SECOND-ORDER DUALITY FOR NONDIFFERENTIABLE MULTIOBJECTIVE PROGRAMMING PROBLEMS

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□ This paper is concerned with second-order duality for a class of nondifferentiable multiobjective programming problems. Usual duality theorems are proved for Mangasarian type and general Mond–Weir type vector duals under generalized convexity assumptions.

Keywords Generalized convexity; Nondifferentiable multiobjective programming; Properly efficient solution; Second-order duality.

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1. INTRODUCTION

Preda [13] introduced the notion of \((F, \rho)\)-convexity, an extension of \(F\)-convexity defined by Hanson and Mond [7] and \(\rho\)-convexity introduced by Vial [14], and he used this concept to obtain multiobjective duality results for efficient solutions. Gulati and Islam [5] derived sufficiency and duality theorems for efficient and properly efficient solutions of a multiobjective nonlinear programming problem under the assumptions taken by Hanson and Mond [7]. In [2], Ahmad obtained a number of sufficiency theorems for efficient and properly efficient solutions of a multiobjective programming problem under various generalized convexity assumptions, and he discussed duality results also for a general Mond–Weir type dual.

Second-order duality was first introduced by Mangasarian [8] for a nonlinear programming problem, which involves second-order derivatives of the objective and the constraint functions. He established duality results involving somewhat complicated assumptions. Mond [11] reproved the second-order duality results under different and less restricted assumptions.
than those previously considered in [8]. Zhang and Mond [16] discussed
duality results for nondifferentiable programs under generalized invexity.

Mishra [9] formulated a second-order Mond–Weir type multiobjective
dual and derived weak and strong duality theorems under generalized

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type I functions. Aghezzaf [1] formulated a second-order multiobjective
mixed type dual and obtained various duality results involving a new
class of generalized second-order \((F, \rho)\)-convex functions. In [6], Hachimi
and Aghezzaf proposed a new class of second-order generalized type
I functions and established multiobjective duality results for a mixed
type dual. Recently, Ahmad and Husain [3] studied a Mond–Weir type
multiobjective dual and derived duality results by defining second-order
\((F, \alpha, \rho, d)\)-convex function and its generalizations.

In this paper, we consider the following nondifferentiable vector
optimization problem:

\[
\begin{align*}
\text{Minimize} & \quad (f_1(x) + (x' B_1 x)^{\frac{1}{2}}, f_2(x) + (x' B_2 x)^{\frac{1}{2}}, \ldots, f_k(x) + (x' B_k x)^{\frac{1}{2}}) \\
\text{subject to} & \quad x \in S = \{x \in X : g(x) \leq 0\},
\end{align*}
\]

where \(X\) is an open subset of \(R^n\), \(f_j : X \rightarrow R, j = 1, 2, \ldots, k\), \(g : X \rightarrow R^m\)
and \(B_j, j = 1, 2, \ldots, k\) is an \(n \times n\) positive semidefinite symmetric matrix.
We formulate Mangasarian type and general Mond–Weir type second-order
duals for (P) and prove weak, strong, and strict converse duality theorems
under bonvexity/generalized bonvexity.

2. NOTATIONS AND PRELIMINARIES

The following conventions for vectors \(u, v \in R^n\) will be used: \(u \geq v \Leftrightarrow u_i \geq v_i, i = 1, 2, \ldots, n; u \geq v \Leftrightarrow u_i \geq v_i, i = 1, 2, \ldots, n, \text{ but } u \neq v; u > v \Leftrightarrow u_i > v_i, i = 1, 2, \ldots, n.\) The index sets are \(K = \{1, 2, \ldots, k\}\) and \(M = \{1, 2, \ldots, m\}.\) For each \(j \in K, K_j = K - \{j\}.

Consider the following vector optimization problem:

\[
\begin{align*}
\text{Minimize} & \quad f(x) = [f_1(x), f_2(x), \ldots, f_k(x)] \\
\text{subject to} & \quad x \in S = \{x \in X : g(x) \leq 0\}. 
\end{align*}
\]

**Definition 2.1.** A point \(x^o \in S\) is said to be an efficient solution of (VOP)
if there exists no other \(x \in S\) such that

\[f_j(x) \leq f_j(x^o), \quad \text{for all } j \in K,\]

and

\[f_i(x) < f_i(x^o), \quad \text{for at least one } i \in K_j.\]
**Definition 2.2.** An efficient solution $x^\circ$ is said to be a properly efficient solution of (VOP) if there exists a scalar $M > 0$ such that for each $j \in K, f_j(x) < f_j(x^\circ)$ and $x \in S$ imply that

$$
\frac{f_j(x^\circ) - f_j(x)}{f_i(x) - f_i(x^\circ)} \leq M,
$$

for at least one $i \in K_j$ such that $f_i(x^\circ) < f_i(x)$.

The concept of so-called second-order convex functions was first introduced by Mond [11]. Later on, Bector and Chandra [4] named these functions as bonvex functions. They also introduced their generalizations.

