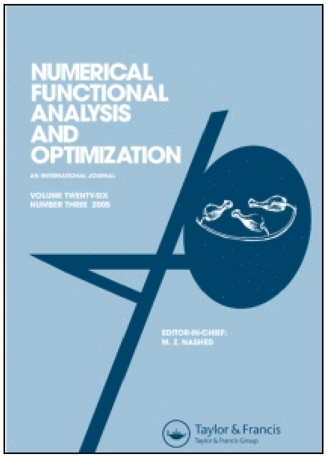

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SECOND-ORDER DUALITY FOR NONDIFFERENTIABLE MULTIOBJECTIVE PROGRAMMING PROBLEMS

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□ *This paper is concerned with second-order duality for a class of nondifferentiable multiobjective programming problems. Usual duality theorems are proved for Mangasarian type and general Mond–Weir type vector duals under generalized bonvexity assumptions.*

Keywords Generalized bonvexity; Nondifferentiable multiobjective programming; Properly efficient solution; Second-order duality.

AMS Subject Classification 90C29; 90C30; 90C46.

1. INTRODUCTION

Preda [13] introduced the notion of (F, ρ)-convexity, an extension of F-convexity defined by Hanson and Mond [7] and ρ-convexity introduced by Vial [14], and he used this concept to obtain multiobjective duality results for efficient solutions. Gulati and Islam [5] derived sufficiency and duality theorems for efficient and properly efficient solutions of a multiobjective nonlinear programming problem under the assumptions taken by Hanson and Mond [7]. In [2], Ahmad obtained a number of sufficiency theorems for efficient and properly efficient solutions of a multiobjective programming problem under various generalized convexity assumptions, and he discussed duality results also for a general Mond–Weir type dual.

Second-order duality was first introduced by Mangasarian [8] for a nonlinear programming problem, which involves second-order derivatives of the objective and the constraint functions. He established duality results involving somewhat complicated assumptions. Mond [11] reproved the second-order duality results under different and less restricted assumptions

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than those previously considered in [8]. Zhang and Mond [16] discussed duality results for nondifferentiable programs under generalized invexity.

Mishra [9] formulated a second-order Mond–Weir type multiobjective dual and derived weak and strong duality theorems under generalized type I functions. Aghezzaf [1] formulated a second-order multiobjective mixed type dual and obtained various duality results involving a new class of generalized second-order (F, ρ)-convex functions. In [6], Hachimi and Aghezzaf proposed a new class of second-order generalized type I functions and established multiobjective duality results for a mixed type dual. Recently, Ahmad and Husain [3] studied a Mond–Weir type multiobjective dual and derived duality results by defining second-order (F, α, ρ, d)-convex function and its generalizations.

In this paper, we consider the following nondifferentiable vector optimization problem:

$$\begin{aligned} &\text{Minimize} \quad \left(f_1(x) + (x^t B_1 x)^{\frac{1}{2}}, f_2(x) + (x^t B_2 x)^{\frac{1}{2}}, \ldots, f_k(x) + (x^t B_k x)^{\frac{1}{2}}\right) \\ &\text{subject to} \quad x \in S = \{x \in X : g(x) \leqq 0\}, \end{aligned} \tag{P}$$

where X is an open subset of R^n, $f_j : X \to R$, $j = 1, 2, \ldots, k$, $g : X \to R^m$ and B_j, $j = 1, 2, \ldots, k$ is an $n \times n$ positive semidefinite symmetric matrix. We formulate Mangasarian type and general Mond–Weir type second-order duals for (P) and prove weak, strong, and strict converse duality theorems under bonvexity/generalized bonvexity.

2. NOTATIONS AND PRELIMINARIES

The following conventions for vectors $u, v \in R^n$ will be used: $u \geqq v \Leftrightarrow u_i \geqq v_i$, $i = 1, 2, \ldots, n$; $u \geq v \Leftrightarrow u_i \geqq v_i$, $i = 1, 2, \ldots, n$, but $u \neq v$; $u > v \Leftrightarrow u_i > v_i$, $i = 1, 2, \ldots, n$. The index sets are $K = \{1, 2, \ldots, k\}$ and $M = \{1, 2, \ldots, m\}$. For each $j \in K, K_j = K - \{j\}$.

Consider the following vector optimization problem:

$$\begin{aligned} &\text{Minimize} \quad f(x) = [f_1(x), f_2(x), \ldots, f_k(x)] \\ &\text{subject to} \quad x \in S = \{x \in X : g(x) \leqq 0\}. \end{aligned} \tag{VOP}$$

Definition 2.1. A point $x^\circ \in S$ is said to be an efficient solution of (VOP) if there exists no other $x \in S$ such that

$$f_j(x) \leqq f_j(x^\circ), \quad \text{for all } j \in K,$$

and

$$f_i(x) < f_i(x^\circ), \quad \text{for at least one } i \in K_j.$$

Definition 2.2. An efficient solution x° is said to be a properly efficient solution of (VOP) if there exists a scalar $M > 0$ such that for each $j \in K, f_j(x) < f_j(x^\circ)$ and $x \in S$ imply that

$$\frac{f_j(x^\circ) - f_j(x)}{f_i(x) - f_i(x^\circ)} \leqq M,$$

for at least one $i \in K_j$ such that $f_i(x^\circ) < f_i(x)$.

