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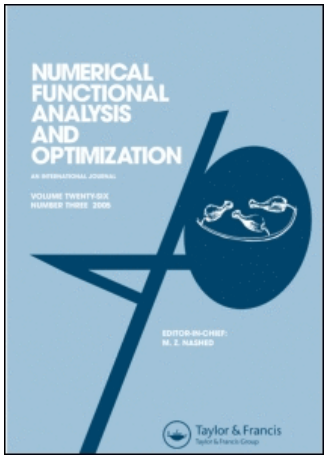

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Higher-Order Duality in Nondifferentiable Multiobjective Programming

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HIGHER-ORDER DUALITY IN NONDIFFERENTIABLE MULTIOBJECTIVE PROGRAMMING

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□ *A new generalized class of higher-order (F, α, ρ, d)–type I functions is introduced, and a general Mond–Weir type higher-order dual is formulated for a nondifferentiable multiobjective programming problem. Based on the concepts introduced, various higher-order duality results are established. At the end, some special cases are also discussed.*

Keywords Higher-order duality; Higher-order generalized type I functions; Multiobjective programming; Properly efficient solutions.

AMS Subject Classification 90C29; 90C26.

1. INTRODUCTION

Zhang [16] considered the following nondifferentiable mathematical programming problem:

$$\begin{aligned} &\text{Minimize} \quad f(x) + (x^t Bx)^{\frac{1}{2}} \\ &\text{subject to} \quad g(x) \geqq 0, \end{aligned} \tag{P}$$

where $f : R^n \mapsto R$ and $g : R^n \mapsto R^m$ are differentiable functions, and B is an $n \times n$ positive semidefinite symmetric matrix. He formulated the following Mangasarian [8] type and Mond–Weir [13] type nondifferentiable higher-order duals associated with (P):

$$\begin{aligned} &\text{Maximize} \quad f(u) + h(u,p) + (u+p)^t Bw - y^t g(u) - y^t k(u,p) \\ &\text{subject to} \quad \nabla_p h(u,p) + Bw - \nabla_p (y^t k(u,p)) = 0, \\ &\qquad\qquad\quad w^t Bw \leqq 1, \\ &\qquad\qquad\quad y \geqq 0. \end{aligned} \tag{MHD}$$

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$$\begin{aligned} \text{Maximize} \quad & f(u) + h(u,p) + u^t Bw - p^t \nabla_p h(u,p) \\ \text{subject to} \quad & \nabla_p h(u,p) + Bw - \nabla_p (y^t k(u,p)) = 0, \\ & y^t g(u) + y^t k(u,p) + p^t \nabla_p (y^t k(u,p)) \leqq 0, \\ & w^t Bw \leqq 1, \\ & y \geqq 0, \end{aligned} \tag{MWHD}$$

where u, w, $p \in R^n$ and $y \in R^m$ and $h : R^n \times R^n \mapsto R$, $k : R^n \times R^n \mapsto R^m$ are differentiable functions, and B is a positive semidefinite symmetric matrix of order n.

In [16], Zhang established various duality results between (P) and duals (MHD) and (MWHD) under higher-order invexity assumptions. Mishra and Rueda [10] generalized the results of Zhang [16] to higher-order type I functions. Later on, Yang et al. [15] discussed higher-order duality results under generalized convexity assumptions for multiobjective programming problems involving support functions.

Based on the various generalizations of convex function, Liang et al. [7] introduced a unified formulation of generalized convex function, called (F, α, ρ, d)-convex function, which was further extended to (F, α, ρ, d)–type I function and its generalizations by Hachimi and Aghezzaf [5]. These two classes of functions were recently generalized to second-order by Ahmad and Husain [2] and Hachimi and Aghezzaf [6].

Consider the following nondifferentiable multiobjective programming problem:

$$\begin{aligned} \text{Minimize} \quad & f(x) = \left(f_1(x) + (x^t B_1 x)^{\frac{1}{2}}, \ldots, f_s(x) + (x^t B_s x)^{\frac{1}{2}}\right) \\ \text{subject to} \quad & x \in Z = \{x \in X : g(x) \leqq 0\}, \end{aligned} \tag{NMP}$$

where $f = (f_1, f_2, \ldots, f_s) : X \mapsto R^s, g = (g_1, g_2, \ldots, g_m) : X \mapsto R^m$ are differentiable functions defined on X, an open subset of R^n, and B_i, $i = 1, 2, \ldots, s$ is $n \times n$ positive semidefinite symmetric matrix.

In this paper, we introduce a new generalized class of higher-order type I functions, named higher-order (F, α, ρ, d)–type I functions. Based on these concepts, weak, strong, and strict converse duality theorems are established in order to relate (NMP) and its corresponding higher-order dual. Our results extend the higher-order duality results discussed in [10, 16] to the multiobjective case. The results, recently derived by Hachimi and Aghezzaf [6], appear as a special case of our results.

2. PREREQUISITES

The following convention for vectors in R^n will be followed throughout the paper: $x \geqq y \Leftrightarrow x_i \geqq y_i$, $i = 1, 2, \ldots, n$; $x \geq y \Leftrightarrow x \geqq y$ and $x \neq y$;

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$x > y \Leftrightarrow x_i > y_i,\ i = 1, 2, \ldots, n$. The index sets are $S = \{1, 2, \ldots, s\}$ and $M = \{1, 2, \ldots, m\}$. For each $r \in S$, the set $S_r = S - \{r\}$.