**Definition 2.3.** A real-valued twice-differentiable function $f_j : X \to \mathbb{R}$, $j \in K$, is said to be bonvex at $x^\circ$ if for every $x \in X$ and $p \in \mathbb{R}^n$, we have

$$
f_j(x) - f_j(x^\circ) + \frac{1}{2} p^T \nabla^2 f_j(x^\circ) p \geq (\nabla f_j(x^\circ) + \nabla^2 f_j(x^\circ) p)(x - x^\circ).
$$

**Definition 2.4.** A real-valued twice-differentiable function $f_j : X \to \mathbb{R}$, $j \in K$, is said to be strictly bonvex at $x^\circ$ if for every $x \in X$, $x \neq x^\circ$, and $p \in \mathbb{R}^n$, we have

$$
f_j(x) - f_j(x^\circ) + \frac{1}{2} p^T \nabla^2 f_j(x^\circ) p > (\nabla f_j(x^\circ) + \nabla^2 f_j(x^\circ) p)(x - x^\circ).
$$

**Definition 2.5.** A real-valued twice-differentiable function $f_j : X \to \mathbb{R}$, $j \in K$, is said to be quasibonvex at $x^\circ$ if for every $x \in X$ and $p \in \mathbb{R}^n$, we have

$$
f_j(x) \leq f_j(x^\circ) - \frac{1}{2} p^T \nabla^2 f_j(x^\circ) p \Rightarrow (\nabla f_j(x^\circ) + \nabla^2 f_j(x^\circ) p)(x - x^\circ) \leq 0.
$$

**Definition 2.6.** A real-valued twice-differentiable function $f_j : X \to \mathbb{R}$, $j \in K$, is said to be pseudobonvex at $x^\circ$ if for every $x \in X$ and $p \in \mathbb{R}^n$, we have

$$
f_j(x) < f_j(x^\circ) - \frac{1}{2} p^T \nabla^2 f_j(x^\circ) p \Rightarrow (\nabla f_j(x^\circ) + \nabla^2 f_j(x^\circ) p)(x - x^\circ) < 0.
$$

**Definition 2.7.** A real-valued twice-differentiable function $f_j : X \to \mathbb{R}$, $j \in K$, is said to be strictly pseudobonvex at $x^\circ$ if for every $x \in X$, $x \neq x^\circ$, and $p \in \mathbb{R}^n$, we have

$$
(\nabla f_j(x^\circ) + \nabla^2 f_j(x^\circ) p)(x - x^\circ) \geq 0 \Rightarrow f_j(x) > f_j(x^\circ) - \frac{1}{2} p^T \nabla^2 f_j(x^\circ) p.
We shall make use of the following generalized Schwartz inequality:

\[ x^t A z \leq (x^t A x)^{\frac{1}{2}} (z^t A z)^{\frac{1}{2}}, \]

where \( x, z \in \mathbb{R}^n \), and \( A \) is a positive semidefinite symmetric matrix of order \( n \).

The following theorem will be needed in the sequel.

**Theorem 2.8** [12]. Let \( x^o \) be a properly efficient solution of \((P)\) at which a constraint qualification [10, 15] is satisfied. Then there exist \( \lambda^o \in \mathbb{R}^k \), \( u^o \in \mathbb{R}^m \) and \( v_j^o \in \mathbb{R}^n \), \( j \in K \) such that

\[
\sum_{j=1}^k \lambda^o_j (\nabla f_j(x^o) + B_j v_j^o) + \nabla u^o g(x^o) = 0,
\]

\[
u^o g(x^o) = 0,
\]

\[
(x^o B_j x^o)^{\frac{1}{2}} = x^o B_j v_j^o, \quad j \in K,
\]

\[
v_j^o B_j v_j^o \leq 1, \quad j \in K,
\]

\[
\lambda^o > 0, \quad \sum_{j=1}^k \lambda^o_j = 1, \quad u^o \geq 0.
\]

### 3. MANGASARIAN TYPE DUALITY

In this section, we propose the following Mangasarian type dual to \((P)\) and prove weak, strong, and strict converse duality theorems.

Maximize

\[
\left( f_1(y) + u^t g(y) + y^t B_1 v_1 - \frac{1}{2} p^t \nabla^2 f_1(y) + u^t g(y) \right)p, \ldots,
\]

\[
f_k(y) + u^t g(y) + y^t B_k v_k - \frac{1}{2} p^t \nabla^2 f_k(y) + u^t g(y) \right)p
\]

