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Sufficiency and duality for multiobjective variational control problems with generalized $(F, \alpha, \rho, \theta)$-V-convexity

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ABSTRACT

Zalmai introduced generalized $(F, \alpha, \rho, \theta)$-V-convex functions, a new class of functions that unifies several concepts of generalized convex functions. In this article, the generalized $(F, \alpha, \rho, \theta)$-V-convex functions are extended to variational control problem. By utilizing the new concepts, we obtain sufficient optimality conditions and prove Wolfe type and Mond–Weir type duality results for multiobjective variational control programming problem.

1. Introduction

During the last two decades, multiobjective control problems have been considered in flight control design, in the control of space structures, in industrial process control and other diverse fields. Mond and Hanson [1] first obtained duality results for control problems under convexity, which were further extended by Mond and Smart [2] to invex functions. Bhatia and Kumar [3] derived duality theorems for multiobjective control problems under generalized ρ-invexity assumptions. Xiuhong [4] proved duality relations through a parametric approach to relate properly efficient solutions of multiobjective control problems under invexity assumptions. The objective and constraint functions in both articles were different. In [5], Gulati et al. established optimality conditions and duality results for multiobjective control problems involving generalized convexity. Nahak and Nanda [6] discussed the efficiency and duality for multiobjective variational control problems with (F, ρ)-convexity. Recently, Nahak and Nanda [7] obtained sufficient optimality criteria and duality results for multiobjective variational control problems under V-invexity assumptions.

Motivated by various concepts of generalized convexity, Liang et al. [8,9] introduced a unified formulation of generalized convexity, which was called (F, α, ρ, d)-convexity and obtained some corresponding optimality conditions and duality results for the single objective fractional and multiobjective fractional problems. In [10], Ahmad and Husain established sufficient optimality conditions and usual duality results for a nondifferentiable minimax fractional programming with (F, α, ρ, d)-convexity. Chinchuluun et al. [11] generalized the results of Liang et al. [9] for nondifferentiable multiobjective fractional problems. Recently, Gulati et al. [12] and Yuan et al. [13] presented unified formulation of generalized convex functions called (F, α, ρ, d)-V-type I functions and (C, α, ρ, d) type I functions, respectively, and discussed sufficient optimality conditions and duality results for multiobjective programming problems. In [14], Yuan et al. investigated the properties of (h, ϕ)-generalized gradient, and relations between (h, ϕ)-generalized monotocity. They also considered monotocity of some averages based on the algebraic operations under the ϕ-convexity assumption.

In this article, by taking the motivation from Liang et al. [9] and Zalmai [15], we extend the class of generalized $(F, \alpha, \rho, \theta)$-V-convex functions for multiobjective variational control problem and obtain sufficient optimality conditions

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under these assumptions. Moreover, weak and strong duality theorems are proved for Wolfe and Mond–Weir type vector duals. This work extends the earlier work of Nahak and Nanda [6,7].

2. Notations and preliminaries

Let $I = [a, b]$ be a real interval, $f_i : I \times R^n \times R^n \times R^m \times R^m \mapsto R$, $i \in P = \{1, 2, \ldots, p\}$, $g_j : I \times R^n \times R^n \times R^m \times R^m \mapsto R$, $j \in M = \{1, 2, \ldots, m\}$ and $h_k : I \times R^n \times R^n \times R^m \times R^m \mapsto R$, $k \in N = \{1, 2, \ldots, n\}$ be continuously differentiable functions. For $i \in P$, the set $P_i = P - \{i\}$. Consider the function $f(t, x(t), \dot{x}(t), u(t), \dot{u}(t))$, where t is the independent variable, $x : I \mapsto R^n$ is the state variable and $u : I \mapsto R^m$ is the control variable. $u(t)$ is related to $x(t)$ through the state equation $h(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) = 0$, where the dot denotes the derivative with respect to t. f_{it}, f_{ix}, $f_{i\dot{x}}$, f_{iu} and $f_{i\dot{u}}$ denote the partial derivative of f_i with respect to t, x, $\dot{x}$, u and $\dot{u}$, respectively. For instance,

$$f_{ix} = \left(\frac{\partial f_i}{\partial x_1}, \frac{\partial f_i}{\partial x_2}, \ldots, \frac{\partial f_i}{\partial x_n}\right), \qquad f_{i\dot{x}} = \left(\frac{\partial f_i}{\partial \dot{x}_1}, \frac{\partial f_i}{\partial \dot{x}_2}, \ldots, \frac{\partial f_i}{\partial \dot{x}_n}\right)$$
$$f_{iu} = \left(\frac{\partial f_i}{\partial u_1}, \frac{\partial f_i}{\partial u_2}, \ldots, \frac{\partial f_i}{\partial u_n}\right), \qquad f_{i\dot{u}} = \left(\frac{\partial f_i}{\partial \dot{u}_1}, \frac{\partial f_i}{\partial \dot{u}_2}, \ldots, \frac{\partial f_i}{\partial \dot{u}_n}\right).$$

