Optim Lett (2008) 2:599–604
DOI 10.1007/s11590-008-0075-8

SHORT COMMUNICATION

Note on Mond–Weir type nondifferentiable second order symmetric duality

Z. Husain · I. Ahmad

Received: 20 July 2007 / Accepted: 11 January 2008 / Published online: 29 January 2008

Abstract In this paper, we point out some inconsistencies in the earlier work of Ahmad and Husain (Appl. Math. Lett. **18**, 721–728, 2005), and present the correct forms of their strong and converse duality theorems.

Keywords Nondifferentiable programming · Multiobjective programming · Second order symmetric duality

1 Introduction

In connection with symmetric duality in multiobjective programming, Ahmad and Husain [1] formulated a pair of Mond–Weir type nondifferentiable multiobjective second order symmetric dual programs and established weak, strong and converse duality theorems under second order F-pseudoconvexity/F-pseudoconcavity assumpions. But, in their strong duality theorem, the assumptions made by them are seemed to be incorrect.

In this paper, we attempt to trace out this incorrectness and to resolve it by making some modifications in the assumptions.

The research of Z. Husain is supported by the Department of Atomic Energy, Government of India, under the NBHM Post-Doctoral Fellowship Program No. 40/9/2005-R&D II/2398.

Z. Husain (✉) · I. Ahmad
Department of Mathematics,
Aligarh Muslim University, Aligarh 202002, India
e-mail: zhusain_amu@hotmail.com

I. Ahmad
e-mail: izharamu@hotmail.com

2 Notations

The following convention for inequalities will be used: If $x, u \in R^n$, then

$$x > u \Leftrightarrow x_i > u_i, \quad i = 1, 2, \ldots n,$$
$$x \geqq u \Leftrightarrow x_i \geqq u_i, \quad i = 1, 2, \ldots n,$$
$$x \geq u \Leftrightarrow x \geqq u \quad \text{and} \quad x \neq u.$$

If $F(x, y)$ is a scalar valued twice differentiable function of x and y, where $x \in R^n$ and $y \in R^m$, then $\nabla_x F$ and $\nabla_y F$ denote the gradient (column) vectors with respect to first and second variables, respectively. Also, $\nabla_{xx} F$, $\nabla_{yx} F$ and $\nabla_{yy} F$ are respectively, the $n \times n$, $n \times m$ and $m \times m$ matrices of second order partial derivatives.

For other definitions and preliminaries, we refer to Ahmad and Husain [1].

3 Symmetric duality

Ahmad and Husain [1] discussed weak, strong and converse duality theorems for the following pair of nondifferentiable multiobjective second order symmetric dual programs:

(MP) Minimize $K(x,y,w,p)=\big(K_1(x,y,w,p), K_2(x,y,w,p), \ldots, K_k(x, y, w, p)\big)$
subject to

$$\sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x, y) - C_i w_i + \nabla_{yy} f_i(x, y) p_i] \leqq 0,$$
$$y^t \sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x, y) - C_i w_i + \nabla_{yy} f_i(x, y) p_i] \geqq 0,$$
$$w_i^t C_i w_i \leqq 1, \quad i = 1, 2, \ldots, k,$$
$$\lambda > 0,$$
$$x \geqq 0.$$

(MD) Maximize $G(u, v, z, r) = (G_1(u, v, z, r), G_2(u, v, z, r), \ldots, G_k(u, v, z, r))$
subject to

$$\sum_{i=1}^{k} \lambda_i [\nabla_x f_i(u, v) + B_i z_i + \nabla_{xx} f_i(u, v) r_i] \geqq 0,$$
$$u^t \sum_{i=1}^{k} \lambda_i [\nabla_x f_i(u, v) + B_i z_i + \nabla_{xx} f_i(u, v) r_i] \leqq 0,$$
$$z_i^t B_i z_i \leqq 1, \quad i = 1, 2, \ldots, k,$$
$$\lambda > 0,$$
$$v \geqq 0,$$

where

$$K_i(x, y, w, p) = f_i(x, y) + (x^t B_i x)^{\frac{1}{2}} - y^t C_i w_i - \frac{1}{2} p_i^t \nabla_{yy} f_i(x, y) p_i,$$

$$G_i(u, v, z, r) = f_i(u, v) - (v^t C_i v)^{\frac{1}{2}} + u^t B_i z_i - \frac{1}{2} r_i^t \nabla_{xx} f_i(u, v) r_i,$$

$\lambda_i \in R$, p_i, $w_i \in R^m$, r_i, $z_i \in R^n$, $i = 1, 2, \ldots, k$, and f_i, $i = 1, 2, \ldots, k$, are twice differentiable functions from $R^n \times R^m$ to R, B_i and C_i, $i = 1, 2, \ldots, k$, are positive semidefinite symmetric matrices of order n and m, respectively. Also, we take $p = (p_1, p_2, \ldots, p_k)$, $r = (r_1, r_2, \ldots, r_k)$, $w = (w_1, w_2, \ldots, w_k)$, and $z = (z_1, z_2, \ldots, z_k)$.