Because the objectives in multiobjective problems generally conflict with one another, an optimal solution is chosen from the set of feasible solutions in the following sense:

Definition 2.1. A feasible point $\bar{x}$ is said to be an efficient solution of the vector minimum problem (NMP) if there exists no other feasible x such that

$$f_r(x) + (x^t B_r x)^{\frac{1}{2}} \leqq f_r(\bar{x}) + (\bar{x}^t B_r \bar{x})^{\frac{1}{2}}, \quad \text{for all } r$$

and

$$f_l(x) + (x^t B_l x)^{\frac{1}{2}} < f_l(\bar{x}) + (\bar{x}^t B_l \bar{x})^{\frac{1}{2}}, \quad \text{for at least one } l \in S_i.$$

Geoffrion [4] defined a restricted concept of efficiency, called proper efficiency, as follows:

Definition 2.2. An efficient solution $\bar{x}$ of (NMP) is said to be properly efficient if there exists a scalar $M > 0$ such that for each $r \in S$ and $x \in Z$ satisfying $f_r(x) + (x^t B_r x)^{\frac{1}{2}} < f_r(\bar{x}) + (\bar{x}^t B_r \bar{x})^{\frac{1}{2}}$, we have

$$\begin{aligned}&[f_r(\bar{x}) + (\bar{x}^t B_r \bar{x})^{\frac{1}{2}}] - [f_r(x) + (x^t B_r x)^{\frac{1}{2}}] \\ &\quad \leqq M\Big[[f_l(x) + (x^t B_l x)^{\frac{1}{2}}] - [f_l(\bar{x}) + (\bar{x}^t B_l \bar{x})^{\frac{1}{2}}]\Big]\end{aligned}$$

for at least one $l \in S_r$ satisfying $f_l(\bar{x}) + (\bar{x}^t B_l \bar{x})^{\frac{1}{2}} < f_l(x) + (x^t B_l x)^{\frac{1}{2}}$.

Definition 2.3. A functional $F : X \times X \times R^n \mapsto R$ is said to be sublinear in its third argument, if for all $x, \bar{x} \in X$,

(i) $F(x, \bar{x}; a + b) \leqq F(x, \bar{x}; a) + F(x, \bar{x}; b)$, for all $a, b \in R^n$; and
(ii) $F(x, \bar{x}; \xi a) = \xi F(x, \bar{x}; a)$, for all $\xi \in R, \xi \geqq 0$, and for all $a \in R^n$.

Let F be a sublinear functional, and let $\rho = (\rho^1, \rho^2)$, where $\rho^1 = (\rho_1^1, \rho_2^1, \ldots, \rho_s^1) \in R^s$, $\rho^2 = (\rho_1^2, \rho_2^2, \ldots, \rho_m^2) \in R^m$. Let $\alpha^1, \alpha^2 : X \times X \mapsto R_+ \backslash \{0\}$, and $d(\cdot, \cdot) : X \times X \mapsto R$.

Definition 2.4. For each $i \in S$ and $j \in M$, $(f_i + (\cdot)^t B_i w_i, g_j)$ is said to be higher-order (F, α, ρ, d)–type I at $\bar{x} \in X$ with respect to $p \in R^n$ if for all $x \in Z$, we have

$$\begin{aligned}f_i(x) + x^t B_i w_i &\geqq f_i(\bar{x}) + \bar{x}^t B_i w_i + h_i(\bar{x}, p) - p^t \nabla_p h_i(\bar{x}, p) \\ &\quad + F(x, \bar{x}; \alpha^1(x, \bar{x})(\nabla_p h_i(\bar{x}, p) + B_i w_i)) + \rho_i^1 d^2(x, \bar{x}), \quad i \in S,\end{aligned}$$

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$$-[g_j(\bar{x}) + k_j(\bar{x}, p) - p^t \nabla_p k_j(\bar{x}, p)]$$
$$\geqq F(x, \bar{x}; \alpha^2(x, \bar{x})(\nabla_p k_j(\bar{x}, p)) + \rho_j^2 d^2(x, \bar{x}), \quad j \in M.$$

Definition 2.5. For each $i \in S$ and $j \in M$, $(f_i + (\cdot)^t B_i w_i, g_j)$ is said to be higher-order (F, α, ρ, d)–pseudoquasi type I at $\bar{x} \in X$ with respect to $p \in R^n$ if for all $x \in Z$, we have

$$f_i(x) + x^t B_i w_i < f_i(\bar{x}) + \bar{x}^t B_i w_i + h_i(\bar{x}, p) - p^t \nabla_p h_i(\bar{x}, p)$$
$$\Rightarrow F(x, \bar{x}; \alpha^1(x, \bar{x})(\nabla_p h_i(\bar{x}, p) + B_i w_i)) < -\rho_i^1 d^2(x, \bar{x}), \quad i \in S,$$
$$-[g_j(\bar{x}) + k_j(\bar{x}, p) - p^t \nabla_p k_j(\bar{x}, p)] \leqq 0$$
$$\Rightarrow F(x, \bar{x}; \alpha^2(x, \bar{x})(\nabla_p k_j(\bar{x}, p)) \leqq -\rho_j^2 d^2(x, \bar{x}), \quad j \in M.$$

Definition 2.6. For each $i \in S$ and $j \in M$, $(f_i + (\cdot)^t B_i w_i, g_j)$ is said to be higher-order (F, α, ρ, d)–strictly pseudoquasi type I at $\bar{x} \in X$ with respect to $p \in R^n$ if for all $x \in Z$, we have

$$F(x, \bar{x}; \alpha^1(x, \bar{x})(\nabla_p h_i(\bar{x}, p) + B_i w_i)) \geqq -\rho_i^1 d^2(x, \bar{x})$$
$$\Rightarrow f_i(x) + x^t B_i w_i > f_i(\bar{x}) + \bar{x}^t B_i w_i + h_i(\bar{x}, p) - p^t \nabla_p h_i(\bar{x}, p), \quad i \in S,$$
$$-[g_j(\bar{x}) + k_j(\bar{x}, p) - p^t \nabla_p k_j(\bar{x}, p)] \leqq 0$$
$$\Rightarrow F(x, \bar{x}; \alpha^2(x, \bar{x})(\nabla_p k_j(\bar{x}, p)) \leqq -\rho_j^2 d^2(x, \bar{x}), \quad j \in M.$$