Similarly, g_{jt}, g_{jx}, $g_{j\dot{x}}$, g_{ju}, $g_{j\dot{u}}$ and h_{kt}, h_{kx}, $h_{k\dot{x}}$, h_{ku}, $h_{k\dot{u}}$ can be defined.

Let $C(I, R^n)$ denote the space of piecewise smooth functions x with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by

$$u = Dx \Leftrightarrow x(t) = \alpha + \int_a^t u(s)\mathrm{d}s,$$

where α is a given boundary value. Therefore, $D = \frac{\mathrm{d}}{\mathrm{d}t}$ except at discontinuities. Let X denotes the space of all piecewise smooth functions $x : I \mapsto R^n$ and Y denotes the space of all piecewise smooth functions $u : I \mapsto R^m$. For notational convenience, we use $\psi(t, x, \dot{x}, u, \dot{u})$ for $\psi(t, x(t), \dot{x}(t), u(t), \dot{u}(t))$ and $\boldsymbol{\alpha}$ for $\alpha(x(t), \dot{x}(t), u(t), \dot{u}(t), \bar{x}(t), \dot{\bar{x}}(t), \bar{u}(t), \dot{\bar{u}}(t))$.

We consider the following multiobjective variational control problem:

$$\text{(VP)}\ \text{Minimize} \int_a^b f(t, x, \dot{x}, u, \dot{u})\mathrm{d}t = \left(\int_a^b f_1(t, x, \dot{x}, u, \dot{u})\mathrm{d}t, \int_a^b f_2(t, x, \dot{x}, u, \dot{u})\mathrm{d}t, \ldots, \int_a^b f_p(t, x, \dot{x}, u, \dot{u})\mathrm{d}t\right)$$

subject to

$$x(a) = \alpha, \quad x(b) = \beta,$$
$$g(t, x, \dot{x}, u, \dot{u}) \leq 0, \quad t \in I, \tag{1}$$
$$h(t, x, \dot{x}, u, \dot{u}) = 0, \quad t \in I. \tag{2}$$

Let S denotes the set of all feasible solutions of (VP),i.e.,

$$S = \{(x, u) \in (X, Y) \mid x(a) = \alpha,\ x(b) = \beta,\ g(t, x, \dot{x}, u, \dot{u}) \leq 0,\ h(t, x, \dot{x}, u, \dot{u}) = 0, t \in I\}.$$

Definition 1. A point $(\bar{x}, \bar{u}) \in S$ is said to be an efficient solution of (VP), if there exists no other point $(x, u) \in S$ such that

$$\int_a^b f_i(t, x, \dot{x}, u, \dot{u})\mathrm{d}t \leq \int_a^b f_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t, \quad \forall\, i \in P$$

and

$$\int_a^b f_l(t, x, \dot{x}, u, \dot{u})\mathrm{d}t < \int_a^b f_l(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t, \quad \forall\, l \in P_i.$$

Definition 2 ([16]). A point $(\bar{x}, \bar{u}) \in S$ is said to be a weakly efficient solution of (VP), if there exists no other point $(x, u) \in S$ such that

$$\int_a^b f_i(t, x, \dot{x}, u, \dot{u})\mathrm{d}t < \int_a^b f_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t, \quad \forall\, i \in P.$$

Definition 3. A function $F : I \times R^n \times R^n \times R^m \times R^m \mapsto R$ is said to be sublinear in its third argument, if for any $x, \dot{x} \in R^n$, $u, \dot{u} \in R^m$,

$$F(t, x, \dot{x}, u, \dot{u}; a_1 + a_2) \leq F(t, x, \dot{x}, u, \dot{u}; a_1) + F(t, x, \dot{x}, u, \dot{u}; a_2),$$
$$F(t, x, \dot{x}, u, \dot{u}; \alpha a) = \alpha F(t, x, \dot{x}, u, \dot{u}; a),$$

for any $a_1, a_2 \in R^n$, $\alpha \in R$, $\alpha \geq 0$ and $a \in R^n$.

