

- Q.1** a) i) Find the Cartesian equation of the curve whose parametric equations are given by $x = 2 \cot t$, $y = 2 \sin^2 t$.
 ii) Find the point where the curve intersects the y -axis.

Solution:

i) $y = 2 \sin^2 t$ or $\sin^2 t = y/2$.

$$x = 2 \cot t \text{ implies that } x^2 = 4 \cot^2 t = \frac{4 \cos^2 t}{\sin^2 t}$$

$$\text{or } x^2 = \frac{4(1 - y/2)}{y/2} = \frac{4(2 - y)}{y}$$

Thus, the cartesian equation of the curve is $x^2 y = 8 - 4y$

- ii) The curve intersects the y -axis at $(0, 2)$.
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- b) Find the points on the curve $x = 6t - t^3$, $y = 3t^2$ where the tangent is parallel to the line with equation $y = 5 - 2x$.

Solution:

$$\frac{dx}{dt} = 6t - t^3; \quad \frac{dy}{dt} = 3t^2$$

Slope of the given curve is

$$\frac{dy}{dx} = \frac{6t}{6 - 3t^2}$$

Slope of the line $y = 5 - 2x$ is equal to -2 . Both slopes are equal. So, we have

$$\frac{6t}{6 - 3t^2} = -2 \quad \text{or} \quad 6t^2 - 6t - 12 = 0 \quad \text{or} \quad t^2 - t - 2 = 0$$

$$t = -1, 2$$

At $t = -1$: point $(-5, 3)$

At $t = 2$: point $(4, 12)$.

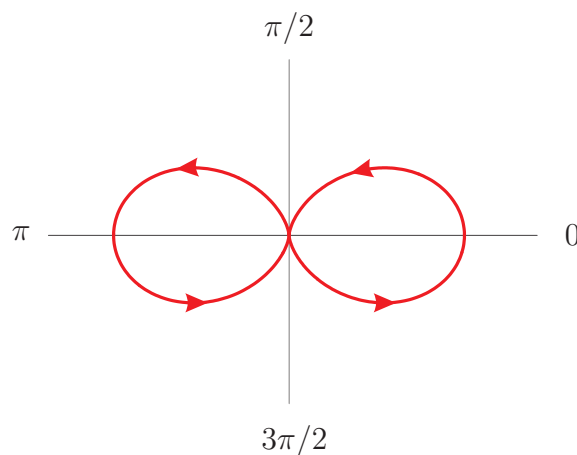
Q.2 Consider the polar equation $r = 2 + 2 \cos 2\theta$.

- i) Sketch the curve of the given polar equation
- ii) Find the slope of the tangent line to the polar curve at $\theta = \pi/4$

Solution:

i)

θ	$0 \rightarrow \frac{\pi}{4}$	$\frac{\pi}{4} \rightarrow \frac{\pi}{2}$	$\frac{\pi}{2} \rightarrow \frac{3\pi}{4}$	$\frac{3\pi}{4} \rightarrow \pi$	$\pi \rightarrow \frac{5\pi}{4}$	$\frac{5\pi}{4} \rightarrow \frac{3\pi}{2}$	$\frac{3\pi}{2} \rightarrow \frac{7\pi}{4}$	$\frac{7\pi}{4} \rightarrow 2\pi$
r	$4 \rightarrow 2$	$2 \rightarrow 0$	$0 \rightarrow 2$	$2 \rightarrow 4$	$4 \rightarrow 2$	$2 \rightarrow 0$	$0 \rightarrow 2$	$2 \rightarrow 4$



$$\text{ii) } \frac{dr}{d\theta} = -4 \sin 2\theta, \quad \frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta = -4 \sin 2\theta \cos \theta - (2 + 2 \cos 2\theta) \sin \theta$$

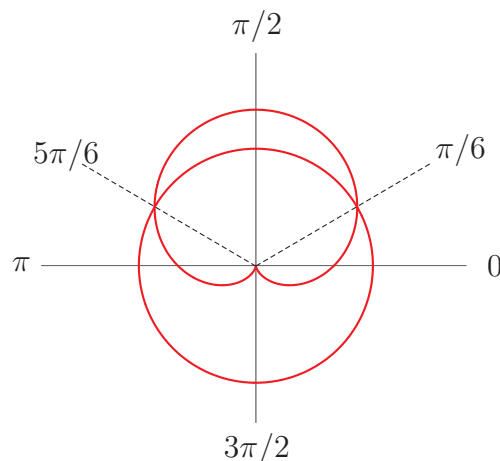
$$\frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta = -4 \sin 2\theta \sin \theta - (2 + 2 \cos 2\theta) \cos \theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-4 \sin 2\theta \sin \theta - (2 + 2 \cos 2\theta) \cos \theta}{-4 \sin 2\theta \cos \theta - (2 + 2 \cos 2\theta) \sin \theta}$$

$$\begin{aligned} \text{At } \theta = \frac{\pi}{4}, \quad \frac{dy}{dx} &= \frac{-4 \sin \pi/2 \sin \pi/4 - (2 + 2 \cos \pi/2) \cos \pi/4}{-4 \sin \pi/2 \cos \pi/4 - (2 + 2 \cos \pi/2) \sin \pi/4} \\ &= \frac{-4(1)(\sqrt{2}/2) + (2)((\sqrt{2}/2))}{-4(1)(\sqrt{2}/2) - (2)((\sqrt{2}/2))} = \frac{-4 + 2}{-4 - 2} = \frac{1}{3} \end{aligned}$$

- Q.3** a) Find the area of the region inside the polar curve $r = 2 + 2 \sin \theta$ and outside the curve $r = 3$.