(MSD)

subject to

\[
\sum_{j=1}^k \lambda_j (\nabla f_j(y) + \nabla^2 f_j(y)p + B_j v_j) + \nabla u^t g(y) + \nabla^2 u^t g(y)p = 0, \quad (3.1)
\]

\[
v_j^t B_j v_j \leq 1, \quad j \in K, \quad (3.2)
\]

\[
\lambda > 0, \quad \sum_{j=1}^k \lambda_j = 1, \quad u \geq 0, \quad (3.3)
\]

where \( y, v_j, p \in \mathbb{R}^n, j \in K \), and \( u \in \mathbb{R}^m \).
Theorem 3.1 (Weak Duality). Let $x$ and $(y, u, v_1, v_2, \ldots, v_k, \lambda, p)$ be feasible solutions of $(P)$ and (MSD), respectively. Suppose $(f_j(\cdot) + (\cdot)^t B_j v_j), j \in K$, is convex at $y$, and $g_i(\cdot), i \in M$, is convex at $y$. Then, the following cannot hold:

$$f_j(x) + (x' B_j x)^{\frac{1}{2}} \leq f_j(y) + u' g(y) + y' B_j v_j - \frac{1}{2} p' \nabla^2 f_j(y) p,$$

for all $j \in K$, (3.4)

and

$$f_i(x) + (x' B_j x)^{\frac{1}{2}} < f_i(y) + u' g(y) + y' B_j v_j - \frac{1}{2} p' \nabla^2 f_i(y) p,$$

for at least one $i \in K_j$. (3.5)

Proof. Suppose to the contrary that (3.4) and (3.5) hold, that is,

$$f_j(x) + (x' B_j x)^{\frac{1}{2}} \leq f_j(y) + u' g(y) + y' B_j v_j - \frac{1}{2} p' \nabla^2 f_j(y) p, \quad \text{for all } j \in K,$$

and

$$f_i(x) + (x' B_j x)^{\frac{1}{2}} < f_i(y) + u' g(y) + y' B_j v_j - \frac{1}{2} p' \nabla^2 f_i(y) p,$$

for at least one $i \in K_j$.

Because $\lambda > 0$ and $\sum_{j=1}^k \lambda_j = 1$, the above inequalities yield

$$\sum_{j=1}^k \lambda_j f_j(x) + (x' B_j x)^{\frac{1}{2}} < \sum_{j=1}^k \lambda_j \left( f_j(y) + y' B_j v_j - \frac{1}{2} p' \nabla^2 f_j(y) p \right) + u' g(y)$$

$$- \frac{1}{2} p' \nabla^2 u' g(y) p.$$  (3.6)

As $(f_j(\cdot) + (\cdot)^t B_j v_j), j \in K$ and $g_i(\cdot), i \in M$, are convex at $y$, we have

$$f_j(x) + x' B_j v_j - f_j(y) - y' B_j v_j + \frac{1}{2} p' \nabla^2 f_j(y) p$$

$$\geq (\nabla f_j(y) + \nabla^2 f_j(y) p + B_j v_j)(x - y), \quad (3.7)$$

and

$$g_i(x) - g_i(y) + \frac{1}{2} p' \nabla^2 g_i(y) p \geq (\nabla g_i(y) + \nabla^2 g_i(y) p)(x - y). \quad (3.8)$$
On multiplying (3.7) by \( \lambda_j > 0, j \in K \) and (3.8) by \( u_i \geq 0, i \in M \), and then summing up to get

\[
\sum_{j=1}^{k} \lambda_j \left( f_j(x) + x^t B_j v_j - f_j(y) - y^t B_j v_j + \frac{1}{2} p^t \nabla^2 f_j(y) p \right)
\]

\[
+ u^t g(x) - u^t g(y) + \frac{1}{2} p^t \nabla^2 u^t g(y) p \geq \left[ \sum_{j=1}^{k} \lambda_j \left( \nabla f_j(y) + \nabla^2 f_j(y) p + B_j v_j \right) + \nabla u^t g(y) + \nabla^2 u^t g(y) p \right] (x - y).
\]

The above inequality, in view of (3.2), (3.6), the generalized Schwartz inequality, and \( u^t g(x) \leq 0 \), gives

\[
\left[ \sum_{j=1}^{k} \lambda_j \left( \nabla f_j(y) + \nabla^2 f_j(y) p + B_j v_j \right) + \nabla u^t g(y) + \nabla^2 u^t g(y) p \right] (x - y) < 0,
\]

which is a contradiction to (3.1).

\[\square\]

**Theorem 3.2 (Strong Duality).** Let \( x^o \) be a properly efficient solution of \((P)\) at which a constraint qualification \([10, 15]\) is satisfied. Then there exist \( \lambda^o \in \mathbb{R}^k \), \( u^o \in \mathbb{R}^m \), \( v^o_j \in \mathbb{R}^n, j \in K \), and \( p^o = 0 \) such that \((x^o, u^o, v^o_1, v^o_2, \ldots, v^o_k, \lambda^o, p^o = 0)\) is feasible for \((MSD)\) and the corresponding objective values of \((P)\) and \((MSD)\) are equal. If, in addition, the assumptions of weak duality (Theorem 3.1) are satisfied, then \((x^o, u^o, v^o_1, v^o_2, \ldots, v^o_k, \lambda^o, p^o = 0)\) is a properly efficient solution of \((MSD)\).