Let F be a sublinear functional, and let $\theta : I \times X \times X \times Y \times Y \mapsto R$.

Definition 4. A vector function $\psi(t, x, \dot{x}, u, \dot{u})$ is said to be $(F, \alpha, \rho, \theta)$-V-convex at $(\bar{x}, \bar{u}) \in (X, Y)$, if there exist functions $\alpha_i : X \times X \times Y \times Y \mapsto R_+ \setminus \{0\}$, and $\rho_i \in R$, $i \in P$ such that for all $(x, u) \in (X, Y)$ and $i \in P$,

$$\int_a^b \psi_i(t, x, \dot{x}, u, \dot{u})\mathrm{d}t - \int_a^b \psi_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t \geq \int_a^b F(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}; \boldsymbol{\alpha}_{\mathbf{i}}(\psi_{ix}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \psi_{iu}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})$$
$$- D(\psi_{i\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \psi_{i\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}))))\mathrm{d}t + \int_a^b \rho_i \theta^2(t, x, \bar{x}, u, \bar{u})\mathrm{d}t.$$

Definition 5. A vector function $\psi(t, x, \dot{x}, u, \dot{u})$ is said to be $(F, \tilde{\alpha}, \tilde{\rho}, \theta)$-V- (strictly) pseudoconvex at $(\bar{x}, \bar{u}) \in (X, Y)$, if there exist functions $\tilde{\alpha}_i : X \times X \times Y \times Y \mapsto R_+ \setminus \{0\}$, $i \in P$ and $\tilde{\rho} \in R$ such that for all $(x, u) \in (X, Y)$,

$$\int_a^b F\Bigg(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}; \sum_{i\in P} \psi_{ix}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \psi_{iu}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})$$
$$- D\left(\sum_{i\in P} \psi_{i\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \psi_{i\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\right)\Bigg)\mathrm{d}t + \int_a^b \tilde{\rho}\theta^2(t, x, \bar{x}, u, \bar{u})\mathrm{d}t \geq 0$$
$$\Rightarrow \int_a^b \sum_{i\in P} \tilde{\boldsymbol{\alpha}}_{\mathbf{i}} \psi_i(t, x, \dot{x}, u, \dot{u})\mathrm{d}t (>) \geq \int_a^b \sum_{i\in P} \tilde{\boldsymbol{\alpha}}_{\mathbf{i}} \psi_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t,$$

or equivalently,

$$\int_a^b \sum_{i\in P} \tilde{\boldsymbol{\alpha}}_{\mathbf{i}} \psi_i(t, x, \dot{x}, u, \dot{u})\mathrm{d}t (\leq) < \int_a^b \sum_{i\in P} \tilde{\boldsymbol{\alpha}}_{\mathbf{i}} \psi_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t$$
$$\Rightarrow \int_a^b F\Bigg(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}; \sum_{i\in P} \psi_{ix}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \psi_{iu}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})$$
$$- D\left(\sum_{i\in P} \psi_{i\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \psi_{i\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\right)\Bigg)\mathrm{d}t + \int_a^b \tilde{\rho}\theta^2(t, x, \bar{x}, u, \bar{u})\mathrm{d}t < 0.$$

Definition 6. A vector function $\psi(t, x, \dot{x}, u, \dot{u})$ is said to be $(F, \hat{\alpha}, \hat{\rho}, \theta)$-V- (strictly) quasiconvex at $(\bar{x}, \bar{u}) \in (X, Y)$, if there exist functions $\hat{\alpha}_i : X \times X \times Y \times Y \mapsto R_+ \setminus \{0\}$, $i \in P$ and $\hat{\rho} \in R$ such that for all $(x, u) \in (X, Y)$,

$$\int_a^b \sum_{i\in P} \hat{\boldsymbol{\alpha}}_{\mathbf{i}} \psi_i(t, x, \dot{x}, u, \dot{u})\mathrm{d}t \leq \int_a^b \sum_{i\in P} \hat{\boldsymbol{\alpha}}_{\mathbf{i}} \psi_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t$$
$$\Rightarrow \int_a^b F\Bigg(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}; \sum_{i\in P} \psi_{ix}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \psi_{iu}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})$$
$$- D\left(\sum_{i\in P} \psi_{i\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \psi_{i\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\right)\Bigg)\mathrm{d}t + \int_a^b \hat{\rho}\theta^2(t, x, \bar{x}, u, \bar{u})\mathrm{d}t (<) \leq 0,$$