Remark 2.7. If we put $B_i = 0$, $h_i(\bar{x}, p) = p^t \nabla f_i(\bar{x}) + \frac{1}{2} p^t \nabla^2 f_i(\bar{x}) p$, $i \in S$, and $k_j(\bar{x}, p) = p^t \nabla g_j(\bar{x}) + \frac{1}{2} p^t \nabla^2 g_j(\bar{x}) p$, $j \in M$, in the above definitions, then for each $i \in S$ and $j \in M$, (f_i, g_j) is second-order (F, α, ρ, p, d)–type I, second-order (F, α, ρ, p, d)–pseudoquasi type I, and second-order (F, α, ρ, p, d)–strictly pseudoquasi type I at $\bar{x}$, respectively, introduced by Hachimi and Aghezzaf [6].

In the analysis to follow, we will require the following generalized Schwartz inequality stated as follows:

Let B be a positive semidefinite symmetric matrix of order n. Then, for all $x, w \in R^n$, $x^t B w \leqq (x^t B x)^{\frac{1}{2}} (w^t B w)^{\frac{1}{2}}$. Equality holds if $Bx = \eta B w$, for some $\eta \geqq 0$.

Mond et al. [12] established the following necessary conditions for proper efficiency of (NMP) as follows:

Theorem 2.8 (Necessary Optimality Conditions). *Suppose that $\bar{x}$ is a properly efficient solution of* (NMP) *and a constraint qualification* [11, 14] *is satisfied at $\bar{x}$, then there exist $\bar{\lambda} \in R^s$, $\bar{y} \in R^m$, and $\bar{w}_i \in R^n$, $i \in S$ satisfying*

$$\sum_{i \in S} \bar{\lambda}_i \{\nabla f_i(\bar{x}) + B_i \bar{w}_i\} + \nabla \bar{y}^t g(\bar{x}) = 0,$$
$$\bar{y}^t g(\bar{x}) = 0,$$

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$$(\bar{x}^t B_i \bar{x})^{\frac{1}{2}} = \bar{x}^t B_i \bar{w}_i, \quad i \in S,$$
$$\bar{w}_i^t B_i \bar{w}_i \leqq 1, \quad i \in S,$$
$$\bar{y} \geqq 0,$$
$$\bar{\lambda} > 0, \quad \sum_{i \in S} \bar{\lambda}_i = 1.$$

3. NONDIFFERENTIABLE HIGHER-ORDER GENERALIZED DUALITY

In this section, we formulate the following nondifferentiable higher-order general Mond–Weir type dual for (NMP) and discuss weak, strong, and strict converse duality theorems:

$$\begin{aligned} \text{Maximize} \quad & f_1(u) + h_1(u,p) - p^t \nabla_p h_1(u,p) + u^t B_1 w_1 \\ & + \sum_{j \in I_0} [y_j g_j(u) + y_j k_j(u,p) - p^t \nabla_p (y_j k_j(u,p))], \ldots, f_s(u) \\ & + h_s(u,p) - p^t \nabla_p h_s(u,p) + u^t B_s w_s \\ & + \sum_{j \in I_0} [y_j g_j(u) + y_j k_j(u,p) - p^t \nabla_p (y_j k_j(u,p))] \end{aligned} \tag{NHMD}$$

$$\text{subject to} \quad \lambda^t \nabla_p h(u,p) + \sum_{i \in S} \lambda_i B_i w_i + \nabla_p (y^t k(u,p)) = 0, \tag{3.1}$$

$$\sum_{j \in I_\beta} [y_j g_j(u) + y_j k_j(u,p) - p^t \nabla_p (y_j k_j(u,p))] \geqq 0, \quad \beta = 1, 2, \ldots, r, \tag{3.2}$$

$$w_i^t B_i w_i \leqq 1, \quad i \in S, \tag{3.3}$$

$$y \geqq 0, \tag{3.4}$$

$$\lambda_i > 0, \quad i \in S, \tag{3.5}$$

$$\sum_{i \in S} \lambda_i = 1, \tag{3.6}$$

where u, $p \in R^n$, $w_i \in R^n$, $i \in S$, $\lambda \in R^s$, $h_i : R^n \times R^n \to R$, $i \in S$, and $k_j : R^n \times R^n \to R$, $j \in M$ are differentiable functions; B_i, for each $i \in S$ is an $n \times n$ positive semidefinite symmetric matrix. $I_\beta \subseteq M$, $\beta = 0, 1, 2, \ldots, r$ with $I_\beta \cap I_\gamma = \emptyset$, $\beta \neq \gamma$ and $\bigcup_{\beta=0}^{r} I_\beta = M$.

