}\right) = 1\left[x - \left(\frac{1}{2} - \frac{\sqrt{3}}{4}\right)\right]$$

or

$$y = x - \frac{5}{4} + \frac{3\sqrt{3}}{4}$$

Q.3 Compute the area that lies inside both curves $r = 2\cos 2\theta$ and $r = 1$.

Solution



$$
\begin{aligned}
A &= 8\left(\int_0^{\pi/6} \frac{1}{2}(1^2)\ d\theta + \int_{\pi/6}^{\pi/4} \frac{1}{2}(4\cos^2 2\theta)d\theta\right) \\
&= 8\left(\frac{\pi}{12} + 2\int_{\pi/6}^{\pi/4} \frac{1}{2}(1+\cos 4\theta)d\theta\right) \\
&= 8\left(\frac{\pi}{12} + \left[\theta + \frac{\sin 4\theta}{4}\right]_{\pi/6}^{\pi/4}\right) \\
&= 8\left(\frac{\pi}{12} + \frac{\pi}{4} - \frac{\pi}{6} + \frac{\sin\pi - \sin 2\pi/3}{4}\right) \\
&= 8\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right) \\
&= \frac{4\pi}{3} - \sqrt{3}
\end{aligned}
$$

Q.4 Let $\vec{a} = \langle -1, -2, 2 \rangle$ and $\vec{b} = \langle 3, 0, 4 \rangle$.

i) Find the vector projection $\vec{v} = \text{proj}_{\vec{a}}\, \vec{b}$.

ii) Evaluate $\vec{v} \cdot (\vec{b} - \vec{v})$.

Solution

i)

$$\begin{aligned} \vec{v} &= \text{proj}_{\vec{a}}\, \vec{b} \\ &= \left(\frac{a \cdot b}{|a|^2} \right) \vec{a} \\ &= \frac{-3+8}{3^2} \langle -1, -2, 2 \rangle \\ &= \left\langle \frac{-5}{9}, \frac{-10}{9}, \frac{10}{9} \right\rangle \end{aligned}$$

ii)

$$\begin{aligned} \vec{v} \cdot (\vec{b} - \vec{v}) &= \left\langle \frac{-5}{9}, \frac{-10}{9}, \frac{10}{9} \right\rangle \cdot \left\langle 3 + \frac{5}{9}, \frac{10}{9}, 4 - \frac{10}{9} \right\rangle \\ &= -\frac{15}{9} - \frac{25}{81} - \frac{100}{81} + \frac{40}{9} - \frac{100}{81} \\ &= \frac{-135 - 25 - 100 - 100 + 360}{81} \\ &= 0 \end{aligned}$$

Q.5 Consider a triangle with vertices $A(0,0,b)$, $B(1,1,0)$ and $C(2,3,0)$.

i) Compute the area of the given triangle.

ii) Find all values of b such that the area of the triangle is equal to 8.

Solution

i) $\overrightarrow{AB} = \langle 1, 1, -b\rangle$ and $\overrightarrow{AC} = \langle 2, 3, -b\rangle$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -b \\ 2 & 3 & -b \end{vmatrix}$$

$$= 2b\vec{i} - b\vec{j} + \vec{k}$$

$$\text{Area of the triangle} = \frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{2}$$

$$= \frac{\sqrt{4b^2 + b^2 + 1}}{2}$$

$$= \frac{\sqrt{5b^2 + 1}}{2}$$

ii) Area of the triangle $= \dfrac{\sqrt{5b^2+1}}{2} = 8$. So,

$$\sqrt{5b^2+1} = 16$$

$$5b^2 + 1 = 256$$

$$b^2 = \frac{255}{5}$$

$$b^2 = 51$$

$$b = \pm\sqrt{51}$$

Q.6 Find an equation of the plane that contains the line $x = y = 3z - 1$ and the point $B(2, 3, 0)$.

Solution

Parametric equations of the line are given by

$$x = t, \quad y = t, \quad z = \frac{1}{3} + \frac{1}{3}t.$$

So, $A(0, 0, 1/3)$ is a point on the line and $\vec{v} = \langle 1, 1, 1/3 \rangle$ is a vector parallel to the line. Since the required plane contains the line, the point A is also on the plane. Thus, the normal vector of the plane is

$$\vec{n} = \overrightarrow{AB} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1/3 \\ 1 & 1 & 1/3 \end{vmatrix}$$

$$= \frac{4}{3}\vec{i} - \vec{j} - \vec{k}.$$

We have $\vec{n} = \langle 4/3, -1, -1 \rangle$ and a point on the plane $B(2, 3, 0)$. Hence, the equation of the plane is given by

$$\frac{4}{3}(x - 2) - (y - 3) - z = 0$$

$$\text{or} \quad \frac{4}{3}x - y - z + \frac{1}{3} = 0$$

$$\text{or} \quad 4x - 3y - 3z + 1 = 0$$