or equivalently,

$$\int_a^b F\Bigg(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}; \sum_{i\in P} \psi_{ix}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \psi_{iu}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})$$
$$- D\left(\sum_{i\in P} \psi_{i\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \psi_{i\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\right)\Bigg)\mathrm{d}t + \int_a^b \hat{\rho}\theta^2(t, x, \bar{x}, u, \bar{u})\mathrm{d}t (\geq) > 0$$
$$\Rightarrow \int_a^b \sum_{i\in P} \hat{\boldsymbol{\alpha}}_{\mathbf{i}} \psi_i(t, x, \dot{x}, u, \dot{u})\mathrm{d}t > \int_a^b \sum_{i\in P} \hat{\boldsymbol{\alpha}}_{\mathbf{i}} \psi_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t.$$

In order to prove strong duality theorem, we will invoke the following results.

Lemma 1 (*Kuhn–Tucker Necessary Optimality Conditions*). *Let $(\bar{x}, \bar{u})$ solves the following single objective problem*

$$\text{Minimize } \int_a^b f(t, x, \dot{x}, u, \dot{u})\mathrm{d}t$$

subject to

$$x(a) = \alpha, \qquad x(b) = \beta,$$
$$g(t, x, \dot{x}, u, \dot{u}) \leq 0, \quad t \in I,$$
$$h(t, x, \dot{x}, u, \dot{u}) = 0, \quad t \in I.$$

If the Frechet derivative $[D - H_x(\bar{x}, \bar{u})]$ is surjective and the optimal solution $(\bar{x}, \bar{u})$ is normal, then there exist piecewise smooth functions $\bar{\mu}^* : I \mapsto R^m$ and $\bar{\nu}^* : I \mapsto R^n$ satisfying the following for all $t \in I$,

$$f_x(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j\in M} \bar{\mu}_j^* g_{jx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k^* h_{kx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})$$
$$= D\left[f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j\in M} \bar{\mu}_j^* g_{j\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k^* h_{k\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\right],$$
$$f_u(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j\in M} \bar{\mu}_j^* g_{ju}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k^* h_{ku}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})$$
$$= D\left[f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j\in M} \bar{\mu}_j^* g_{j\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k^* h_{k\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\right],$$
$$\bar{\mu}^* g(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t = 0,$$
$$\bar{\mu}^* \geq 0.$$

Lemma 2 ([17]). *$(\bar{x}, \bar{u})$ is an efficient solution for (VP) if and only if $(\bar{x}, \bar{u})$ solves*

$$(\mathbf{VP})_s \text{ Minimize } \int_a^b f_k(t, x, \dot{x}, u, \dot{u})\mathrm{d}t$$
$$\text{subject to } x(a) = \alpha, \qquad x(b) = \beta,$$
$$\int_a^b f_i(t, x, \dot{x}, u, \dot{u})\mathrm{d}t \leq \int_a^b f_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t, \quad i \neq k,$$
$$g(t, x, \dot{x}, u, \dot{u}) \leq 0, \quad t \in I,$$
$$h(t, x, \dot{x}, u, \dot{u}) = 0, \quad t \in I,$$

for all $i \in P$.

3. Sufficiency

We now present the sufficient optimality conditions for (VP) under $(F, \alpha, \rho, \theta)$-V-convexity/generalized convexity assumptions. In this section and in Section 5, $\int_a^b f^{\bar{\lambda}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t$ denotes the vector $(\int_a^b \bar{\lambda}_1 f_1(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t, \int_a^b \bar{\lambda}_2 f_2(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t, \dots, \int_a^b \bar{\lambda}_p f_p(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t)$ and $\int_a^b g^{\bar{\mu}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t$ denotes the vector $(\int_a^b \bar{\mu}_1 g_1(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t, \int_a^b \bar{\mu}_2 g_2(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t, \dots, \int_a^b \bar{\mu}_m g_m(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t)$. Similarly, $\int_a^b h^{\bar{\nu}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t$ can be defined.