**Proof.** Because \( x^o \) is a properly efficient solution of \((P)\) at which a constraint qualification \([10, 15]\) is satisfied, by Theorem 2.8, there exist \( \lambda^o \in \mathbb{R}^k \), \( u^o \in \mathbb{R}^m \) and \( v^o_j \in \mathbb{R}^n, j \in K \) such that

\[
\sum_{j=1}^{k} \lambda^o_j \left( \nabla f_j(x^o) + B_j v^o_j \right) + \nabla u^o g(x^o) = 0,
\]

\[
u^o g(x^o) = 0,
\]

\[
(x^o B_j x^o)^{\frac{1}{2}} = x^o B_j v^o_j, \quad j \in K,
\]

\[
v^o_j B_j v^o_j \leq 1, \quad j \in K,
\]

\[
\lambda^o > 0, \quad \sum_{j=1}^{k} \lambda^o_j = 1, \quad u^o \geq 0.
\]
Thus, \((x^*, u^*, v^1, v^2, \ldots, v^K, \lambda^*, p^* = 0)\) is feasible for (MSD) and the corresponding objective values of (P) and (MSD) are equal. Now, we show that \((x^*, u^*, v^1, v^2, \ldots, v^K, \lambda^*, p^* = 0)\) is an efficient solution of (MSD).

Suppose that it is not efficient, then there exists a feasible solution \((y^*, u^*, v^1, v^2, \ldots, v^K, \lambda^*, p^*)\) such that

\[
\begin{align*}
&f_i(y^*) + u^*_i g(y^*) + y^*_i B^*_i v^*_j - \frac{1}{2} p^*_i \nabla^2 [f_i(y^*) + u^*_i g(y^*)] p^*_i \\
&\geq f_i(x^*) + u^*_i g(x^*) + x^*_i B^*_i v^*_j, \quad \text{for all } j \in K, \\
\end{align*}
\]

and

\[
\begin{align*}
&f_i(y^*) + u^*_i g(y^*) + y^*_i B^*_i v^*_j - \frac{1}{2} p^*_i \nabla^2 [f_i(y^*) + u^*_i g(y^*)] p^*_i \\
&> f_i(x^*) + u^*_i g(x^*) + x^*_i B^*_i v^*_j, \quad \text{for at least one } i \in K_i.
\end{align*}
\]

Using \((x^*_i B^*_i x^o)^{\frac{1}{2}} = x^*_i B^*_i v^*_j, j \in K,\) and \(u^*_i g(x^o) = 0,\) the above inequalities give

\[
\begin{align*}
f_j(x^o) + (x^*_i B^*_i x^o)^{\frac{1}{2}} &\leq f_j(y^*) + u^*_i g(y^*) + y^*_i B^*_i v^*_j \\
&\quad - \frac{1}{2} p^*_i \nabla^2 [f_j(y^*) + u^*_i g(y^*)] p^*_i, \quad \text{for all } j \in K, \\
\end{align*}
\]

and

\[
\begin{align*}
f_j(x^o) + (x^*_i B^*_i x^o)^{\frac{1}{2}} &< f_j(y^*) + u^*_i g(y^*) + y^*_i B^*_i v^*_j \\
&\quad - \frac{1}{2} p^*_i \nabla^2 [f_j(y^*) + u^*_i g(y^*)] p^*_i, \quad \text{for at least one } i \in K_i,
\end{align*}
\]

which is a contradiction to weak duality (Theorem 3.1). Hence, \((x^*, u^*, v^1, v^2, \ldots, v^K, \lambda^*, p^* = 0)\) is an efficient solution of (MSD). Now assume that it is not properly efficient, then for each scalar \(M > 0,\) there exists a feasible solution \((y^*, u^*, v^1, v^2, \ldots, v^K, \lambda^*, p^*)\) and some \(j \in K,\) such that

\[
\begin{align*}
f_j(y^*) + u^*_i g(y^*) + y^*_i B^*_i v^*_j - \frac{1}{2} p^*_i \nabla^2 [f_j(y^*) + u^*_i g(y^*)] p^*_i \\
&> f_j(x^*) + u^*_i g(x^*) + x^*_i B^*_i v^*_j,
\end{align*}
\]

implies

\[
\begin{align*}
&\left[ f_j(y^*) + u^*_i g(y^*) + y^*_i B^*_i v^*_j - \frac{1}{2} p^*_i \nabla^2 [f_j(y^*) + u^*_i g(y^*)] p^*_i \right] \\
&- \left[ f_j(x^*) + u^*_i g(x^*) + x^*_i B^*_i v^*_j \right]
\end{align*}
\]
for all $i \in K_j$ satisfying

\[
\begin{align*}
[f(x^o) + u^{o'}g(x^o) + x^{o'}B_i v^o_i] \\
> [f(y^*) + u^{o'}g(y^*) + y^{o'}B_i v^*_i - \frac{1}{2} p^{o'}\nabla^2[f(y^*) + u^{o'}g(y^*)]p^*],
\end{align*}
\]