The concept of so-called second-order convex functions was first introduced by Mond [11]. Later on, Bector and Chandra [4] named these functions as bonvex functions. They also introduced their generalizations.

Definition 2.3. A real-valued twice-differentiable function $f_j : X \to R$, $j \in K$, is said to be bonvex at x° if for every $x \in X$ and $p \in R^n$, we have

$$f_j(x) - f_j(x^\circ) + \frac{1}{2}p^t \nabla^2 f_j(x^\circ)p \geqq (\nabla f_j(x^\circ) + \nabla^2 f_j(x^\circ)p)(x - x^\circ).$$

Definition 2.4. A real-valued twice-differentiable function $f_j : X \to R$, $j \in K$, is said to be strictly bonvex at x° if for every $x \in X$, $x \neq x^\circ$, and $p \in R^n$, we have

$$f_j(x) - f_j(x^\circ) + \frac{1}{2}p^t \nabla^2 f_j(x^\circ)p > (\nabla f_j(x^\circ) + \nabla^2 f_j(x^\circ)p)(x - x^\circ).$$

Definition 2.5. A real-valued twice-differentiable function $f_j : X \to R$, $j \in K$, is said to be quasibonvex at x° if for every $x \in X$ and $p \in R^n$, we have

$$f_j(x) \leqq f_j(x^\circ) - \frac{1}{2}p^t \nabla^2 f_j(x^\circ)p \Rightarrow (\nabla f_j(x^\circ) + \nabla^2 f_j(x^\circ)p)(x - x^\circ) \leqq 0.$$

Definition 2.6. A real-valued twice-differentiable function $f_j : X \to R$, $j \in K$, is said to be pseudobonvex at x° if for every $x \in X$ and $p \in R^n$, we have

$$f_j(x) < f_j(x^\circ) - \frac{1}{2}p^t \nabla^2 f_j(x^\circ)p \Rightarrow (\nabla f_j(x^\circ) + \nabla^2 f_j(x^\circ)p)(x - x^\circ) < 0.$$

Definition 2.7. A real-valued twice-differentiable function $f_j : X \to R$, $j \in K$, is said to be strictly pseudobonvex at x° if for every $x \in X$, $x \neq x^\circ$, and $p \in R^n$, we have

$$(\nabla f_j(x^\circ) + \nabla^2 f_j(x^\circ)p)(x - x^\circ) \geqq 0 \Rightarrow f_j(x) > f_j(x^\circ) - \frac{1}{2}p^t \nabla^2 f_j(x^\circ)p.$$

We shall make use of the following generalized Schwartz inequality:

$$x^tAz \leqq (x^tAx)^{\frac{1}{2}}(z^tAz)^{\frac{1}{2}},$$

where $x, z \in R^n$, and A is a positive semidefinite symmetric matrix of order n.

The following theorem will be needed in the sequel.

Theorem 2.8 [12]. *Let x° be a properly efficient solution of (P) at which a constraint qualification [10, 15] is satisfied. Then there exist $\lambda^\circ \in R^k$, $u^\circ \in R^m$ and $v_j^\circ \in R^n, j \in K$ such that*

$$\sum_{j=1}^{k} \lambda_j^\circ\big(\nabla f_j(x^\circ) + B_j v_j^\circ\big) + \nabla u^{\circ t} g(x^\circ) = 0,$$

$$u^{\circ t} g(x^\circ) = 0,$$

$$(x^{\circ t} B_j x^\circ)^{\frac{1}{2}} = x^{\circ t} B_j v_j^\circ, \quad j \in K,$$

$$v_j^{\circ t} B_j v_j^\circ \leqq 1, \quad j \in K,$$

$$\lambda^\circ > 0, \quad \sum_{j=1}^{k} \lambda_j^\circ = 1, \quad u^\circ \geqq 0.$$

3. MANGASARIAN TYPE DUALITY

In this section, we propose the following Mangasarian type dual to (P) and prove weak, strong, and strict converse duality theorems.

$$\text{Maximize} \quad \Big(f_1(y) + u^tg(y) + y^tB_1v_1 - \frac{1}{2}p^t\nabla^2\{f_1(y) + u^tg(y)\}p, \ldots, \\ f_k(y) + u^tg(y) + y^tB_kv_k - \frac{1}{2}p^t\nabla^2\{f_k(y) + u^tg(y)\}p\Big) \qquad \text{(MSD)}$$

subject to

$$\sum_{j=1}^{k} \lambda_j(\nabla f_j(y) + \nabla^2 f_j(y)p + B_jv_j) + \nabla u^tg(y) + \nabla^2 u^tg(y)p = 0, \qquad (3.1)$$

$$v_j^tB_jv_j \leqq 1, \quad j \in K, \qquad (3.2)$$

$$\lambda > 0, \quad \sum_{j=1}^{k} \lambda_j = 1, \quad u \geqq 0, \qquad (3.3)$$

where $y, v_j, p \in R^n, j \in K$, and $u \in R^m$.