Theorem 1. *Let $(\bar{x}, \bar{u})$ be a feasible solution of (VP). Then, there exist $\bar{\lambda} : I \mapsto R^p$, $\bar{\mu} : I \mapsto R^m$ and $\bar{\nu} : I \mapsto R^n$ such that*

(i) $\sum_{i\in P} \bar{\lambda}_i f_{ix}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j\in M} \bar{\mu}_j g_{jx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k h_{kx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) = D[\sum_{i\in P} \bar{\lambda}_i f_{i\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j\in M} \bar{\mu}_j g_{j\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k h_{k\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})]$,

(ii) $\sum_{i\in P} \bar{\lambda}_i f_{iu}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j\in M} \bar{\mu}_j g_{ju}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k h_{ku}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) = D[\sum_{i\in P} \bar{\lambda}_i f_{i\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j\in M} \bar{\mu}_j g_{j\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k h_{k\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})]$,

(iii) $\int_a^b \sum_{j\in M} \bar{\mu}_j g_j(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t = 0$,

(iv) $\int_a^b \sum_{k\in N} \bar{\nu}_k h_k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t = 0$,
(v) $\bar{\lambda} \geq 0$, $\sum_{i\in P} \bar{\lambda}_i = 1$, $\bar{\mu} \geq 0$.

If $(\int_a^b f_1, \int_a^b f_2, \ldots, \int_a^b f_p)(t, x, \dot{x}, u, \dot{u})\mathrm{d}t$ is $(F, \hat{\alpha}, \hat{\rho}, \theta)$-V-convex at $(\bar{x}, \bar{u})$, $(\int_a^b g_1, \int_a^b g_2, \ldots, \int_a^b g_m)(t, x, \dot{x}, u, \dot{u})\mathrm{d}t$ is $(F, \tilde{\alpha}, \tilde{\rho}, \theta)$-V-convex at $(\bar{x}, \bar{u})$ and $(\int_a^b h_1, \int_a^b h_2, \ldots, \int_a^b h_n)(t, x, \dot{x}, u, \dot{u})\mathrm{d}t$ is $(F, \alpha^*, \rho^*, \theta)$-V-convex at $(\bar{x}, \bar{u})$ with $\hat{\alpha}_1 = \hat{\alpha}_2 = \cdots = \hat{\alpha}_p = \tilde{\alpha}_1 = \tilde{\alpha}_2 = \cdots = \tilde{\alpha}_m = \alpha_1^* = \alpha_2^* = \cdots = \alpha_k^* = \gamma > 0$, and $\sum_{i\in P} \bar{\lambda}_i \hat{\rho}_i + \sum_{j\in M} \bar{\mu}_j \tilde{\rho}_j + \sum_{k\in N} \bar{\nu}_k \rho_k^* \geq 0$, then $(\bar{x}, \bar{u})$ is a weakly efficient solution of (VP).

Proof. Suppose contrary that $(\bar{x}, \bar{u})$ is not a weakly efficient solution of (VP), then there exists a feasible solution (x, u) such that

$$\int_a^b f_i(t, x, \dot{x}, u, \dot{u})\mathrm{d}t < \int_a^b f_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t, \quad \forall\, i \in P,$$

which in view of $\bar{\lambda} \geq 0$, $\sum_{i\in P} \bar{\lambda}_i = 1$, gives

$$\int_a^b \sum_{i\in P} \bar{\lambda}_i f_i(t, x, \dot{x}, u, \dot{u})\mathrm{d}t < \int_a^b \sum_{i\in P} \bar{\lambda}_i f_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t. \tag{3}$$

The $(F, \hat{\alpha}, \hat{\rho}, \theta)$-V-convexity of $(\int_a^b f_1, \int_a^b f_2, \ldots, \int_a^b f_p)(t, x, \dot{x}, u, \dot{u})\mathrm{d}t$ at $(\bar{x}, \bar{u})$, $\bar{\lambda} \geq 0$ and $\hat{\alpha}_1 = \hat{\alpha}_2 = \cdots = \hat{\alpha}_p = \gamma$ implies

$$\begin{aligned}
&\int_a^b \sum_{i\in P} \bar{\lambda}_i f_i(t, x, \dot{x}, u, \dot{u})\mathrm{d}t - \int_a^b \sum_{i\in P} \bar{\lambda}_i f_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t \\
&\quad \geq \int_a^b F\Bigg(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}; \gamma\Bigg(\sum_{i\in P} \bar{\lambda}_i f_{ix}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \bar{\lambda}_i f_{iu}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \\
&\qquad - D\Bigg(\sum_{i\in P} \bar{\lambda}_i f_{i\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \bar{\lambda}_i f_{i\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\Bigg)\Bigg)\Bigg)\mathrm{d}t + \int_a^b \sum_{i\in P} \bar{\lambda}_i \hat{\rho}_i \theta^2(t, x, \bar{x}, u, \bar{u})\mathrm{d}t.
\end{aligned}$$

