Q.1 Consider the surface with equation $36x^2 + 9y^2 + 4z^2 = 36$.

i) If $P_0(x_0, y_0, z_0)$ is a point on the given surface, show that the equation of the tangent plane to the surface at P_0 is

$$x_0 x + \frac{y_0 y}{4} + \frac{z_0 z}{9} = 1$$

ii) Identify and sketch the surface.

Solution:

i) Dividing by 36, the equation of the surface reads $x^2+y^2/4+z^2/9 = 1$ Let $F(x, y, z) = x^2 + y^2/4 + z^2/9$. We have $\nabla F = \langle 2x, y/2, 2z/9\rangle$. The equation of the tangent plane to the surface at (x_0, y_0, z_0) is

$$\nabla F(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$$

$$\text{or} \quad \langle 2x_0, y_0/2, 2z_0/9\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$$

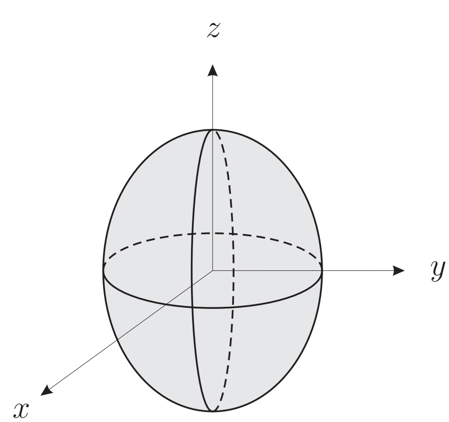
$$\text{or} \quad 2x_0(x - x_0) + \frac{y_0}{2}(y - y_0) + \frac{2z_0}{9}(z - z_0) = 0$$

$$\text{or} \quad 2x_0 x + \frac{y_0 y}{2} + \frac{2z_0 z}{9} = 2x_0^2 + \frac{y_0^2}{2} + \frac{2z_0^2}{9}$$

$$\text{or} \quad x_0 x + \frac{y_0 y}{4} + \frac{z_0 z}{9} = x_0^2 + \frac{y_0^2}{4} + \frac{z_0^2}{9}$$

$$\text{or} \quad x_0 x + \frac{y_0 y}{4} + \frac{z_0 z}{9} = 1$$

ii) From i) we know that the surface is an ellipsoid.



Q.2 Consider the function $f(x,y) = \sqrt{3} + 4xy - x^4 - y^4$.

i) Find all critical points of f.

ii) Find the relative (local) maximum and minimum values and saddle points of f.

Solution:

i) The first derivatives of f are given by

$$f_x(x,y) = 4y - 4x^3, \qquad f_y(x,y) = 4x - 4y^3$$

From the equation $4y - 4x^3 = 0$, we get $y = x^3$. From equation $4x - 4y^3 = 0$, we get

$$4x - 4x^9 = 0 \quad \text{or} \quad x(1 - x^8) = 0.$$

So, we have $x = 0, 1, -1$. The critical points are $(0,0)$, $(1,1)$ and $(-1,-1)$.

ii) The second derivatives of f are

$$f_{xx} = 12x^2, \qquad f_{yy} = 12y^2, \qquad f_{xy} = 4.$$

So

$$\begin{aligned} D(x,y) &= f_{xx} f_{yy} - [f_{xy}]^2 \\ &= 12x^2(12y^2) - (4)^2 = 144x^2y^2 - 16. \end{aligned}$$

$D(0,0) = -16 < 0$,
$(0,0)$ is a saddle point.
$D(1,1) = 144 - 16 > 0$, $f_{xx}(1,1) = 12 > 0$,
$(1,1)$ is a maximum point.
$D(-1,-1) = 144 - 16 > 0$, $f_{xx}(1,1) = 12 > 0$,
$(-1,-1)$ is a maximum point.

Q.3 Let $f(x,y) = \dfrac{e^{\sqrt{x^2+y^2-1}}}{3+\sqrt{4-x^2-y^2}}$.

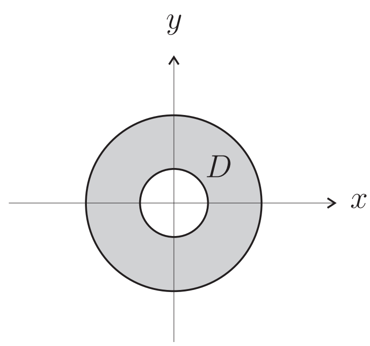
i) Find and sketch the domain of f.

ii) Find the range of f

Solution:

i) The domain of f is

$$D = \{(x,y)|\ 1 \le x^2 + y^2 \le 4\}.$$



ii) The range of f is

$$R = \{z \mid \frac{1}{3+\sqrt{3}} < z < \frac{e^{\sqrt{3}}}{5}\}$$

Q.4 (a) Let $f(x,y) = \dfrac{xy^3}{x^2+y^6}$.

i) Evaluate $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ along the line $y = mx$, where m is a constant.

ii) Evaluate $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ along the curve $y^3 = x$.

iii) Does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist? Justify your answer.