Solution:



Points of intersection of the curves:

$$3 = r = 2 + 2 \sin \theta \implies \sin \theta = 1/2 \implies \theta = \pi/6, 5\pi/6.$$

By symmetry; $2(\pi/6 \leq \theta \leq \pi/2)$

$$\begin{aligned} A &= 2 \int_{\pi/6}^{\pi/2} \frac{1}{2} [(2 + 2 \sin \theta)^2 - 9] d\theta = \int_{\pi/6}^{\pi/2} [4 \sin^2 \theta + 8 \sin \theta - 5] d\theta \\ &= \int_{\pi/6}^{\pi/2} \left[4 \left(\frac{1 - \cos 2\theta}{2} \right) + 8 \sin \theta - 5 \right] d\theta \\ &= \int_{\pi/6}^{\pi/2} [-2 \cos 2\theta + 8 \sin \theta - 3] d\theta \\ &= \left[-\sin 2\theta - 8 \cos \theta - 3\theta \right]_{\pi/6}^{\pi/2} = -\pi + 4\sqrt{3} + \frac{\sqrt{3}}{2} \end{aligned}$$

- b) Set up an integral to find the area of one loop of the rose $r = 3 \cos 6\theta$.
(Do Not Evaluate the Integral)

Solution:

$$0 = r = 3 \cos 6\theta \implies 6\theta = \pm \frac{\pi}{2} \implies \theta = \pm \frac{\pi}{12}.$$

$$\text{Area of one loop} = \frac{1}{2} \int_{-\pi/12}^{\pi/12} 9 \cos^2 6\theta d\theta$$

Q.4 A sphere has equation $x^2 + y^2 + z^2 = 10y - 16z + C$, where C is a constant.

- i) Find the center of the sphere
- ii) Find the radius of the sphere in terms of C .
- iii) If the radius of the sphere is equal to 10, find the points where the sphere intersects the y -axis.

Solution:

i) $x^2 + (y - 5)^2 + (z + 8)^2 = 25 + 64 + C = 89 + C$.

The center of the sphere is $(0, 5, -8)$.

ii) Radius of the sphere is $\sqrt{89 + C}$

iii) $\sqrt{89 + C} = 10 \implies C = 100 - 89 = 11$.

Equation becomes $x^2 + y^2 + z^2 = 10y - 16z + 11$.

The sphere intersects the y -axis when $x = z = 0$

So that $y^2 - 10y - 11 = 0$.

This gives $(y - 11)(y + 1) = 0$ i.e. $y = 11$, $y = -1$.

The required points are $(0, 11, 0)$ and $(0, -1, 0)$.

Q.5 Let $\vec{a} = \langle \sqrt{2}, 1, 1 \rangle$ and $\vec{b} = \langle -\sqrt{2}, 4, -1 \rangle$ be two vectors in \mathbb{R}^3 .

- i) Find the scalar projection and vector projection of \vec{b} onto \vec{a} .
- ii) Find the angle between the vectors \vec{a} and $\vec{a} + \vec{b}$.
- iii) If $\vec{r} = \langle x, y, z \rangle$, show that the vector equation $(\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{b}) = 0$ represents a sphere.

Solution:

i)

$$\text{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{-2 + 4 - 1}{\sqrt{2 + 1 + 1}} = \frac{1}{2}$$

$$\text{proj}_{\vec{a}} \vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a} = \frac{1}{4} \langle \sqrt{2}, 1, 1 \rangle = \left\langle \frac{\sqrt{2}}{4}, \frac{1}{4}, \frac{1}{4} \right\rangle$$

ii) $\vec{a} + \vec{b} = \langle 0, 5, 0 \rangle$

$$\vec{a} \cdot (\vec{a} + \vec{b}) = |\vec{a}| |\vec{a} + \vec{b}| \cos \theta$$

$$5 = (2)(5) \cos \theta$$

$$\cos \theta = \frac{1}{2} \implies \theta = \frac{\pi}{3}$$

iii)

$$(\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{b}) = 0$$

$$(x - \sqrt{2})(x - \sqrt{2}) + (y - 1)(y - 4) + (z - 1)(z + 1) = 0$$

$$x^2 - 2 + y^2 + 5y + 4 + z^2 - 1 = 0$$

$$x^2 + (y - 5/2)^2 + z^2 = \frac{21}{4}$$

This is a sphere with radius $\sqrt{21}/2$ and center $(0, 5/2, 0)$.

Q.6 Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$ and $R(1, -1, 1)$.

Solution:

Let

$$\vec{a} = \vec{PQ} = \langle -3, 1, -7 \rangle$$

$$\vec{b} = \vec{PR} = \langle 0, -5, -5 \rangle$$

Area of triangle $PQR = \frac{1}{2}|\vec{a} \times \vec{b}|$

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} \\ &= -40\vec{i} - 15\vec{j} + 15\vec{k} \\ &= 5\langle -8, -3, 3 \rangle\end{aligned}$$

$$|\vec{a} \times \vec{b}| = 5\sqrt{64 + 9 + 9} = 5\sqrt{82}$$

Area of the triangle is $\frac{5}{2}\sqrt{82}$