The above inequality along with (3) gives

$$\begin{aligned}
&\int_a^b F\Bigg(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}; \gamma\Bigg(\sum_{i\in P} \bar{\lambda}_i f_{ix}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \bar{\lambda}_i f_{iu}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \\
&\qquad - D\Bigg(\sum_{i\in P} \bar{\lambda}_i f_{i\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \bar{\lambda}_i f_{i\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\Bigg)\Bigg)\Bigg)\mathrm{d}t \\
&\quad < -\int_a^b \sum_{i\in P} \bar{\lambda}_i \hat{\rho}_i \theta^2(t, x, \bar{x}, u, \bar{u})\mathrm{d}t.
\end{aligned} \tag{4}$$

By $(F, \alpha^*, \rho^*, \theta)$-V-convexity of $(\int_a^b h_1, \int_a^b h_2, \ldots, \int_a^b h_n)(t, x, \dot{x}, u, \dot{u})\mathrm{d}t$ at $(\bar{x}, \bar{u})$, $\bar{\nu} \neq 0$ and $\alpha_1^* = \alpha_2^* = \cdots = \alpha_k^* = \gamma$, we have

$$\begin{aligned}
&\int_a^b \sum_{k\in N} \bar{\nu}_k h_k(t, x, \dot{x}, u, \dot{u})\mathrm{d}t - \int_a^b \sum_{k\in N} \bar{\nu}_k h_k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t \\
&\quad \geq \int_a^b F\Bigg(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}; \gamma\Bigg(\sum_{k\in N} \bar{\nu}_k h_{kx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k h_{ku}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \\
&\qquad - D\Bigg(\sum_{k\in N} \bar{\nu}_k h_{k\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k h_{k\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\Bigg)\Bigg)\Bigg)\mathrm{d}t + \int_a^b \sum_{k\in N} \bar{\nu}_k \rho_k^* \theta^2(t, x, \bar{x}, u, \bar{u})\mathrm{d}t,
\end{aligned}$$

which by $\int_a^b \sum_{k\in N} \bar{\nu}_k h_k(t, x, \dot{x}, u, \dot{u})\mathrm{d}t = \int_a^b \sum_{k\in N} \bar{\nu}_k h_k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t$ yields

$$\int_a^b F\Bigg(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}; \gamma\Bigg(\sum_{k\in N} \bar{\nu}_k h_{kx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k h_{ku}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})$$

$$-D\left(\sum_{k\in N}\bar{\nu}_k h_{k\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{k\in N}\bar{\nu}_k h_{k\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\Bigg)\Bigg)\mathrm{d}t$$

$$\leq -\int_a^b \sum_{k\in N}\bar{\nu}_k\rho_k^*\theta^2(t,x,\bar{x},u,\bar{u})\mathrm{d}t. \tag{5}$$

Summing (4) and (5) and using the sublinearity of F, we obtain

$$\int_a^b F\Bigg(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}};\gamma\Bigg(\sum_{i\in P}\bar{\lambda}_i f_{ix}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{i\in P}\bar{\lambda}_i f_{iu}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$-D\left(\sum_{i\in P}\bar{\lambda}_i f_{i\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{i\in P}\bar{\lambda}_i f_{i\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)+\sum_{k\in N}\bar{\nu}_k h_{kx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$+\sum_{k\in N}\bar{\nu}_k h_{ku}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-D\left(\sum_{k\in N}\bar{\nu}_k h_{k\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{k\in N}\bar{\nu}_k h_{k\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\Bigg)\Bigg)\mathrm{d}t$$

$$< -\int_a^b\left(\sum_{i\in P}\bar{\lambda}_i\hat{\rho}_i+\sum_{k\in N}\bar{\nu}_k\rho_k^*\right)\theta^2(t,x,\bar{x},u,\bar{u})\mathrm{d}t.$$