Ahmad and Husain [1] proved the following strong duality theorem for (MP) and (MD):

Theorem 1 (Strong duality) *Let* $f : R^n \times R^m \to R^k$ *be thrice differentiable. Let* $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p})$ *be a weak efficient solution for (MP), and* $\lambda = \bar{\lambda}$ *fixed in (MD). Assume that*

(i) $\nabla_{yy} f_i$ *is nonsingular for all* $i = 1, 2, \ldots, k$,
(ii) *the matrix* $\sum_{i=1}^{k} \bar{\lambda}_i (\nabla_{yy} f_i \bar{p}_i)_y$ *is positive or negative definite; and*
(iii) *the set* $\{\nabla_y f_1 - C_1 \bar{w}_1 + \nabla_{yy} f_1 \bar{p}_1, \nabla_y f_2 - C_2 \bar{w}_2 + \nabla_{yy} f_2 \bar{p}_2, \ldots, \nabla_y f_k - C_k \bar{w}_k + \nabla_{yy} f_k \bar{p}_k\}$ *is linearly independent,*

where $f_i = f_i(\bar{x}, \bar{y})$, $i = 1, 2, \ldots, k$. *Then* $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{r} = 0)$ *is feasible for (MD), and the two objectives have the same values. Also, if the hypotheses of Theorem 3.1 in* [1] *are satisfied for all feasible solutions of (MP) and (MD), then* $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{r} = 0)$ *is an efficient solution for (MD).*

One can observe that the assumption of positive or negative definiteness of the matrix $\sum_{i=1}^{k} \bar{\lambda}_i (\nabla_{yy} f_i \bar{p}_i)_y$ in the above theorem, and the fact that $\bar{p}_i = 0$, $i = 1, 2, \ldots, k$ (as proved in Theorem 3.2 [1]), are not compatible. In the following theorem an attempt is made in order to eliminate this inconsistency by making some modifications in the assumptions.

Theorem 2 (Strong duality) *Let* $f : R^n \times R^m \to R^k$ *be thrice differentiable. Let* $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p})$ *be a weakly efficient solution for (MP), and* $\lambda = \bar{\lambda}$ *fixed in (MD). Assume that*

(a) *either* (i) *the matrix* $\nabla_{yy} f_i$ *is positive definite for all* $i = 1, 2, \ldots, k$, *and* $\sum_{i=1}^{k} \bar{\lambda}_i \bar{p}_i^t [\nabla_y f_i - C_i \bar{w}_i] \geqq 0$; *or* (ii) *the matrix* $\nabla_{yy} f_i$ *is negative definite for all* $i = 1, 2, \ldots, k$, *and* $\sum_{i=1}^{k} \bar{\lambda}_i \bar{p}_i^t [\nabla_y f_i - C_i \bar{w}_i] \leqq 0$; *and*
(b) *the set* $\{\nabla_y f_1 - C_1 \bar{w}_1 + \nabla_{yy} f_1 \bar{p}_1, \nabla_y f_2 - C_2 \bar{w}_2 + \nabla_{yy} f_2 \bar{p}_2, \ldots, \nabla_y f_k - C_k \bar{w}_k + \nabla_{yy} f_k \bar{p}_k\}$ *is linearly independent,*

where $f_i = f_i(\bar{x}, \bar{y})$, $i = 1, 2, \ldots, k$. *Then* $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{r} = 0)$ *is feasible for (MD), and the two objectives have the same values. Also, if the hypotheses of weak duality* (*Theorem 3.1 in* [1]) *are satisfied for all feasible solutions of (MP) and (MD), then* $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{r} = 0)$ *is an efficient solution for (MD).*

Proof Since $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p})$ is a weakly efficient solution for (MP), by the Fritz John type conditions [2], there exist $\alpha \in R^k$, $\beta \in R^m$, $\gamma \in R$, $\nu \in R^k$, $\delta \in R^k$, and $\xi \in R^n$, such that

$$\sum_{i=1}^{k} \alpha_i \left[\nabla_x f_i + B_i \bar{z}_i - \frac{1}{2} (\nabla_{yy} f_i \bar{p}_i)_x \bar{p}_i \right] + \sum_{i=1}^{k} \bar{\lambda}_i [\nabla_{yx} f_i + (\nabla_{yy} f_i \bar{p}_i)_x](\beta - \gamma \bar{y}) - \xi = 0, \tag{1.1}$$