Theorem 3.1 (Weak Duality). *Assume that for all feasible x for* (NMP) *and for all feasible* $(u, y, \lambda, w_1, w_2, \ldots, w_s, p)$ *for* (NHMD),

$$\left(\sum_{i \in S} \lambda_i \{f_i(\cdot) + (\cdot)^t B_i w_i\} + \sum_{j \in I_0} y_j g_j(\cdot), \sum_{j \in I_\beta} y_j g_j(\cdot), \beta = 1, 2, \ldots, r \right)$$

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is higher-order (F, α, ρ, d)–pseudoquasi type I at u, and

$$\left(\frac{\rho_1^1}{\alpha^1(x,u)} + \frac{1}{\alpha^2(x,u)}\sum_{\beta=1}^{r}\rho_\beta^2\right) \geqq 0.$$

Then, the following cannot hold:

$$\begin{aligned} f_i(x) + (x^t B_i x)^{\frac{1}{2}} \leqq f_i(u) + h_i(u,p) - p^t\nabla_p h_i(u,p) + u^t B_i w_i \\ + \sum_{j\in I_0}[y_j g_j(u) + y_j k_j(u,p) - p^t\nabla_p(y_j k_j(u,p))], \quad \textit{for all } i, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} f_l(x) + (x^t B_l x)^{\frac{1}{2}} < f_l(u) + h_l(u,p) - p^t\nabla_p h_l(u,p) + u^t B_l w_l \\ + \sum_{j\in I_0}[y_j g_j(u) + y_j k_j(u,p) - p^t\nabla_p(y_j k_j(u,p))], \\ \textit{for at least one } l \in S_i. \end{aligned} \tag{3.8}$$

Proof. Suppose contrary to the result that (3.7) and (3.8) hold. Because x is feasible for (NMP), $\lambda > 0$, $\sum_{i\in S}\lambda_i = 1$ and $y \geqq 0$, then using the generalized Schwartz inequality along with (3.3), it follows that

$$\begin{aligned} &\sum_{i\in S}\lambda_i[f_i(x) + x^t B_i w_i] + \sum_{j\in I_0} y_j g_j(x) \\ &\quad < \sum_{i\in S}\lambda_i[f_i(u) + h_i(u,p) - p^t\nabla_p h_i(u,p) + u^t B_i w_i] \\ &\qquad + \sum_{j\in I_0}[y_j g_j(u) + y_j k_j(u,p) - p^t\nabla_p(y_j k_j(u,p))]. \end{aligned} \tag{3.9}$$

Also, the feasibility of $(u, y, \lambda, w_1, w_2, \ldots, w_s, p)$ for (NHMD) gives

$$-\sum_{j\in I_\beta}[y_j g_j(u) + y_j k_j(u,p) - p^t\nabla_p(y_j k_j(u,p))] \leqq 0, \quad \beta = 1, 2, \ldots, r. \tag{3.10}$$

Because

$$\left(\sum_{i\in S}\lambda_i\{f_i(\cdot) + (\cdot)^t B_i w_i\} + \sum_{j\in I_0} y_j g_j(\cdot), \sum_{j\in I_\beta} y_j g_j(\cdot), \beta = 1, 2, \ldots, r\right)$$

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is higher-order (F, α, ρ, d)–pseudoquasi type I at u, (3.9) and (3.10) imply

$$F\left(x, u; \alpha^1(x, u)\left\{\nabla_p \lambda^t h(u, p) + \sum_{i \in S} \lambda_i B_i w_i + \nabla_p\left(\sum_{j \in I_0} y_j k_j(u, p)\right)\right\}\right)$$
$$< -\rho_1^1 d^2(x, u),$$
$$F\left(x, u; \alpha^2(x, u)\left\{\nabla_p\left(\sum_{j \in I_\beta} y_j k_j(u, p)\right)\right\}\right) \leqq -\rho_\beta^2 d^2(x, u), \quad \beta = 1, 2, \ldots, r.$$

By the sublinearity of F and $\alpha^1(x, u) > 0$, $\alpha^2(x, u) > 0$, we summarize to get

$$F\left(x, u; \nabla_p \lambda^t h(u, p) + \sum_{i \in S} \lambda_i B_i w_i + \nabla_p\left(\sum_{j \in M} y_j k_j(u, p)\right)\right)$$
$$< -\left(\frac{\rho_1^1}{\alpha^1(x, u)} + \frac{1}{\alpha^2(x, u)} \sum_{\beta=1}^{r} \rho^2{}_\beta\right) d^2(x, u). \tag{3.11}$$

Because $\left(\frac{\rho_1^1}{\alpha^1(x,u)} + \frac{1}{\alpha^2(x,u)} \sum_{\beta=1}^{r} \rho_\beta^2\right) \geqq 0$, (3.11) reduces to

$$F\left(x, u; \nabla_p \lambda^t h(u, p) + \sum_{i \in S} \lambda_i B_i w_i + \nabla_p\left(\sum_{j \in M} y_j k_j(u, p)\right)\right) < 0. \tag{3.12}$$

The first dual constraint and the sublinearity of F yield

$$F\left(x, u; \nabla_p \lambda^t h(u, p) + \sum_{i \in S} \lambda_i B_i w_i + \nabla_p\left(\sum_{j \in M} y_j k_j(u, p)\right)\right) = 0,$$

which contradicts (3.12). □

Theorem 3.2 (Strong Duality). *Let $\bar{x}$ be a properly efficient solution of* (NMP) *at which a constraint qualification* [11, 14] *is satisfied, and*

$$\begin{cases} h_i(\bar{x}, 0) = 0, \quad i \in S; & \nabla_p h(\bar{x}, 0) = \nabla f(\bar{x}), \\ k_j(\bar{x}, 0) = 0, \quad j \in M; & \nabla_p k(\bar{x}, 0) = \nabla g(\bar{x}). \end{cases} \tag{3.13}$$

Then, there exist $\bar{\lambda} \in R^s$, $\bar{y} \in R^m$, $\bar{w}_i \in R^n$, $i \in S$, and $\bar{p} \in R^n$ such that $(\bar{u} = \bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_s, \bar{p} = 0)$ is a feasible solution of (NHMD) *and the corresponding objective values of* (NMP) *and* (NHMD) *are equal.*

If the weak duality (Theorem 3.1) holds for all feasible $(u, y, \lambda, w_1, w_2, \ldots, w_s, p)$, then $(\bar{u} = \bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_s, \bar{p} = 0)$ is a properly efficient solution of (NHMD).