Since $\sum_{i\in P}\bar{\lambda}_i\hat{\rho}_i+\sum_{j\in M}\bar{\mu}_j\tilde{\rho}_j+\sum_{k\in N}\bar{\nu}_k\rho_k^*\geq 0$ and $\gamma>0$, we have

$$\int_a^b F\Bigg(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}};\sum_{i\in P}\bar{\lambda}_i f_{ix}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{i\in P}\bar{\lambda}_i f_{iu}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$-D\left(\sum_{i\in P}\bar{\lambda}_i f_{i\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{i\in P}\bar{\lambda}_i f_{i\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)+\sum_{k\in N}\bar{\nu}_k h_{kx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$+\sum_{k\in N}\bar{\nu}_k h_{ku}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-D\left(\sum_{k\in N}\bar{\nu}_k h_{k\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{k\in N}\bar{\nu}_k h_{k\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\Bigg)\mathrm{d}t$$

$$<\int_a^b\sum_{j\in M}\frac{\bar{\mu}_j\tilde{\rho}_j}{\gamma}\theta^2(t,x,\bar{x},u,\bar{u})\mathrm{d}t. \tag{6}$$

Now assumptions (i) and (ii) together with sublinearity of F yield

$$0=\int_a^b F\Bigg(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}};\sum_{i\in P}\bar{\lambda}_i f_{ix}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{i\in P}\bar{\lambda}_i f_{iu}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$-D\left(\sum_{i\in P}\bar{\lambda}_i f_{i\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{i\in P}\bar{\lambda}_i f_{i\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)+\sum_{k\in N}\bar{\nu}_k h_{kx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$+\sum_{k\in N}\bar{\nu}_k h_{ku}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-D\left(\sum_{k\in N}\bar{\nu}_k h_{k\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{k\in N}\bar{\nu}_k h_{k\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)$$

$$+\sum_{j\in M}\bar{\mu}_j g_{jx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\bar{\mu}_j g_{ju}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-D\left(\sum_{j\in M}\bar{\mu}_j g_{j\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\bar{\mu}_j g_{j\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\Bigg)\mathrm{d}t$$

$$\leq\int_a^b F\Bigg(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}};\sum_{i\in P}\bar{\lambda}_i f_{ix}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{i\in P}\bar{\lambda}_i f_{iu}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$-D\left(\sum_{i\in P}\bar{\lambda}_i f_{i\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{i\in P}\bar{\lambda}_i f_{i\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)+\sum_{k\in N}\bar{\nu}_k h_{kx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$+\sum_{k\in N}\bar{\nu}_k h_{ku}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-D\left(\sum_{k\in N}\bar{\nu}_k h_{k\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{k\in N}\bar{\nu}_k h_{k\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\Bigg)\mathrm{d}t$$
$$+\int_a^b F\Bigg(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}};\sum_{j\in M}\bar{\mu}_j g_{jx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\bar{\mu}_j g_{ju}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$
$$-D\left(\sum_{j\in M}\bar{\mu}_j g_{j\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\bar{\mu}_j g_{j\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\Bigg)\mathrm{d}t. \tag{7}$$

In view of (6), the above inequality implies

$$\int_a^b F\Bigg(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}};\sum_{j\in M}\bar{\mu}_j g_{jx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\bar{\mu}_j g_{ju}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$
$$-D\left(\sum_{j\in M}\bar{\mu}_j g_{j\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\bar{\mu}_j g_{j\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\Bigg)\mathrm{d}t+\int_a^b\sum_{j\in M}\frac{\bar{\mu}_j\tilde{\rho}_j}{\gamma}\theta^2(t,x,\bar{x},u,\bar{u})\mathrm{d}t>0. \tag{8}$$

The $(F,\tilde{\alpha},\tilde{\rho},\theta)$-V-convexity of $(\int_a^b g_1,\int_a^b g_2,\ldots,\int_a^b g_m)(t,x,\dot{x},u,\dot{u})\mathrm{d}t$ at $(\bar{x},\bar{u})$, $\bar{\mu}\geq 0$ and $\tilde{\alpha}_1=\tilde{\alpha}_2=\cdots=\tilde{\alpha}_m=\gamma$ yields

$$\int_a^b\sum_{j\in M}\bar{\mu}_j g_j(t,x,\dot{x},u,\dot{u})\mathrm{d}t-\int_a^b\sum_{j\in M}\bar{\mu}_j g_j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t$$
$$\geq\int_a^b F\Bigg(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}};\gamma\Bigg(\sum_{j\in M}\bar{\mu}_j g_{jx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\bar{\mu}_j g_{ju}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$
$$-D\left(\sum_{j\in M}\bar{\mu}_j g_{j\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\bar{\mu}_j g_{j\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\Bigg)\Bigg)\mathrm{d}t+\int_a^b\sum_{j\in M}\bar{\mu}_j\tilde{\rho}_j\theta^2(t,x,\bar{x},u,\bar{u})\mathrm{d}t.$$