Again, by $(x^{o'}B_j x^o)^{\frac{1}{2}} = x^{o'}B_j v^*_j, j \in K, \text{ and } u^{o'}g(x^o) = 0$, we get

\[
\begin{align*}
[f(y^*) + u^{o'}g(y^*) + y^{o'}B_j v^*_j - \frac{1}{2} p^{o'}\nabla^2[f(y^*) + u^{o'}g(y^*)]p^*] \\
- [f(x^o) + (x^{o'}B_j x^o)^{\frac{1}{2}}] \\
> M\left[[f(x^o) + (x^{o'}B_j x^o)^{\frac{1}{2}}] \\
- [f(y^*) + u^{o'}g(y^*) + y^{o'}B_j v^*_j - \frac{1}{2} p^{o'}\nabla^2[f(y^*) + u^{o'}g(y^*)]p^*] \right].
\end{align*}
\]

This means that $[f(y^*) + u^{o'}g(y^*) + y^{o'}B_j v^*_j - \frac{1}{2} p^{o'}\nabla^2[f(y^*) + u^{o'}g(y^*)]p^*]$ is infinitely better than $[f(x^o) + (x^{o'}B_j x^o)^{\frac{1}{2}}]$ for some $j \in K$, whereas $[f(x^o) + (x^{o'}B_j x^o)^{\frac{1}{2}}]$ is at most finitely better than $[f(y^*) + u^{o'}g(y^*) + y^{o'}B_j v^*_j - \frac{1}{2} p^{o'}\nabla^2[f(y^*) + u^{o'}g(y^*)]p^*]$ for all $i \in K_j$. Therefore,

\[
f(x^o) + (x^{o'}B_j x^o)^{\frac{1}{2}} \leq f(y^*) + u^{o'}g(y^*) + y^{o'}B_j v^*_j \\
- \frac{1}{2} p^{o'}\nabla^2[f(y^*) + u^{o'}g(y^*)]p^*, \text{ for all } j \in K,
\]

and

\[
f(x^o) + (x^{o'}B_j x^o)^{\frac{1}{2}} < f(y^*) + u^{o'}g(y^*) + y^{o'}B_j v^*_j \\
- \frac{1}{2} p^{o'}\nabla^2[f(y^*) + u^{o'}g(y^*)]p^*, \text{ for at least one } i \in K_j,
\]

which again contradicts weak duality (Theorem 3.1). Hence, $(x^o, u^o, v^o_i, v^o_2, \ldots, v^o_k, \lambda^o, p^o = 0)$ is a properly efficient solution of (MSD). \qed
Theorem 3.3 (Strict Converse Duality). Let \( x^* \) and \( (y^*, u^*, v^1, v^2, \ldots, v^k, \lambda^*, p^\circ) \) be feasible solutions of (P) and (MSD), respectively, such that
\[
\sum_{j=1}^{k} \lambda_j^* [f_j(x^*) + y^* B_j v^j] \leq \sum_{j=1}^{k} \lambda_j^* \left[ f_j(y^*) + y^* B_j v^j \right] - \frac{1}{2} p^\circ \nabla^2 f_j(y^*) p^\circ + u^* g(y^*) - \frac{1}{2} p^\circ \nabla^2 u^* f(y^*) p^\circ. \tag{3.9}
\]
Suppose \( (f_j(\cdot) + (\cdot)' B_j v^j), j \in K \) is strictly convex at \( y^* \), and \( g_i(\cdot), i \in M \) is convex at \( y^* \), then \( x^* = y^* \).

Proof. We assume that \( x^* \neq y^* \) and exhibit a contradiction. The strict convexity of \( (f_j(\cdot) + (\cdot)' B_j v^j), j \in K \) at \( y^* \) and convexity of \( g_i(\cdot), i \in M \) at \( y^* \), imply
\[
\left( f_j(x^*) + x^* B_j v^j - f_j(y^*) - y^* B_j v^j + \frac{1}{2} p^\circ \nabla^2 f_j(y^*) p^\circ \right) > \left( \nabla f_j(y^*) + \nabla^2 f_j(y^*) p^\circ + B_j v^j \right) (x^* - y^*), \tag{3.10}
\]
and
\[
g(x^*) - g(y^*) + \frac{1}{2} p^\circ \nabla^2 g(y^*) p^\circ \geq \left( \nabla g(y^*) + \nabla^2 g(y^*) p^\circ \right) (x^* - y^*). \tag{3.11}
\]
On multiplying (3.10) by \( \lambda_j^* > 0, j \in K \) and (3.11) by \( u_i^* \geq 0, i \in M \), and then summing up to get
\[
\sum_{j=1}^{k} \lambda_j^* [f_j(x^*) + x^* B_j v^j] + u^* g(x^*) - \sum_{j=1}^{k} \lambda_j^* [f_j(y^*) + y^* B_j v^j] - \frac{1}{2} p^\circ \nabla^2 f_j(y^*) p^\circ - u^* g(y^*) + \frac{1}{2} p^\circ \nabla^2 u^* f(y^*) p^\circ > \sum_{j=1}^{k} \lambda_j^* \left( \nabla f_j(y^*) + \nabla^2 f_j(y^*) p^\circ + B_j v^j \right) + \nabla u^* g(y^*) + \nabla^2 u^* g(y^*) p^\circ \right) (x^* - y^*).
\]
The above inequality on using (3.1) and \( u^* g(x^*) \leq 0 \), gives
\[
\sum_{j=1}^{k} \lambda_j^* [f_j(x^*) + x^* B_j v^j] > \sum_{j=1}^{k} \lambda_j^* \left[ f_j(y^*) + y^* B_j v^j \right] - \frac{1}{2} p^\circ \nabla^2 f_j(y^*) p^\circ + u^* g(y^*) - \frac{1}{2} p^\circ \nabla^2 u^* g(y^*) p^\circ,
\]
which is a contradiction to (3.9). Hence, \( x^* = y^* \).
4. GENERAL MOND–WEIR TYPE DUALITY