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Proof. Since $\bar{x}$ is a properly efficient solution of (NMP) at which a constraint qualification [11, 14] is satisfied. Then, by Theorem 2.8, there exist $\bar{\lambda} \in R^s, \bar{y} \in R^m$ and $\bar{w}_i \in R^n, i \in S$ such that

$$\sum_{i \in S} \bar{\lambda}_i \{\nabla f_i(\bar{x}) + B_i \bar{w}_i\} + \nabla \bar{y}^t g(\bar{x}) = 0, \tag{3.14}$$

$$\bar{y}^t g(\bar{x}) = 0, \tag{3.15}$$

$$(\bar{x}^t B_i \bar{x})^{\frac{1}{2}} = \bar{x}^t B_i \bar{w}_i, \quad i \in S, \tag{3.16}$$

$$\bar{w}_i^t B_i \bar{w}_i \leqq 1, \quad i \in S, \tag{3.17}$$

$$\bar{y} \geqq 0, \tag{3.18}$$

$$\bar{\lambda} > 0, \tag{3.19}$$

$$\sum_{i \in S} \bar{\lambda}_i = 1. \tag{3.20}$$

The relations (3.14), (3.15), (3.17)–(3.20) along with (3.13) imply that $(\bar{u} = \bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_s, \bar{p} = 0)$ is a feasible solution of (NHMD). Equality of objective values follows from (3.15) and (3.16). We first show that $(\bar{u} = \bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_s, \bar{p} = 0)$ is an efficient solution of (NHMD). Assume that it is not efficient; there exists a feasible solution $(\hat{u}, \hat{y}, \hat{\lambda}, \hat{w}_1, \hat{w}_2, \ldots, \hat{w}_s, \hat{p})$ such that

$$\begin{aligned} & f_i(\hat{u}) + h_i(\hat{u}, \hat{p}) - \hat{p}^t \nabla_p h_i(\hat{u}, \hat{p}) + \hat{u}^t B_i \hat{w}_i \\ & \quad + \sum_{j \in I_0} [\hat{y}_j g_j(\hat{u}) + \hat{y}_j k_j(\hat{u}, \hat{p}) - \hat{p}^t \nabla_p (\hat{y}_j k_j(\hat{u}, \hat{p}))] \\ & \quad \geqq f_i(\bar{x}) + h_i(\bar{x}, 0) + \bar{x}^t B_i \bar{w}_i + \sum_{j \in I_0} [\bar{y}_j g_j(\bar{x}) + \bar{y}_j k_j(\bar{x}, 0)], \end{aligned}$$

for all i, and

$$\begin{aligned} & f_l(\hat{u}) + h_l(\hat{u}, \hat{p}) - \hat{p}^t \nabla_p h_l(\hat{u}, \hat{p}) + \hat{u}^t B_l \hat{w}_l \\ & \quad + \sum_{j \in I_0} [\hat{y}_j g_j(\hat{u}) + \hat{y}_j k_j(\hat{u}, \hat{p}) - \hat{p}^t \nabla_p (\hat{y}_j k_j(\hat{u}, \hat{p}))] \\ & \quad > f_l(\bar{x}) + h_l(\bar{x}, 0) + \bar{x}^t B_l \bar{w}_l + \sum_{j \in I_0} [\bar{y}_j g_j(\bar{x}) + \bar{y}_j k_j(\bar{x}, 0)], \end{aligned}$$

for at least one $l \in S_i$.

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Thus, from (3.13), (3.15), and (3.16), we have

$$
\begin{aligned}
&f_i(\bar{x}) + (\bar{x}^t B_i \bar{x})^{\frac{1}{2}} \\
&\quad \leqq f_i(\hat{u}) + h_i(\hat{u}, \hat{p}) - \hat{p}^t \nabla_p h_i(\hat{u}, \hat{p}) + \hat{u}^t B_i \hat{w}_i \\
&\qquad + \sum_{j \in I_0} [\hat{y}_j g_j(\hat{u}) + \hat{y}_j k_j(\hat{u}, \hat{p}) - \hat{p}^t \nabla_p (\hat{y}_j k_j(\hat{u}, \hat{p}))], \quad \text{for all } i, \text{ and} \\
&f_l(\bar{x}) + (\bar{x}^t B_l \bar{x})^{\frac{1}{2}} \\
&\quad < f_l(\hat{u}) + h_l(\hat{u}, \hat{p}) - \hat{p}^t \nabla_p h_l(\hat{u}, \hat{p}) + \hat{u}^t B_l \hat{w}_l \\
&\qquad + \sum_{j \in I_0} [\hat{y}_j g_j(\hat{u}) + \hat{y}_j k_j(\hat{u}, \hat{p}) - \hat{p}^t \nabla_p (\hat{y}_j k_j(\hat{u}, \hat{p}))], \quad \text{for at least one } l \in S_i,
\end{aligned}
$$

which contradicts weak duality (Theorem 3.1). Hence $(\bar{u} = \bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_s, \bar{p} = 0)$ is an efficient solution of (NHMD).