Theorem 3.1 (Weak Duality). *Let x and $(y, u, v_1, v_2, \ldots, v_k, \lambda, p)$ be feasible solutions of* (P) *and* (MSD), *respectively. Suppose $(f_j(\cdot) + (\cdot)^t B_j v_j), j \in K$, is bonvex at y, and $g_i(\cdot)$, $i \in M$, is bonvex at y. Then, the following cannot hold:*

$$f_j(x) + (x^t B_j x)^{\frac{1}{2}} \leqq f_j(y) + u^t g(y) + y^t B_j v_j - \frac{1}{2} p^t \nabla^2 \{f_j(y) + u^t g(y)\} p, \quad \text{for all } j \in K, \tag{3.4}$$

and

$$f_i(x) + (x^t B_i x)^{\frac{1}{2}} < f_i(y) + u^t g(y) + y^t B_i v_i - \frac{1}{2} p^t \nabla^2 \{f_i(y) + u^t g(y)\} p, \quad \text{for at least one } i \in K_j. \tag{3.5}$$

Proof. Suppose to the contrary that (3.4) and (3.5) hold, that is,

$$f_j(x) + (x^t B_j x)^{\frac{1}{2}} \leqq f_j(y) + u^t g(y) + y^t B_j v_j - \frac{1}{2} p^t \nabla^2 \{f_j(y) + u^t g(y)\} p, \quad \text{for all } j \in K,$$

and

$$f_i(x) + (x^t B_i x)^{\frac{1}{2}} < f_i(y) + u^t g(y) + y^t B_i v_i - \frac{1}{2} p^t \nabla^2 \{f_i(y) + u^t g(y)\} p, \quad \text{for at least one } i \in K_j.$$

Because $\lambda > 0$ and $\sum_{j=1}^{k} \lambda_j = 1$, the above inequalities yield

$$\sum_{j=1}^{k} \lambda_j \big(f_j(x) + (x^t B_j x)^{\frac{1}{2}}\big) < \sum_{j=1}^{k} \lambda_j \left(f_j(y) + y^t B_j v_j - \frac{1}{2} p^t \nabla^2 f_j(y) p\right) + u^t g(y) - \frac{1}{2} p^t \nabla^2 u^t g(y) p. \tag{3.6}$$

As $(f_j(\cdot) + (\cdot)^t B_j v_j)$, $j \in K$ and $g_i(\cdot)$, $i \in M$, are bonvex at y, we have

$$f_j(x) + x^t B_j v_j - f_j(y) - y^t B_j v_j + \frac{1}{2} p^t \nabla^2 f_j(y) p \geqq \big(\nabla f_j(y) + \nabla^2 f_j(y) p + B_j v_j\big)(x - y), \tag{3.7}$$

and

$$g_i(x) - g_i(y) + \frac{1}{2} p^t \nabla^2 g_i(y) p \geqq \big(\nabla g_i(y) + \nabla^2 g_i(y) p\big)(x - y). \tag{3.8}$$

On multiplying (3.7) by $\lambda_j > 0$, $j \in K$ and (3.8) by $u_i \geqq 0$, $i \in M$, and then summing up to get

$$\sum_{j=1}^{k} \lambda_j \left(f_j(x) + x^t B_j v_j - f_j(y) - y^t B_j v_j + \frac{1}{2} p^t \nabla^2 f_j(y) p \right)$$
$$+ u^t g(x) - u^t g(y) + \frac{1}{2} p^t \nabla^2 u^t g(y) p$$
$$\geqq \left[\sum_{j=1}^{k} \lambda_j \left(\nabla f_j(y) + \nabla^2 f_j(y) p + B_j v_j \right) + \nabla u^t g(y) + \nabla^2 u^t g(y) p \right] (x - y).$$

The above inequality, in view of (3.2), (3.6), the generalized Schwartz inequality, and $u^t g(x) \leqq 0$, gives

$$\left[\sum_{j=1}^{k} \lambda_j \left(\nabla f_j(y) + \nabla^2 f_j(y) p + B_j v_j \right) + \nabla u^t g(y) + \nabla^2 u^t g(y) p \right] (x - y) < 0,$$

which is a contradiction to (3.1). □

Theorem 3.2 (Strong Duality). *Let x° be a properly efficient solution of* (P) *at which a constraint qualification* [10, 15] *is satisfied. Then there exist $\lambda^\circ \in R^k$, $u^\circ \in R^m$, $v_j^\circ \in R^n$, $j \in K$, and $p^\circ \in R^n$ such that $(x^\circ, u^\circ, v_1^\circ, v_2^\circ, \ldots, v_k^\circ, \lambda^\circ, p^\circ = 0)$ is feasible for* (MSD) *and the corresponding objective values of* (P) *and* (MSD) *are equal. If, in addition, the assumptions of weak duality (Theorem 3.1) are satisfied, then $(x^\circ, u^\circ, v_1^\circ, v_2^\circ, \ldots, v_k^\circ, \lambda^\circ, p^\circ = 0)$ is a properly efficient solution of (*MSD*).*

Proof. Because x° is a properly efficient solution of (P) at which a constraint qualification [10, 15] is satisfied, by Theorem 2.8, there exist $\lambda^\circ \in R^k$, $u^\circ \in R^m$ and $v_j^\circ \in R^n$, $j \in K$ such that

$$\sum_{j=1}^{k} \lambda_j^\circ \left(\nabla f_j(x^\circ) + B_j v_j^\circ \right) + \nabla u^{\circ t} g(x^\circ) = 0,$$
$$u^{\circ t} g(x^\circ) = 0,$$
$$\left(x^{\circ t} B_j x^\circ \right)^{\frac{1}{2}} = x^{\circ t} B_j v_j^\circ, \quad j \in K,$$
$$v_j^{\circ t} B_j v_j^\circ \leqq 1, \quad j \in K,$$
$$\lambda^\circ > 0, \quad \sum_{j=1}^{k} \lambda_j^\circ = 1, \quad u^\circ \geqq 0.$$