For (P), we now formulate the following general Mond–Weir type second-order dual:

\[
\text{Maximize } f_1(y) + y^t Bh_1 v_1 + \sum_{i \in I_0} u_ig_i(y) - \frac{1}{2} p^t \nabla^2 \left[ f_1(y) + \sum_{i \in I_0} u_ig_i(y) \right] p, \ldots,
\]

\[
f_k(y) + y^t Bh_k v_k + \sum_{i \in I_0} u_ig_i(y) - \frac{1}{2} p^t \nabla^2 \left[ f_k(y) + \sum_{i \in I_0} u_ig_i(y) \right] p
\]

subject to

\[
\sum_{j=1}^k \lambda_j (\nabla f_j(y) + \nabla^2 f_j(y)p + Bh_j) + \nabla u_i g_i(y) + \nabla^2 u_i g_i(y)p = 0, \quad (4.1)
\]

\[
\sum_{i \in I_0} u_ig_i(y) - \frac{1}{2} p^t \nabla^2 \sum_{i \in I_0} u_ig_i(y)p \geq 0, \quad \alpha = 1, 2, \ldots, r, \quad (4.2)
\]

\[
v_j^t B_j v_j \leq 1, \quad j \in K, \quad (4.3)
\]

\[
\lambda > 0, \quad \sum_{j=1}^k \lambda_j = 1, \quad u \geq 0, \quad (4.4)
\]

where \(I_0 \subseteq M, \alpha = 0, 1, 2, \ldots, r, \) with \(\bigcup_{\alpha=0}^r I_\alpha = M \) and \(I_\alpha \cap I_\beta = \emptyset, \) if \(\alpha \neq \beta.\)

**Remark 4.1.** If \(I_0 = \emptyset, \alpha = 1, 2, \ldots, r, \) and \(I_0 = M, \) then (GMD) reduces to (MSD).

**Theorem 4.2 (Weak Duality).** Let \(x \) and \((y, u, v_1, v_2, \ldots, v_k, \lambda, p)\) be feasible solutions of (P) and (GMD), respectively. Suppose \(\sum_{j=1}^k \lambda_j \left( f_j(\cdot) + \cdot^t Bh_j \right) + \sum_{i \in I_0} u_ig_i(\cdot) \) is pseudomonvex at \(y, \) and \(\sum_{i \in I_0} u_ig_i(\cdot), \) \(\alpha = 1, 2, \ldots, r, \) is quasibonvex at \(y.\) Then, the following cannot hold:

\[
f_j(x) + (x^t Bh_j)^2 \leq f_j(y) + y^t Bh_j v_j + \sum_{i \in I_0} u_ig_i(y)
\]

\[-\frac{1}{2} p^t \nabla^2 \left[ f_j(y) + \sum_{i \in I_0} u_ig_i(y) \right] p, \quad \text{for all } j \in K, \quad (4.5)\]
and
\[ f_i(x) + (x^TB_ix)^{\frac{1}{2}} < f_i(y) + y^TB_iv_l + \sum_{i \in I_0} u_i g_i(y) - \frac{1}{2} p^T \nabla^2 \left[ f_i(y) + \sum_{i \in I_0} u_i g_i(y) \right] p, \]
for at least one \( l \in K_j \).