Suppose that it is not properly efficient, then for each scalar $M > 0$, there exists a feasible solution $(\hat{u}, \hat{y}, \hat{\lambda}, \hat{w}_1, \hat{w}_2, \ldots, \hat{w}_s, \hat{p})$ and some $i \in S$ such that

$$
\begin{aligned}
&\Bigg[f_i(\hat{u}) + h_i(\hat{u}, \hat{p}) - \hat{p}^t \nabla_p h_i(\hat{u}, \hat{p}) + \hat{u}^t B_i \hat{w}_i \\
&\qquad + \sum_{j \in I_0} [\hat{y}_j g_j(\hat{u}) + \hat{y}_j k_j(\hat{u}, \hat{p}) - \hat{p}^t \nabla_p (\hat{y}_j k_j(\hat{u}, \hat{p}))] \Bigg] \\
&\quad - \Bigg[f_i(\bar{x}) + h_i(\bar{x}, 0) + \bar{x}^t B_i \bar{w}_i + \sum_{j \in I_0} [\bar{y}_j g_j(\bar{x}) + \bar{y}_j k_j(\bar{x}, 0)] \Bigg] \\
&\quad > M \Bigg[\Bigg[f_l(\bar{x}) + h_l(\bar{x}, 0) + \bar{x}^t B_l \bar{w}_l + \sum_{j \in I_0} [\bar{y}_j g_j(\bar{x}) + \bar{y}_j k_j(\bar{x}, 0)] \Bigg] \\
&\qquad - \Bigg[f_l(\hat{u}) + h_l(\hat{u}, \hat{p}) - \hat{p}^t \nabla_p h_l(\hat{u}, \hat{p}) + \hat{u}^t B_l \hat{w}_l \\
&\qquad\quad + \sum_{j \in I_0} [\hat{y}_j g_j(\hat{u}) + \hat{y}_j k_j(\hat{u}, \hat{p}) - \hat{p}^t \nabla_p (\hat{y}_j k_j(\hat{u}, \hat{p}))] \Bigg] \Bigg],
\end{aligned}
$$

for all $l \in S_i$ satisfying

$$
\begin{aligned}
&\Bigg[f_l(\bar{x}) + h_l(\bar{x}, 0) + \bar{x}^t B_l \bar{w}_l + \sum_{j \in I_0} [\bar{y}_j g_j(\bar{x}) + \bar{y}_j k_j(\bar{x}, 0)] \Bigg] \\
&\quad > \Bigg[f_l(\hat{u}) + h_l(\hat{u}, \hat{p}) - \hat{p}^t \nabla_p h_l(\hat{u}, \hat{p}) + \hat{u}^t B_l \hat{w}_l \\
&\qquad + \sum_{j \in I_0} [\hat{y}_j g_j(\hat{u}) + \hat{y}_j k_j(\hat{u}, \hat{p}) - \hat{p}^t \nabla_p (\hat{y}_j k_j(\hat{u}, \hat{p}))] \Bigg].
\end{aligned}
$$

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Therefore, from (3.13), (3.15) and (3.16), we get

$$\Bigg[f_i(\hat{u}) + h_i(\hat{u},\hat{p}) - \hat{p}^t\nabla_p h_i(\hat{u},\hat{p}) + \hat{u}^t B_i\hat{w}_i + \sum_{j\in I_0}[\hat{y}_j g_j(\hat{u}) + \hat{y}_j k_j(\hat{u},\hat{p}) - \hat{p}^t\nabla_p(\hat{y}_j k_j(\hat{u},\hat{p}))]\Bigg] - [f_i(\bar{x}) + (\bar{x}^t B_i\bar{x})^{\frac{1}{2}}]$$
$$> M\Bigg[[f_l(\bar{x}) + (\bar{x}^t B_l\bar{x})^{\frac{1}{2}}] - [f_l(\hat{u}) + h_l(\hat{u},\hat{p}) - \hat{p}^t\nabla_p h_l(\hat{u},\hat{p}) + \hat{u}^t B_l\hat{w}_l + \sum_{j\in I_0}[\hat{y}_j g_j(\hat{u}) + \hat{y}_j k_j(\hat{u},\hat{p}) - \hat{p}^t\nabla_p(\hat{y}_j k_j(\hat{u},\hat{p}))]]\Bigg].$$

This means that

$$f_i(\hat{u}) + h_i(\hat{u},\hat{p}) - \hat{p}^t\nabla_p h_i(\hat{u},\hat{p}) + \hat{u}^t B_i\hat{w}_i + \sum_{j\in I_0}[\hat{y}_j g_j(\hat{u}) + \hat{y}_j k_j(\hat{u},\hat{p}) - \hat{p}^t\nabla_p(\hat{y}_j k_j(\hat{u},\hat{p}))]$$

is infinitely better than $f_i(\bar{x}) + (\bar{x}^t B_i\bar{x})^{\frac{1}{2}}$ for some i, whereas $f_l(\bar{x}) + (\bar{x}^t B_l\bar{x})^{\frac{1}{2}}$ is at most finitely better than

$$f_l(\hat{u}) + h_l(\hat{u},\hat{p}) - \hat{p}^t\nabla_p h_l(\hat{u},\hat{p}) + \hat{u}^t B_l\hat{w}_l + \sum_{j\in I_0}[\hat{y}_j g_j(\hat{u}) + \hat{y}_j k_j(\hat{u},\hat{p}) - \hat{p}^t\nabla_p(\hat{y}_j k_j(\hat{u},\hat{p}))]$$

for all $l \in S_i$; that is,

$$f_i(\bar{x}) + (\bar{x}^t B_i\bar{x})^{\frac{1}{2}} \leqq f_i(\hat{u}) + h_i(\hat{u},\hat{p}) - \hat{p}^t\nabla_p h_i(\hat{u},\hat{p}) + \hat{u}^t B_i\hat{w}_i + \sum_{j\in I_0}[\hat{y}_j g_j(\hat{u}) + \hat{y}_j k_j(\hat{u},\hat{p}) - \hat{p}^t\nabla_p(\hat{y}_j k_j(\hat{u},\hat{p}))], \quad \text{for all } i$$

and

$$f_l(\bar{x}) + (\bar{x}^t B_l\bar{x})^{\frac{1}{2}} < f_l(\hat{u}) + h_l(\hat{u},\hat{p}) - \hat{p}^t\nabla_p h_l(\hat{u},\hat{p}) + \hat{u}^t B_l\hat{w}_l + \sum_{j\in I_0}[\hat{y}_j g_j(\hat{u}) + \hat{y}_j k_j(\hat{u},\hat{p}) - p^t\nabla_p(\hat{y}_j k_j(\hat{u},\hat{p}))],$$
$$\text{for at least one } l \in S_i,$$

which again contradicts weak duality (Theorem 3.1). Hence, $(\bar{u} = \bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_s, \bar{p} = 0)$ is a properly efficient solution of (NHMD). □