Thus, $(x^\circ, u^\circ, v_1^\circ, v_2^\circ, \ldots, v_k^\circ, \lambda^\circ, p^\circ = 0)$ is feasible for (MSD) and the corresponding objective values of (P) and (MSD) are equal. Now, we show that $(x^\circ, u^\circ, v_1^\circ, v_2^\circ, \ldots, v_k^\circ, \lambda^\circ, p^\circ = 0)$ is an efficient solution of (MSD). Suppose that it is not efficient, then there exists a feasible solution $(y^*, u^*, v_1^*, v_2^*, \ldots, v_k^*, \lambda^*, p^*)$ such that

$$f_j(y^*) + u^{*t}g(y^*) + y^{*t}B_jv_j^* - \frac{1}{2}p^{*t}\nabla^2[f_j(y^*) + u^{*t}g(y^*)]p^*$$
$$\geqq f_j(x^\circ) + u^{\circ t}g(x^\circ) + x^{\circ t}B_jv_j^\circ, \quad \text{for all } j \in K,$$

and

$$f_i(y^*) + u^{*t}g(y^*) + y^{*t}B_iv_i^* - \frac{1}{2}p^{*t}\nabla^2[f_i(y^*) + u^{*t}g(y^*)]p^*$$
$$> f_i(x^\circ) + u^{\circ t}g(x^\circ) + x^{\circ t}B_iv_i^\circ, \quad \text{for at least one } i \in K_j.$$

Using $(x^{\circ t}B_jx^\circ)^{\frac{1}{2}} = x^{\circ t}B_jv_j^\circ$, $j \in K$, and $u^{\circ t}g(x^\circ) = 0$, the above inequalities give

$$f_j(x^\circ) + \left(x^{\circ t}B_jx^\circ\right)^{\frac{1}{2}} \leqq f_j(y^*) + u^{*t}g(y^*) + y^{*t}B_jv_j^*$$
$$- \frac{1}{2}p^{*t}\nabla^2[f_j(y^*) + u^{*t}g(y^*)]p^*, \quad \text{for all } j \in K,$$

and

$$f_i(x^\circ) + \left(x^{\circ t}B_ix^\circ\right)^{\frac{1}{2}} < f_i(y^*) + u^{*t}g(y^*) + y^{*t}B_iv_i^*$$
$$- \frac{1}{2}p^{*t}\nabla^2[f_i(y^*) + u^{*t}g(y^*)]p^*, \quad \text{for at least one } i \in K_j,$$

which is a contradiction to weak duality (Theorem 3.1). Hence, $(x^\circ, u^\circ, v_1^\circ, v_2^\circ, \ldots, v_k^\circ, \lambda^\circ, p^\circ = 0)$ is an efficient solution of (MSD). Now assume that it is not properly efficient, then for each scalar $M > 0$, there exists a feasible solution $(y^*, u^*, v_1^*, v_2^*, \ldots, v_k^*, \lambda^*, p^*)$ and some $j \in K$, such that

$$f_j(y^*) + u^{*t}g(y^*) + y^{*t}B_jv_j^* - \frac{1}{2}p^{*t}\nabla^2[f_j(y^*) + u^{*t}g(y^*)]p^*$$
$$> f_j(x^\circ) + u^{\circ t}g(x^\circ) + x^{\circ t}B_jv_j^\circ,$$

implies

$$\left[f_j(y^*) + u^{*t}g(y^*) + y^{*t}B_jv_j^* - \frac{1}{2}p^{*t}\nabla^2[f_j(y^*) + u^{*t}g(y^*)]p^*\right]$$
$$- [f_j(x^\circ) + u^{\circ t}g(x^\circ) + x^{\circ t}B_jv_j^\circ]$$

$$> M\bigg[[f_i(x^\circ) + u^{\circ t}g(x^\circ) + x^{\circ t}B_i v_i^\circ] - \bigg[f_i(y^*) + u^{*t}g(y^*) + y^{*t}B_i v_i^* - \frac{1}{2}p^{*t}\nabla^2[f_i(y^*) + u^{*t}g(y^*)]p^*\bigg]\bigg],$$

for all $i \in K_j$ satisfying

$$[f_i(x^\circ) + u^{\circ t}g(x^\circ) + x^{\circ t}B_i v_i^\circ] > \bigg[f_i(y^*) + u^{*t}g(y^*) + y^{*t}B_i v_i^* - \frac{1}{2}p^{*t}\nabla^2[f_i(y^*) + u^{*t}g(y^*)]p^*\bigg].$$