**Proof.** Because \( x \) and \((y, u, v_1, v_2, \ldots, v_k, \lambda, p)\) are feasible solutions of \((P)\) and \((GMD)\), respectively, then
\[ \sum_{i \in I_0} u_i g_i(x) \leq 0 \leq \sum_{i \in I_0} u_i g_i(y) - \frac{1}{2} p^T \nabla^2 \sum_{i \in I_0} u_i g_i(y) p, \]
which by the quasibonvexity of \( \sum_{i \in I_0} u_i g_i(\cdot) \), \( \lambda = 1, 2, \ldots, r \), at \( y \) yields
\[ \left( \nabla \sum_{i \in I_0} u_i g_i(y) + \nabla^2 \sum_{i \in I_0} u_i g_i(y) p \right) (x - y) \leq 0, \quad \lambda = 1, 2, \ldots, r. \] (4.7)

Now, suppose to the contrary that (4.5) and (4.6) hold, that is,
\[ f_i(x) + (x^TB_ix)^{\frac{1}{2}} \leq f_i(y) + y^TB_iv_l + \sum_{i \in I_0} u_i g_i(y) \]
\[ - \frac{1}{2} p^T \nabla^2 \left[ f_i(y) + \sum_{i \in I_0} u_i g_i(y) \right] p, \quad \text{for all } j \in K, \]
and
\[ f_i(x) + (x^TB_ix)^{\frac{1}{2}} < f_i(y) + y^TB_iv_l + \sum_{i \in I_0} u_i g_i(y) \]
\[ - \frac{1}{2} p^T \nabla^2 \left[ f_i(y) + \sum_{i \in I_0} u_i g_i(y) \right] p, \quad \text{for at least one } l \in K_j. \]

Because \( \lambda > 0 \) and \( \sum_{j=1}^k \lambda_j = 1 \), we get
\[ \sum_{j=1}^k \lambda_j \left( f_i(x) + (x^TB_ix)^{\frac{1}{2}} \right) < \sum_{j=1}^k \lambda_j \left( f_i(y) + y^TB_iv_l - \frac{1}{2} p^T \nabla^2 f_i(y) p \right) \]
\[ + \sum_{i \in I_0} u_i g_i(y) - \frac{1}{2} p^T \nabla^2 \sum_{i \in I_0} u_i g_i(y) p, \]
which on using the generalized Schwartz inequality along with (4.3) and the feasibility of \((P)\), implies

\[
\sum_{j=1}^{k} \lambda_j(f_j(x) + (x^i B_j v_j)) + \sum_{i \in L} u_i g_i(x)
\]

\[
< \sum_{j=1}^{k} \lambda_j \left( f_j(y) + y^i B_j v_j - \frac{1}{2} p^i \nabla^2 f_j(y) p \right) + \sum_{i \in L} u_i g_i(y) - \frac{1}{2} p^i \nabla^2 \sum_{i \in L} u_i g_i(y) p.
\]

The pseudobonvexity of \(\left[ \sum_{j=1}^{k} \lambda_j(f_j(\cdot) + (\cdot)^i B_j v_j) + \sum_{i \in L} u_i g_i(\cdot) \right]\) at \(y\), implies

\[
\left[ \sum_{j=1}^{k} \lambda_j(\nabla f_j(y) + \nabla^2 f_j(y) p + B_j v_j) + \nabla \sum_{i \in L} u_i g_i(y) + \nabla^2 \sum_{i \in L} u_i g_i(y) p \right] \cdot (x - y) < 0.
\]

(4.8)

On combining (4.7) and (4.8), we obtain

\[
\left[ \sum_{j=1}^{k} \lambda_j(\nabla f_j(y) + \nabla^2 f_j(y) p + B_j v_j) + \nabla u^i g_i(y) + \nabla^2 u^i g_i(y) p \right] \cdot (x - y) < 0,
\]

which is a contradiction to (4.1). \(\Box\)

**Theorem 4.3** (Strong Duality). Let \(x^o\) be a properly efficient solution of \((P)\) at which a constraint qualification \([10, 15]\) is satisfied. Then there exist \(\lambda^o \in \mathbb{R}^k\), \(u^o \in \mathbb{R}^m, v^o_j \in \mathbb{R}^n, j \in K\) and \(p^o \in \mathbb{R}^2\) such that \((x^o, u^o, v^o_1, v^o_2, \ldots, v^o_k, \lambda^o, p^o = 0)\) is feasible for \((GMD)\) and the corresponding objective values of \((P)\) and \((GMD)\) are equal. If, in addition, the assumptions of weak duality (Theorem 4.2) are satisfied, then \((x^o, u^o, v^o_1, v^o_2, \ldots, v^o_k, \lambda^o, p^o = 0)\) is a properly efficient solution of \((GMD)\).