$$\sum_{i=1}^{k} \alpha_i \left[\nabla_y f_i - C_i \bar{w}_i - \frac{1}{2} (\nabla_{yy} f_i \bar{p}_i)_y \bar{p}_i \right] + \sum_{i=1}^{k} \bar{\lambda}_i [\nabla_{yy} f_i + (\nabla_{yy} f_i \bar{p}_i)_y](\beta - \gamma \bar{y}) - \gamma \sum_{i=1}^{k} \bar{\lambda}_i [\nabla_y f_i - C_i \bar{w}_i + \nabla_{yy} f_i \bar{p}_i] = 0, \tag{1.2}$$

$$(\beta - \gamma \bar{y})^t \left[\nabla_y f_i - C_i \bar{w}_i + \nabla_{yy} f_i \bar{p}_i \right] - \delta_i = 0, \quad i = 1, 2, \ldots, k, \tag{1.3}$$

$$\alpha_i C_i \bar{y} + \bar{\lambda}_i C_i (\beta - \gamma \bar{y}) = 2\nu_i C_i \bar{w}_i, \quad i = 1, 2, \ldots, k, \tag{1.4}$$

$$\nabla_{yy} f_i \left[(\beta - \gamma \bar{y}) \bar{\lambda}_i - \alpha_i \bar{p}_i \right] = 0, \quad i = 1, 2, \ldots, k, \tag{1.5}$$

$$\bar{x}^t B_i \bar{z}_i = (\bar{x}^t B_i \bar{x})^{\frac{1}{2}}, \quad i = 1, 2, \ldots, k, \tag{1.6}$$

$$\beta^t \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_y f_i - C_i \bar{w}_i + \nabla_{yy} f_i \bar{p}_i) = 0, \tag{1.7}$$

$$\gamma \bar{y}^t \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_y f_i - C_i \bar{w}_i + \nabla_{yy} f_i \bar{p}_i) = 0, \tag{1.8}$$

$$\nu_i (\bar{w}_i^t C_i \bar{w}_i - 1) = 0, \quad i = 1, 2, \ldots, k, \tag{1.9}$$

$$\delta^t \bar{\lambda} = 0, \tag{1.10}$$

$$\bar{x}^t \xi = 0, \tag{1.11}$$

$$\bar{z}_i^t B_i \bar{z}_i \leqq 1, \quad i = 1, 2, \ldots, k, \tag{1.12}$$

$$(\alpha, \beta, \gamma, \nu, \delta, \xi) \geqq 0, \tag{1.13}$$

$$(\alpha, \beta, \gamma, \nu, \delta, \xi) \neq 0. \tag{1.14}$$

Since $\bar{\lambda} > 0$ and $\delta \geqq 0$, (1.10) implies $\delta = 0$. Consequently, (1.3) yields

$$(\beta - \gamma \bar{y})^t \left[\nabla_y f_i - C_i \bar{w}_i + \nabla_{yy} f_i \bar{p}_i \right] = 0, \quad i = 1, 2, \ldots, k. \tag{1.15}$$

As $\nabla_{yy} f_i$ is positive definite or negative definite for $i = 1, 2, \ldots, k$, by hypothesis (a) and (1.5), we get

$$(\beta - \gamma \bar{y}) \bar{\lambda}_i = \alpha_i \bar{p}_i, \quad i = 1, 2, \ldots, k. \tag{1.16}$$

Now, we claim that $\alpha_i \neq 0$, $i = 1, 2, \ldots, k$. Indeed, if for some i, $\alpha_i = 0$, then it follows from $\bar{\lambda}_i > 0$ for some i, and (1.16) that $\beta = \gamma \bar{y}$.

From (1.2), we get

$$\sum_{i=1}^{k}(\alpha_i - \gamma\bar{\lambda}_i)(\nabla_y f_i - C_i\bar{w}_i) + \sum_{i=1}^{k}\bar{\lambda}_i \nabla_{yy} f_i(\beta - \gamma\bar{y} - \gamma\bar{p}_i)$$
$$+\sum_{i=1}^{k}(\nabla_{yy} f_i\bar{p}_i)_y\left[(\beta - \gamma\bar{y})\bar{\lambda}_i - \frac{1}{2}\alpha_i\bar{p}_i\right] = 0.$$

By using (1.16), it follows that

$$\sum_{i=1}^{k}(\alpha_i - \gamma\bar{\lambda}_i)(\nabla_y f_i - C_i\bar{w}_i + \nabla_{yy} f_i\bar{p}_i) + \frac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{yy} f_i\bar{p}_i)_y(\beta - \gamma\bar{y}) = 0,$$

which on using hypothesis (b) and $\beta = \gamma\bar{y}$ yields

$$\alpha_i = \gamma\bar{\lambda}_i, \quad i = 1, 2, \ldots, k.$$

As $\bar{\lambda}_i > 0, i = 1, 2, \ldots, k$, and $\alpha_i = 0$ for some i, the equations $\alpha_i = \gamma\bar{\lambda}_i, i = 1, 2, \ldots, k$, and $\beta = \gamma\bar{y}$ show $\gamma = 0 = \beta$. Therefore, (1.4) and (1.9) give $\nu = 0$, and (1.1) gives $\xi = 0$. Thus, $(\alpha, \beta, \gamma, \delta, \nu, \xi) = 0$, a contradiction to (1.14). Hence $\alpha_i > 0, i = 1, 2, \ldots, k$.