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Theorem 3.3 (Strict Converse Duality). *Assume that $\bar{x}$ is a feasible solution of* (NMP) *and* $(\bar{u}, \bar{y}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_s, \bar{\lambda}, \bar{p})$ *is a feasible solution of* (NHMD) *such that*

$$\sum_{i\in S} \bar{\lambda}_i\{f_i(\bar{x}) + \bar{x}^t B_i \bar{w}_i\} \leqq \sum_{i\in S} \bar{\lambda}_i\{f_i(\bar{u}) + h_i(\bar{u}, \bar{p}) - \bar{p}^t \nabla_p h_i(\bar{u}, \bar{p}) + \bar{u}^t B_i \bar{w}_i\} + \sum_{j\in I_0} [\bar{y}_j g_j(\bar{u}) + \bar{y}_j k_j(\bar{u}, \bar{p}) - \bar{p}^t \nabla_p(\bar{y}_j k_j(\bar{u}, \bar{p}))]. \tag{3.21}$$

Also, suppose that

$$\left(\sum_{i\in S} \bar{\lambda}_i\{f_i(\cdot) + (\cdot)^t B_i \bar{w}_i\} + \sum_{j\in I_0} \bar{y}_j g_j(\cdot), \sum_{j\in I_\beta} \bar{y}_j g_j(\cdot), \quad \beta = 1, 2, \ldots, r \right)$$

is higher-order (F, α, ρ, d)–strictly pseudoquasi type I at $\bar{u}$, and

$$\left(\frac{\rho_1^1}{\alpha^1(\bar{x}, \bar{u})} + \frac{1}{\alpha^2(\bar{x}, \bar{u})} \sum_{\beta=1}^{r} \rho_\beta^2 \right) \geqq 0.$$

Then $\bar{x} = \bar{u}$.

Proof. We assume that $\bar{x} \neq \bar{u}$ and exhibit a contradiction. By imposing the higher-order (F, α, ρ, d)–strictly pseudoquasi type I assumption on

$$\left(\sum_{i\in S} \bar{\lambda}_i\{f_i(\cdot) + (\cdot)^t B_i \bar{w}_i\} + \sum_{j\in I_0} \bar{y}_j g_j(\cdot), \sum_{j\in I_\beta} \bar{y}_j g_j(\cdot), \beta = 1, 2, \ldots, r \right),$$

we have

$$\begin{aligned}
&F\left(\bar{x}, \bar{u}; \alpha^1(\bar{x}, \bar{u}) \left\{ \sum_{i\in S} \bar{\lambda}_i\{\nabla_p h_i(\bar{u}, \bar{p}) + B_i \bar{w}_i\} + \nabla_p \left(\sum_{j\in I_0} \bar{y}_j k_j(\bar{u}, \bar{p}) \right) \right\} \right) \\
&\quad \geqq -\rho_1^1 d^2(\bar{x}, \bar{u}) \\
&\quad \Rightarrow \sum_{i\in S} \bar{\lambda}_i\{f_i(\bar{x}) + \bar{x}^t B_i \bar{w}_i\} + \sum_{j\in I_0} \bar{y}_j g_j(\bar{x}) \\
&\quad > \sum_{i\in S} \bar{\lambda}_i\{f_i(\bar{u}) + h_i(\bar{u}, \bar{p}) - \bar{p}^t \nabla_p h_i(\bar{u}, \bar{p}) + \bar{u}^t B_i \bar{w}_i\} \\
&\qquad + \sum_{j\in I_0} [\bar{y}_j g_j(\bar{u}) + \bar{y}_j k_j(\bar{u}, \bar{p}) - \bar{p}^t \nabla_p(\bar{y}_j k_j(\bar{u}, \bar{p}))],
\end{aligned} \tag{3.22}$$

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and

$$-\sum_{j\in I_\beta}[\bar{y}_j g_j(\bar{u}) + \bar{y}_j k_j(\bar{u},\bar{p}) - \bar{p}^t\nabla_p(\bar{y}_j k_j(\bar{u},\bar{p}))] \leqq 0$$
$$\Rightarrow F\left(\bar{x},\bar{u};\alpha^2(\bar{x},\bar{u})\left\{\nabla_p\left(\sum_{j\in I_\beta}\bar{y}_j k_j(\bar{u},\bar{p})\right)\right\}\right) \leqq -\rho_\beta^2 d^2(\bar{x},\bar{u}),$$
$$\beta = 1,2,\ldots,r. \tag{3.23}$$

Because $(\bar{u},\bar{y},\bar{\lambda},\bar{w}_1,\bar{w}_2,\ldots,\bar{w}_s,\bar{p})$ is feasible for (NHMD), it follows that

$$-\sum_{j\in I_\beta}[\bar{y}_j g_j(\bar{u}) + \bar{y}_j k_j(\bar{u},\bar{p}) - \bar{p}^t\nabla_p(\bar{y}_j k_j(\bar{u},\bar{p}))] \leqq 0, \quad \beta = 1,2,\ldots,r,$$

which along with (3.23) gives

$$F\left(\bar{x},\bar{u};\alpha^2(\bar{x},\bar{u})\left\{\nabla_p\left(\sum_{j\in I_\beta}\bar{y}_j k_j(\bar{u},\bar{p})\right)\right\}\right) \leqq -\rho_\beta^2 d^2(\bar{x},\bar{u}), \quad \beta = 1,2,\ldots,r. \tag{3.24}$$