**Proof.** The proof follows on the similar lines of Theorem 3.2.

**Theorem 4.4** (Strict Converse Duality). Let \(x^o\) and \((y^o, u^o, v^o_1, v^o_2, \ldots, v^o_k, \lambda^o, p^o)\) be the feasible solutions of \((P)\) and \((GMD)\), respectively, such that

\[
\sum_{j=1}^{k} \lambda^o_j [f_j(x^o) + x^o^i B_j v^o_j] \leq \sum_{j=1}^{k} \lambda^o_j \left[ f_j(y^o) + y^o^i B_j v^o_j - \frac{1}{2} p^o^i \nabla^2 f_j(y^o) p^o \right]
\]

\[
+ \sum_{i \in L} u^o_i g_i(y^o) - \frac{1}{2} p^o^i \nabla^2 \sum_{i \in L} u^o_i g_i(y^o) p^o.
\]

(4.9)

If \(\left[ \sum_{j=1}^{k} \lambda^o_j(f_j(\cdot) + (\cdot)^i B_j v_j) + \sum_{i \in L} u^o_i g_i(\cdot) \right]\) is strictly pseudobonvex at \(y^o\), and \(\sum_{i \in L} u^o_i g_i(\cdot), i = 1, 2, \ldots, r\) is quasibonvex at \(y^o\), then \(x^o = y^o\).
**Proof.** We assume that $x^o \neq y^o$ and exhibit a contradiction. Because $x^o$ and $(y^o, u^o, v_{i}^o, v_{i}^2, \ldots, v_{k}^o, \lambda^o, \hat{b}^o)$ are feasible solutions of (P) and (GMD), respectively, we have

$$\sum_{i \in I} u_i^o g_i(x^o) \leq 0 \leq \sum_{i \in I} u_i^o g_i(y^o) - \frac{1}{2} \hat{b}^o \nabla^2 \sum_{i \in I} u_i^o g_i(y^o) \beta, \quad \alpha = 1, 2, \ldots, r.$$ 

By the quasibonvexity of $\sum_{i \in I} u_i^o g_i(\cdot)$, $\alpha = 1, 2, \ldots, r$ at $y^o$, it follows that

$$\left(\nabla \sum_{i \in I} u_i^o g_i(y^o) + \nabla^2 \sum_{i \in I} u_i^o g_i(y^o) \beta\right)(x^o - y^o) \leq 0.$$ 

The above inequality together with (4.1) gives

$$\sum_{j=1}^{k} \lambda_j^o \left(\nabla f_j(y^o) + \nabla^2 f_j(y^o) \beta + B_j v_j^o\right) + \nabla \sum_{i \in I} u_i^o g_i(y^o) + \nabla^2 \sum_{i \in I} u_i^o g_i(y^o) \beta \right)(x^o - y^o) \geq 0,$$

which on using the strict pseudobonvexity of $\left[\sum_{j=1}^{k} \lambda_j^o (f_j(\cdot) + (\cdot)' B_j v_j^o\right] + \sum_{i \in I} u_i^o g_i(\cdot)$ at $y^o$, yields

$$\sum_{j=1}^{k} \lambda_j^o \left[f_j(x^o) + x^o' B_j v_j^o\right] + \sum_{i \in I} u_i^o g_i(x^o)$$

$$> \sum_{j=1}^{k} \lambda_j^o \left[f_j(y^o) + y^o' B_j v_j^o - \frac{1}{2} \hat{b}^o \nabla^2 f_j(y^o) \beta\right]$$

$$+ \sum_{i \in I} u_i^o g_i(y^o) - \frac{1}{2} \hat{b}^o \nabla^2 \sum_{i \in I} u_i^o g_i(y^o) \beta,$$

which by $\sum_{i \in I} u_i^o g_i(x^o) \leq 0$ implies

$$\sum_{j=1}^{k} \lambda_j^o \left[f_j(x^o) + x^o' B_j v_j^o\right] > \sum_{j=1}^{k} \lambda_j^o \left[f_j(y^o) + y^o' B_j v_j^o - \frac{1}{2} \hat{b}^o \nabla^2 f_j(y^o) \beta\right]$$

$$+ \sum_{i \in I} u_i^o g_i(y^o) - \frac{1}{2} \hat{b}^o \nabla^2 \sum_{i \in I} u_i^o g_i(y^o) \beta,$$

a contradiction to (4.9). Hence $x^o = y^o$. \hfill \Box
5. CONCLUSION

In this paper, we have discussed the second-order duality results for nondifferentiable multiobjective Mangasarian type and general Mond–Weir type duals involving convexity/generalized convexity. The current results can be further extended for the following nondifferentiable multiobjective fractional programming problem:

\[
\text{Minimize } \left( \frac{f_1(x) + (x'B_1x)^{1/2}}{h_1(x) - (x'D_1x)^{1/2}}, \frac{f_2(x) + (x'B_2x)^{1/2}}{h_2(x) - (x'D_2x)^{1/2}}, \ldots, \frac{f_k(x) + (x'B_kx)^{1/2}}{h_k(x) - (x'D_kx)^{1/2}} \right)
\]

subject to \( x \in S = \{ x \in X : g(x) \leq 0 \} \),

(\( \bar{P} \))

where \( f_j : X \to R \), \( h_j : X \to R \), \( j = 1, 2, \ldots, k \), \( g : X \to R^n \); \( B_j \) and \( D_j \), \( j = 1, 2, \ldots, k \) are \( n \times n \) positive semidefinite symmetric matrices.

REFERENCES