On premultiplying (1.15) by $\bar{\lambda}_i$, and using (1.16) with $\alpha_i > 0, i = 1, 2, \ldots, k$, we obtain

$$\bar{p}_i^t(\nabla_y f_i - C_i\bar{w}_i + \nabla_{yy} f_i\bar{p}_i) = 0, \quad i = 1, 2, \ldots, k.$$

By $\bar{\lambda}_i > 0, i = 1, 2, \ldots, k$, it follows that

$$\sum_{i=1}^{k}\bar{\lambda}_i\bar{p}_i^t(\nabla_y f_i - C_i\bar{w}_i) + \sum_{i=1}^{k}\bar{\lambda}_i(\bar{p}_i^t\nabla_{yy} f_i\bar{p}_i) = 0. \tag{1.17}$$

We now prove that $\bar{p}_i = 0, i = 1, 2, \ldots, k$. Otherwise, either (i) or (ii) of hypothesis (a) implies that

$$\sum_{i=1}^{k}\bar{\lambda}_i\bar{p}_i^t(\nabla_y f_i - C_i\bar{w}_i) + \sum_{i=1}^{k}\bar{\lambda}_i(\bar{p}_i^t\nabla_{yy} f_i\bar{p}_i) \neq 0,$$

contradicting (1.17). Hence $\bar{p}_i = 0, i = 1, 2, \ldots, k$. So, (1.16) gives

$$\beta = \gamma\bar{y}. \tag{1.18}$$

Substituting (1.18) in (1.2) with $\bar{p}_i = 0, i = 1, 2, \ldots k$, and on using hypothesis (b), we get

$$\alpha_i = \gamma \bar{\lambda}_i, \quad i = 1, 2, \ldots, k,$$

and therefore $\gamma > 0$.

Also, from (1.18), we have

$$\bar{y} = \frac{\beta}{\gamma} \geqq 0.$$

The remaining part follows on the lines of Theorem 3.2 [1].

Similarly the converse duality theorem in [1] can be correctly stated as:

Theorem 3 (Converse duality) *Let $f : R^n \times R^m \to R^k$ be thrice differentiable. Let $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}, \bar{r})$ be a weakly efficient solution for (MD), and $\lambda = \bar{\lambda}$ fixed in (MP). Assume that*

(a) *either* (i) *the matrix $\nabla_{xx} f_i$ is negative definite for all $i = 1, 2, \ldots, k$, and $\sum_{i=1}^k \bar{\lambda}_i \bar{r}_i^t [\nabla_x f_i + B_i \bar{z}_i] \leqq 0$; or (ii) $\nabla_{xx} f_i$ is positive definite for all $i = 1, 2, \ldots, k$, and $\sum_{i=1}^k \bar{\lambda}_i \bar{r}_i^t [\nabla_x f_i + B_i \bar{z}_i] \geqq 0$; and*

(b) *the set $\{\nabla_x f_1 + B_1 \bar{z}_1 + \nabla_{xx} f_1 \bar{r}_1, \nabla_x f_2 + B_2 \bar{z}_2 + \nabla_{xx} f_2 \bar{r}_2, \ldots, \nabla_x f_k + B_k \bar{z}_k + \nabla_{xx} f_k \bar{r}_k\}$ is linearly independent,*

where $f_i = f_i(\bar{u}, \bar{v}), i = 1, 2, \ldots, k$. Then $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{w}, \bar{p} = 0)$ is feasible for (MP), and the two objectives have the same values. Also, if the hypotheses of weak duality (Theorem 3.1 in [1]) are satisfied for all feasible solutions of (MP) and (MD), then $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{w}, \bar{p} = 0)$ is an efficient solution for (MP).

Acknowledgments The authors express their deep gratitude to the anonymous referees for their valuable suggestions which have improved the presentation of the paper.

References

1. Ahmad, I., Husain, Z.: Nondifferentiable second order symmetric duality in multiobjective programming. Appl. Math. Lett. **18**, 721–728 (2005)
2. Craven, B.D.: Lagrangean conditions and quasiduality. Bull. Austral. Math. Soc. **16**, 325–339 (1977)