The first dual constraint and (3.24) along with the sublinearity of F and $\alpha^2(\bar{x},\bar{u}) > 0$ imply

$$F\left(\bar{x},\bar{u};\nabla_p\bar{\lambda}^t h(\bar{u},\bar{p}) + \sum_{i\in S}\bar{\lambda}_i B_i\bar{w}_i + \nabla_p\left(\sum_{j\in I_0}\bar{y}_j k_j(\bar{u},\bar{p})\right)\right)$$
$$\geqq \frac{1}{\alpha^2(\bar{x},\bar{u})}\left(\sum_{\beta=1}^{r}\rho_\beta^2\right)d^2(\bar{x},\bar{u}).$$

Because $\left(\frac{\rho_1^1}{\alpha^1(\bar{x},\bar{u})} + \frac{1}{\alpha^2(\bar{x},\bar{u})}\sum_{\beta=1}^{r}\rho_\beta^2\right) \geqq 0$, the above inequality changes to

$$F\left(\bar{x},\bar{u};\nabla_p\bar{\lambda}^t h(\bar{u},\bar{p}) + \sum_{i\in S}\bar{\lambda}_i B_i\bar{w}_i + \nabla_p\left(\sum_{j\in I_0}\bar{y}_j k_j(\bar{u},\bar{p})\right)\right) \geqq -\frac{\rho_1^1}{\alpha^1(\bar{x},\bar{u})}d^2(\bar{x},\bar{u}).$$

That is,

$$F\left(\bar{x},\bar{u};\alpha^1(\bar{x},\bar{u})\left\{\nabla_p\bar{\lambda}^t h(\bar{u},\bar{p}) + \sum_{i\in S}\bar{\lambda}_i B_i\bar{w}_i + \nabla_p\left(\sum_{j\in I_0}\bar{y}_j k_j(\bar{u},\bar{p})\right)\right\}\right)$$
$$\geqq -\rho_1^1 d^2(\bar{x},\bar{u}). \tag{3.25}$$

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The inequality (3.22) together with (3.25) becomes

$$\begin{aligned}\sum_{i\in S}\bar{\lambda}_i\{f_i(\bar{x})+\bar{x}^tB_i\bar{w}_i\}+\sum_{j\in I_0}\bar{y}_jg_j(\bar{x})\\ >\sum_{i\in S}\bar{\lambda}_i\{f_i(\bar{u})+h_i(\bar{u},\bar{p})-\bar{p}^t\nabla_ph_i(\bar{u},\bar{p})+\bar{u}^tB_i\bar{w}_i\}\\ +\sum_{j\in I_0}[\bar{y}_jg_j(\bar{u})+\bar{y}_jk_j(\bar{u},\bar{p})-\bar{p}^t\nabla_p(\bar{y}_jk_j(\bar{u},\bar{p}))].\end{aligned}$$

Because $\bar{x}$ is feasible for (NMP) and $\bar{y}\geqq 0$, we have

$$\begin{aligned}\sum_{i\in S}\bar{\lambda}_i\{f_i(\bar{x})+\bar{x}^tB_i\bar{w}_i\}>\sum_{i\in S}\bar{\lambda}_i\{f_i(\bar{u})+h_i(\bar{u},\bar{p})-\bar{p}^t\nabla_ph_i(\bar{u},\bar{p})+\bar{u}^tB_i\bar{w}_i\}\\ +\sum_{j\in I_\circ}[\bar{y}_jg_j(\bar{u})+\bar{y}_jk_j(\bar{u},\bar{p})-\bar{p}^t\nabla_p(\bar{y}_jk_j(\bar{u},\bar{p}))],\end{aligned}$$

which contradicts (3.21). Hence $\bar{x}=\bar{u}$. □

4. SPECIAL CASES

(i) If we put $h_i(u,p)=p^t\nabla f_i(u)+\frac{1}{2}p^t\nabla^2f_i(u)p$, $i\in S$, $k_j(u,p)=p^t\nabla g_j(u)+\frac{1}{2}p^t\nabla^2g_j(u)p$, $j\in M$ and $I_\beta=\emptyset$, $\beta=1,2,\ldots,r$ in (NHMD), then the dual reduces to the Mangasarian type second-order dual [3].

(ii) Let $h_i(u,p)=p^t\nabla f_i(u)+\frac{1}{2}p^t\nabla^2f_i(u)p$, $i\in S$, and $k_j(u,p)=p^t\nabla g_j(u)+\frac{1}{2}p^t\nabla^2g_j(u)p$, $j\in M$. Then (NHMD) becomes the second-order general Mond–Weir type dual presented in [3]. If, in addition, $B_i=0$, $i\in S$, $I_\circ=J\subseteq M$ and $I_\beta=M\backslash J$, $\beta=1,2,\ldots,r$, then we get the second-order mixed type dual discussed in [6].

(iii) Let $s=1, h_j(u,p)=p^t\nabla f_j(u)+\frac{1}{2}p^t\nabla^2f_j(u)p$, and $k_j(u,p)=p^t\nabla g_j(u)+\frac{1}{2}p^t\nabla^2g_j(u)p$, in (NHMD). Then we get the nondifferentiable general Mond–Weir type second-order dual of Zhang and Mond [17]. If in addition $p=0$, $I_0=J\subseteq M$ and $I_\beta=M\backslash J$, then we obtain the first-order mixed type dual discussed in [1].

(iv) If $s=1$, then (NHMD) reduces to the nondifferentiable general Mond–Weir type higher-order dual considered in [10]. If, in addition, $B_1=0$, then we obtain the general Mond–Weir type higher-order dual given by Mishra and Rueda [9].

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