Again, by $(x^{\circ t}B_j x^\circ)^{\frac{1}{2}} = x^{\circ t}B_j v_j^\circ$, $j \in K$, and $u^{\circ t}g(x^\circ) = 0$, we get

$$\bigg[f_j(y^*) + u^{*t}g(y^*) + y^{*t}B_j v_j^* - \frac{1}{2}p^{*t}\nabla^2[f_j(y^*) + u^{*t}g(y^*)]p^*\bigg] - \Big[f_j(x^\circ) + \big(x^{\circ t}B_j x^\circ\big)^{\frac{1}{2}}\Big]$$
$$> M\bigg[\Big[f_i(x^\circ) + \big(x^{\circ t}B_i x^\circ\big)^{\frac{1}{2}}\Big] - \bigg[f_i(y^*) + u^{*t}g(y^*) + y^{*t}B_i v_i^* - \frac{1}{2}p^{*t}\nabla^2[f_i(y^*) + u^{*t}g(y^*)]p^*\bigg]\bigg].$$

This means that $[f_j(y^*) + u^{*t}g(y^*) + y^{*t}B_j v_j^* - \frac{1}{2}p^{*t}\nabla^2[f_j(y^*) + u^{*t}g(y^*)]p^*]$ is infinitely better than $[f_j(x^\circ) + (x^{\circ t}B_j x^\circ)^{\frac{1}{2}}]$ for some $j \in K$, whereas $[f_j(x^\circ) + (x^{\circ t}B_i x^\circ)^{\frac{1}{2}}]$ is at most finitely better than $[f_i(y^*) + u^{*t}g(y^*) + y^{*t}B_i v_i^* - \frac{1}{2}p^{*t}\nabla^2[f_i(y^*) + u^{*t}g(y^*)]p^*]$ for all $i \in K_j$. Therefore,

$$f_j(x^\circ) + (x^{\circ t}B_j x^\circ)^{\frac{1}{2}} \leqq f_j(y^*) + u^{*t}g(y^*) + y^{*t}B_j v_j^* - \frac{1}{2}p^{*t}\nabla^2[f_j(y^*) + u^{*t}g(y^*)]p^*, \quad \text{for all } j \in K,$$

and

$$f_i(x^\circ) + (x^{\circ t}B_i x^\circ)^{\frac{1}{2}} < f_i(y^*) + u^{*t}g(y^*) + y^{*t}B_i v_i^* - \frac{1}{2}p^{*t}\nabla^2[f_i(y^*) + u^{*t}g(y^*)]p^*, \quad \text{for at least one } i \in K_j,$$

which again contradicts weak duality (Theorem 3.1). Hence, $(x^\circ, u^\circ, v_1^\circ, v_2^\circ, \ldots, v_k^\circ, \lambda^\circ, p^\circ = 0)$ is a properly efficient solution of (MSD). □

Theorem 3.3 (Strict Converse Duality). *Let* x° *and* $(y^\circ, u^\circ, v_1^\circ, v_2^\circ, \ldots, v_k^\circ, \lambda^\circ, p^\circ)$ *be feasible solutions of* (P) *and* (MSD), *respectively, such that*

$$\sum_{j=1}^{k} \lambda_j^\circ [f_j(x^\circ) + x^{\circ t} B_j v_j^\circ] \leqq \sum_{j=1}^{k} \lambda_j^\circ \left[f_j(y^\circ) + y^{\circ t} B_j v_j^\circ - \frac{1}{2} p^{\circ t} \nabla^2 f_j(y^\circ) p^\circ \right] + u^{\circ t} g(y^\circ) - \frac{1}{2} p^{\circ t} \nabla^2 u^{\circ t} f(y^\circ) p^\circ. \quad (3.9)$$

Suppose $(f_j(\cdot) + (\cdot)^t B_j v_j^\circ)$, $j \in K$ *is strictly bonvex at* y°, *and* $g_i(\cdot)$, $i \in M$ *is bonvex at* y°, *then* $x^\circ = y^\circ$.

Proof. We assume that $x^\circ \neq y^\circ$ and exhibit a contradiction. The strict bonvexity of $(f_j(\cdot) + (\cdot)^t B_j v_j^\circ)$, $j \in K$ at y° and bonvexity of $g_i(\cdot)$, $i \in M$ at y°, imply

$$\left(f_j(x^\circ) + x^{\circ t} B_j v_j^\circ - f_j(y^\circ) - y^{\circ t} B_j v_j^\circ + \frac{1}{2} p^{\circ t} \nabla^2 f_j(y^\circ) p^\circ \right) > \left(\nabla f_j(y^\circ) + \nabla^2 f_j(y^\circ) p^\circ + B_j v_j^\circ \right)(x^\circ - y^\circ), \quad (3.10)$$

and

$$g_i(x^\circ) - g_i(y^\circ) + \frac{1}{2} p^{\circ t} \nabla^2 g_i(y^\circ) p^\circ \geqq (\nabla g_i(y^\circ) + \nabla^2 g_i(y^\circ) p^\circ)(x^\circ - y^\circ). \quad (3.11)$$