Solution:

i) Along the line $y = mx$,

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{(x,y)\to(0,0)} \frac{x(mx)^3}{x^2+(mx)^6} \\ &= \lim_{(x,y)\to(0,0)} \frac{m^3x^4}{x^2+m^6x^6} \\ &= \lim_{(x,y)\to(0,0)} \frac{m^3x^2}{1+m^6x^4} = 0. \end{aligned}$$

ii) Along the curve $y^3 = x$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{y^6}{y^6+y^6} = \lim_{(x,y)\to(0,0)} \frac{y^6}{2y^6} = \frac{1}{2}$$

iii) $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist, since two different paths give two different limits.

(b) The equation of a surface in spherical coordinates is $\rho = \sin 2\phi \cos 2\theta$. Find the equation of the given surface in rectangular coordinates.

Solution:

$$\begin{aligned} \rho &= \sin 2\phi \cos 2\theta = 2\sin\phi\cos\phi(\cos^2\theta - \sin^2\theta) \\ \rho^3 &= 2\rho^2 \sin\phi\cos\phi(\cos^2\theta - \sin^2\theta) \\ (x^2+y^2+z^2)^{3/2} &= 2rz\left(\frac{x^2}{r^2} - \frac{y^2}{r^2}\right) = 2z\left(\frac{x^2}{r} - \frac{y^2}{r}\right) \\ (x^2+y^2+z^2)^{3/2} &= 2z\left(\frac{x^2-y^2}{\sqrt{x^2+y^2}}\right) \end{aligned}$$

Q.5 (a) Let $w = \sin(\sqrt{x} + \sqrt{y}), \quad x = \sqrt{u^2 + v^2}, \quad y = e^{uv}$.

i) Find $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$.

ii) Evaluate $\dfrac{\partial w}{\partial u} + 2\dfrac{\partial w}{\partial v}$ at the point $(u, v) = (\pi, 0)$.

Solution:

i) Using the chain rule, we get

$$\begin{aligned}\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} \\ &= \frac{u\cos(\sqrt{x} + \sqrt{y})}{2\sqrt{x}\sqrt{u^2 + y^2}} + \frac{ve^{uv}\cos(\sqrt{x} + \sqrt{y})}{2\sqrt{y}}\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} \\ &= \frac{v\cos(\sqrt{x} + \sqrt{y})}{2\sqrt{x}\sqrt{u^2 + y^2}} + \frac{ue^{uv}\cos(\sqrt{x} + \sqrt{y})}{2\sqrt{y}}\end{aligned}$$

ii) At $(u, v) = (\pi, 0)$, $x = \pi, \quad y = e^0 = 1$. So

$$\frac{\partial w}{\partial u} = \frac{\pi\cos(\sqrt{\pi} + 1)}{2\pi\sqrt{\pi}}, \qquad \frac{\partial w}{\partial v} = \frac{\pi\cos(\sqrt{\pi} + 1)}{2}$$

Thus $\dfrac{\partial w}{\partial u} + 2\dfrac{\partial w}{\partial v} = \left(\dfrac{1}{2\sqrt{\pi}} + \pi\right)\cos(\sqrt{\pi} + 1)$

(b) Let $z = \ln(\sqrt{x^2 + y^2})$ and (x, y) changes from $(3, 4)$ to $(2.95, 4.1)$.

Use differentials to estimate the change Δz of z.

Solution: $dx = \Delta x = -0.05, \quad dy = \Delta y = 0.1, \quad x = 3, \quad y = 4$

$$\begin{aligned}\Delta z \approx dz &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \\ &= \frac{x}{x^2 + y^2}dx + \frac{y}{x^2 + y^2}dy = \frac{3}{25}(-0.05) + \frac{4}{25}(0.1) = 0.01.\end{aligned}$$

Q.6 (a) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if the equation $z\sin x + y\cos z^2 - yz^2 + 4 = 0$ defines z as a function of two independent variables x and y.

Solution: Let $F(x,y,z) = z\sin x + y\cos z^2 - yz^2 + 4$.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{-z\cos x}{\sin x - 2yz\sin z^2 - 2yz}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{z^2 - \cos z^2}{\sin x - 2yz\sin z^2 - 2yz}$$

(b) Let $h(u,v) = u^2 v\tan(uv)$. Find the value of $\left(\dfrac{\partial h}{\partial u}\right)^2 + \dfrac{\partial^2 h}{\partial v^2} - \dfrac{\partial^2 h}{\partial t \partial v}$ at the point $(\pi, 1)$.

Solution:

$$\frac{\partial h}{\partial u} = 2uv\tan(uv) + u^2v^2\sec^2(uv)$$

$$\frac{\partial h}{\partial v} = u^2\tan(uv) + u^3 v\sec^2(uv)$$

$$\frac{\partial^2 h}{\partial v^2} = 2u^3\sec^2(uv) + 2u^4 v\tan(uv)\sec^2(uv)$$

$$\frac{\partial^2 h}{\partial u \partial v} = 2u\tan(uv) + u^2 v\sec^2(uv) + 3u^2 v\sec^2(uv) + u^3v^2\tan(uv)\sec^2(uv)$$

At $(\pi, 1)$,

$$\frac{\partial h}{\partial u} = \pi^2, \quad \frac{\partial^2 h}{\partial v^2} = 2\pi^3, \quad \frac{\partial^2 h}{\partial u \partial v} = 4\pi^2.$$

So, at the given point we have

$$\left(\frac{\partial h}{\partial u}\right)^2 + \frac{\partial^2 h}{\partial v^2} - \frac{\partial^2 h}{\partial t \partial v} = \pi^4 + 2\pi^3 - 4\pi^2.$$