On multiplying (3.10) by $\lambda_j^\circ > 0$, $j \in K$ and (3.11) by $u_i^\circ \geqq 0$, $i \in M$, and then summing up to get

$$\sum_{j=1}^{k} \lambda_j^\circ (f_j(x^\circ) + x^{\circ t} B_j v_j^\circ) + u^{\circ t} g(x^\circ) - \sum_{j=1}^{k} \lambda_j^\circ (f_j(y^\circ) + y^{\circ t} B_j v_j^\circ - \frac{1}{2} p^{\circ t} \nabla^2 f_j(y^\circ) p^\circ) - u^{\circ t} g(y^\circ) + \frac{1}{2} p^{\circ t} \nabla^2 u^{\circ t} g(y^\circ) p^\circ > \left[\sum_{j=1}^{k} \lambda_j^\circ \left(\nabla f_j(y^\circ) + \nabla^2 f_j(y^\circ) p^\circ + B_j v_j^\circ \right) + \nabla u^{\circ t} g(y^\circ) + \nabla^2 u^{\circ t} g(y^\circ) p^\circ \right] (x^\circ - y^\circ).$$

The above inequality on using (3.1) and $u^{\circ t} g(x^\circ) \leqq 0$, gives

$$\sum_{j=1}^{k} \lambda_j^\circ [f_j(x^\circ) + x^{\circ t} B_j v_j^\circ] > \sum_{j=1}^{k} \lambda_j^\circ \left[f_j(y^\circ) + y^{\circ t} B_j v_j^\circ - \frac{1}{2} p^{\circ t} \nabla^2 f_j(y^\circ) p^\circ \right] + u^{\circ t} g(y^\circ) - \frac{1}{2} p^{\circ t} \nabla^2 u^{\circ t} g(y^\circ) p^\circ,$$

which is a contradiction to (3.9). Hence, $x^\circ = y^\circ$. □

4. GENERAL MOND–WEIR TYPE DUALITY

For (P), we now formulate the following general Mond–Weir type second-order dual:

$$\text{Maximize}\left(f_1(y) + y^t B_1 v_1 + \sum_{i\in I_\circ} u_i g_i(y) - \frac{1}{2}p^t\nabla^2\left[f_1(y) + \sum_{i\in I_\circ} u_i g_i(y)\right]p, \ldots, \right.$$
$$\left. f_k(y) + y^t B_k v_k + \sum_{i\in I_\circ} u_i g_i(y) - \frac{1}{2}p^t\nabla^2\left[f_k(y) + \sum_{i\in I_\circ} u_i g_i(y)\right]p\right) \tag{GMD}$$

subject to

$$\sum_{j=1}^{k} \lambda_j(\nabla f_j(y) + \nabla^2 f_j(y)p + B_j v_j) + \nabla u^t g(y) + \nabla^2 u^t g(y)p = 0, \tag{4.1}$$

$$\sum_{i\in I_\alpha} u_i g_i(y) - \frac{1}{2}p^t\nabla^2 \sum_{i\in I_\alpha} u_i g_i(y)p \geqq 0, \quad \alpha = 1, 2, \ldots, r, \tag{4.2}$$

$$v_j{}^t B_j v_j \leqq 1, \quad j \in K, \tag{4.3}$$

$$\lambda > 0, \quad \sum_{j=1}^{k} \lambda_j = 1, \quad u \geqq 0, \tag{4.4}$$

where $I_\alpha \subseteq M$, $\alpha = 0, 1, 2, \ldots, r$, with $\bigcup_{\alpha=0}^{r} I_\alpha = M$ and $I_\alpha \cap I_\beta = \emptyset$, if $\alpha \neq \beta$.

Remark 4.1. If $I_\alpha = \emptyset$, $\alpha = 1, 2, \ldots, r$, and $I_\circ = M$, then (GMD) reduces to (MSD).

Theorem 4.2 (Weak Duality). *Let x and $(y, u, v_1, v_2, \ldots, v_k, \lambda, p)$ be feasible solutions of* (P) *and* (GMD)*, respectively. Suppose $[\sum_{j=1}^{k} \lambda_j(f_j(\cdot) + (\cdot)^t B_j v_j) + \sum_{i\in I_\circ} u_i g_i(\cdot)]$ is pseudobonvex at y, and $\sum_{i\in I_\alpha} u_i g_i(\cdot)$, $\alpha = 1, 2, \ldots, r$, is quasibonvex at y. Then, the following cannot hold:*

$$f_j(x) + (x^t B_j x)^{\frac{1}{2}} \leqq f_j(y) + y^t B_j v_j + \sum_{i\in I_\circ} u_i g_i(y)$$
$$- \frac{1}{2}p^t\nabla^2\left[f_j(y) + \sum_{i\in I_\circ} u_i g_i(y)\right]p, \quad \textit{for all } j \in K, \tag{4.5}$$

and

$$f_l(x) + (x^t B_l x)^{\frac{1}{2}} < f_l(y) + y^t B_l v_l + \sum_{i\in I_\circ} u_i g_i(y) - \frac{1}{2} p^t \nabla^2 \left[f_l(y) + \sum_{i\in I_\circ} u_i g_i(y) \right] p,$$

$$\textit{for at least one } l \in K_j. \tag{4.6}$$

Proof. Because x and $(y, u, v_1, v_2, \ldots, v_k, \lambda, p)$ are feasible solutions of (P) and (GMD), respectively, then

$$\sum_{i\in I_\alpha} u_i g_i(x) \leqq 0 \leqq \sum_{i\in I_\alpha} u_i g_i(y) - \frac{1}{2} p^t \nabla^2 \sum_{i\in I_\alpha} u_i g_i(y) p, \quad \alpha = 1, 2, \ldots, r,$$

which by the quasibonvexity of $\sum_{i\in I_\alpha} u_i g_i(\cdot)$, $\alpha = 1, 2, \ldots, r$, at y yields

$$\left(\nabla \sum_{i\in I_\alpha} u_i g_i(y) + \nabla^2 \sum_{i\in I_\alpha} u_i g_i(y) p \right)(x - y) \leqq 0, \quad \alpha = 1, 2, \ldots, r. \tag{4.7}$$

Now, suppose to the contrary that (4.5) and (4.6) hold, that is,

$$\begin{aligned} f_j(x) + (x^t B_j x)^{\frac{1}{2}} \leqq{}& f_j(y) + y^t B_j v_j + \sum_{i\in I_\circ} u_i g_i(y) \\ &- \frac{1}{2} p^t \nabla^2 \left[f_j(y) + \sum_{i\in I_\circ} u_i g_i(y) \right] p, \quad \text{for all } j \in K, \end{aligned}$$

and

$$\begin{aligned} f_l(x) + (x^t B_l x)^{\frac{1}{2}} <{}& f_l(y) + y^t B_l v_l + \sum_{i\in I_\circ} u_i g_i(y) \\ &- \frac{1}{2} p^t \nabla^2 \left[f_l(y) + \sum_{i\in I_\circ} u_i g_i(y) \right] p, \quad \text{for at least one } l \in K_j. \end{aligned}$$

Because $\lambda > 0$ and $\sum_{j=1}^k \lambda_j = 1$, we get

$$\begin{aligned} \sum_{j=1}^k \lambda_j (f_j(x) + (x^t B_j x)^{\frac{1}{2}}) <{}& \sum_{j=1}^k \lambda_j \left(f_j(y) + y^t B_j v_j - \frac{1}{2} p^t \nabla^2 f_j(y) p \right) \\ &+ \sum_{i\in I_\circ} u_i g_i(y) - \frac{1}{2} p^t \nabla^2 \sum_{i\in I_\circ} u_i g_i(y) p, \end{aligned}$$

which on using the generalized Schwartz inequality along with (4.3) and the feasibility of (P), implies

$$\sum_{j=1}^{k}\lambda_j\big(f_j(x)+(x^tB_jv_j)\big)+\sum_{i\in I_\circ}u_ig_i(x)$$
$$<\sum_{j=1}^{k}\lambda_j\left(f_j(y)+y^tB_jv_j-\frac{1}{2}p^t\nabla^2 f_j(y)p\right)+\sum_{i\in I_\circ}u_ig_i(y)-\frac{1}{2}p^t\nabla^2\sum_{i\in I_\circ}u_ig_i(y)p.$$

The pseudobonvexity of $[\sum_{j=1}^{k}\lambda_j(f_j(\cdot)+(\cdot)^tB_jv_j)+\sum_{i\in I_\circ}u_ig_i(\cdot)]$ at y, implies

$$\left[\sum_{j=1}^{k}\lambda_j(\nabla f_j(y)+\nabla^2 f_j(y)p+B_jv_j)+\nabla\sum_{i\in I_\circ}u_ig_i(y)+\nabla^2\sum_{i\in I_\circ}u_ig_i(y)p\right](x-y)<0. \tag{4.8}$$

On combining (4.7) and (4.8), we obtain

$$\left[\sum_{j=1}^{k}\lambda_j(\nabla f_j(y)+\nabla^2 f_j(y)p+B_jv_j)+\nabla u^tg(y)+\nabla^2u^tg(y)p\right](x-y)<0,$$

which is a contradiction to (4.1). □

Theorem 4.3 (Strong Duality). *Let x° be a properly efficient solution of* (P) *at which a constraint qualification* [10, 15] *is satisfied. Then there exist $\lambda^\circ\in R^k$, $u^\circ\in R^m$, $v_j^\circ\in R^n$, $j\in K$ and $p^\circ\in R^n$ such that $(x^\circ,u^\circ,v_1^\circ,v_2^\circ,\dots,v_k^\circ,\lambda^\circ,p^\circ=0)$ is feasible for* (GMD) *and the corresponding objective values of* (P) *and* (GMD) *are equal. If, in addition, the assumptions of weak duality* (*Theorem* 4.2) *are satisfied, then $(x^\circ,u^\circ,v_1^\circ,v_2^\circ,\dots,v_k^\circ,\lambda^\circ,p^\circ=0)$ is a properly efficient solution of* (GMD).

Proof. The proof follows on the similar lines of Theorem 3.2.

Theorem 4.4 (Strict Converse Duality). *Let x° and $(y^\circ,u^\circ,v_1^\circ,v_2^\circ,\dots,v_k^\circ,\lambda^\circ,p^\circ)$ be the feasible solutions of* (P) *and* (GMD), *respectively, such that*

$$\sum_{j=1}^{k}\lambda_j^\circ[f_j(x^\circ)+x^{\circ t}B_jv_j^\circ]\leqq\sum_{j=1}^{k}\lambda_j^\circ\left[f_j(y^\circ)+y^{\circ t}B_jv_j^\circ-\frac{1}{2}p^{\circ t}\nabla^2 f_j(y^\circ)p^\circ\right]$$
$$+\sum_{i\in I_\circ}u_i^\circ g_i(y^\circ)-\frac{1}{2}p^{\circ t}\nabla^2\sum_{i\in I_\circ}u_i^\circ g_i(y^\circ)p^\circ. \tag{4.9}$$

If $[\sum_{j=1}^{k}\lambda_j^\circ(f_j(\cdot)+(\cdot)^tB_jv_j^\circ)+\sum_{i\in I_\circ}u_i^\circ g_i(\cdot)]$ is strictly pseudobonvex at y°, and $\sum_{i\in I_\alpha}u_i^\circ g_i(\cdot),\alpha=1,2,\dots,r$, is quasibonvex at y°, then $x^\circ=y^\circ$.

Proof. We assume that $x^\circ \neq y^\circ$ and exhibit a contradiction. Because x° and $(y^\circ, u^\circ, v_1^\circ, v_2^\circ, \ldots, v_k^\circ, \lambda^\circ, p^\circ)$ are feasible solutions of (P) and (GMD), respectively, we have

$$\sum_{i\in I_\alpha} u_i^\circ g_i(x^\circ) \leqq 0 \leqq \sum_{i\in I_\alpha} u_i^\circ g_i(y^\circ) - \frac{1}{2} p^{\circ t} \nabla^2 \sum_{i\in I_\alpha} u_i^\circ g_i(y^\circ) p^\circ, \quad \alpha = 1, 2, \ldots, r.$$

By the quasibonvexity of $\sum_{i\in I_\alpha} u_i^\circ g_i(\cdot), \alpha = 1, 2, \ldots, r$ at y°, it follows that

$$\left(\nabla \sum_{i\in I_\alpha} u_i^\circ g_i(y^\circ) + \nabla^2 \sum_{i\in I_\alpha} u_i^\circ g_i(y^\circ) p^\circ\right)(x^\circ - y^\circ) \leqq 0.$$

The above inequality together with (4.1) gives

$$\left[\sum_{j=1}^{k} \lambda_j^\circ (\nabla f_j(y^\circ) + \nabla^2 f_j(y^\circ) p^\circ + B_j v_j^\circ) + \nabla \sum_{i\in I_\circ} u_i^\circ g_i(y^\circ) + \nabla^2 \sum_{i\in I_\circ} u_i^\circ g_i(y^\circ) p^\circ\right](x^\circ - y^\circ) \geqq 0,$$

which on using the strict pseudobonvexity of $[\sum_{j=1}^{k} \lambda_j^\circ (f_j(\cdot) + (\cdot)^t B_j v_j^\circ) + \sum_{i\in I_\circ} u_i^\circ g_i(\cdot)]$ at y°, yields

$$\sum_{j=1}^{k} \lambda_j^\circ [f_j(x^\circ) + x^{\circ t} B_j v_j^\circ] + \sum_{i\in I_\circ} u_i^\circ g_i(x^\circ) > \sum_{j=1}^{k} \lambda_j^\circ \left[f_j(y^\circ) + y^{\circ t} B_j v_j^\circ - \frac{1}{2} p^{\circ t} \nabla^2 f_j(y^\circ) p^\circ\right] + \sum_{i\in I_\circ} u_i^\circ g_i(y^\circ) - \frac{1}{2} p^{\circ t} \nabla^2 \sum_{i\in I_\circ} u_i^\circ g_i(y^\circ) p^\circ,$$

which by $\sum_{i\in I_\circ} u_i^\circ g_i(x^\circ) \leqq 0$ implies

$$\sum_{j=1}^{k} \lambda_j^\circ [f_j(x^\circ) + x^{\circ t} B_j v_j^\circ] > \sum_{j=1}^{k} \lambda_j^\circ \left[f_j(y^\circ) + y^{\circ t} B_j v_j^\circ - \frac{1}{2} p^{\circ t} \nabla^2 f_j(y^\circ) p^\circ\right] + \sum_{i\in I_\circ} u_i^\circ g_i(y^\circ) - \frac{1}{2} p^{\circ t} \nabla^2 \sum_{i\in I_\circ} u_i^\circ g_i(y^\circ) p^\circ,$$

a contradiction to (4.9). Hence $x^\circ = y^\circ$. □

5. CONCLUSION

In this paper, we have discussed the second-order duality results for nondifferentiable multiobjective Mangasarian type and general Mond–Weir type duals involving bonvexity/generalized bonvexity. The current results can be further extended for the following nondifferentiable multiobjective fractional programming problem:

$$\text{Minimize } \left(\frac{f_1(x) + (x^t B_1 x)^{\frac{1}{2}}}{h_1(x) - (x^t D_1 x)^{\frac{1}{2}}}, \frac{f_2(x) + (x^t B_2 x)^{\frac{1}{2}}}{h_2(x) - (x^t D_2 x)^{\frac{1}{2}}}, \ldots, \frac{f_k(x) + (x^t B_k x)^{\frac{1}{2}}}{h_k(x) - (x^t D_k x)^{\frac{1}{2}}} \right)$$

$$\text{subject to } x \in S = \{x \in X : g(x) \leqq 0\}, \qquad (\overline{P})$$

where $f_j : X \to R$, $h_j : X \to R$, $j = 1, 2, \ldots, k$, $g : X \to R^m$; B_j and D_j, $j = 1, 2, \ldots, k$ are $n \times n$ positive semidefinite symmetric matrices.

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