Since $\int_a^b\sum_{j\in M}\bar{\mu}_j g_j(t,x,\dot{x},u,\dot{u})\mathrm{d}t\leq\int_a^b\sum_{j\in M}\bar{\mu}_j g_j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t$ and $\gamma>0$, we get

$$\int_a^b F\Bigg(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}};\sum_{j\in M}\bar{\mu}_j g_{jx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\bar{\mu}_j g_{ju}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$
$$-D\left(\sum_{j\in M}\bar{\mu}_j g_{j\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\bar{\mu}_j g_{j\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\Bigg)\mathrm{d}t+\int_a^b\sum_{j\in M}\frac{\bar{\mu}_j\tilde{\rho}_j}{\gamma}\theta^2(t,x,\bar{x},u,\bar{u})\mathrm{d}t\leq 0,$$

which contradicts (8). Hence, $(\bar{x},\bar{u})$ is a weakly efficient solution of (VP). □

Remark 1. If we replace $(F,\hat{\alpha},\hat{\rho},\theta)$-V-convexity of $(\int_a^b f_1,\int_a^b f_2,\ldots,\int_a^b f_p)(t,x,\dot{x},u,\dot{u})\mathrm{d}t$ by strict $(F,\hat{\alpha},\hat{\rho},\theta)$-V-convexity of $(\int_a^b f_1,\int_a^b f_2,\ldots,\int_a^b f_p)(t,x,\dot{x},u,\dot{u})\mathrm{d}t$, then we get stronger result that $(\bar{x},\bar{u})$ is an efficient solution of (VP). The theorem is stated below and the proof follows on the similar lines of above theorem.

Theorem 2. *Let $(\bar{x},\bar{u})$ be a feasible solution of (VP). Then, there exist $\bar{\lambda}:I\mapsto R^p$, $\bar{\mu}:I\mapsto R^m$ and $\bar{\nu}:I\mapsto R^n$ satisfying* (i) *to* (v). *If $(\int_a^b f_1,\int_a^b f_2,\ldots,\int_a^b f_p)(t,x,\dot{x},u,\dot{u})\mathrm{d}t$ is strictly $(F,\hat{\alpha},\hat{\rho},\theta)$-V-convex at $(\bar{x},\bar{u})$, $(\int_a^b g_1,\int_a^b g_2,\ldots,\int_a^b g_m)(t,x,\dot{x},u,\dot{u})\mathrm{d}t$ is $(F,\tilde{\alpha},\tilde{\rho},\theta)$-V-convex at $(\bar{x},\bar{u})$, and $(\int_a^b h_1,\int_a^b h_2,\ldots,\int_a^b h_n)(t,x,\dot{x},u,\dot{u})\mathrm{d}t$ is $(F,\alpha^*,\rho^*,\theta)$-V-convex at $(\bar{x},\bar{u})$ with $\hat{\alpha}_1=\hat{\alpha}_2=\cdots=\hat{\alpha}_p=\tilde{\alpha}_1=\tilde{\alpha}_2=\cdots=\tilde{\alpha}_m=\alpha_1^*=\alpha_2^*=\cdots=\alpha_k^*=\gamma>0$, and $\sum_{i\in P}\bar{\lambda}_i\hat{\rho}_i+\sum_{j\in M}\bar{\mu}_j\tilde{\rho}_j+\sum_{k\in N}\bar{\nu}_k\rho_k^*\geq 0$, then $(\bar{x},\bar{u})$ is an efficient solution of (VP).*

Theorem 3. *Let $(\bar{x},\bar{u})$ be a feasible solution of (VP). Then, there exist $\bar{\lambda}:I\mapsto R^p$, $\bar{\mu}:I\mapsto R^m$ and $\bar{\nu}:I\mapsto R^n$ satisfying* (i) *to* (v). *If $\int_a^b f^{\bar{\lambda}}(t,x,\dot{x},u,\dot{u})\mathrm{d}t$ is $(F,\tilde{\alpha},\tilde{\rho},\theta)$-V-pseudoconvex at $(\bar{x},\bar{u})$, $\int_a^b g^{\bar{\mu}}(t,x,\dot{x},u,\dot{u})\mathrm{d}t$ is $(F,\hat{\alpha},\hat{\rho},\theta)$-V-quasiconvex at $(\bar{x},\bar{u})$ and $\int_a^b h^{\bar{\nu}}(t,x,\dot{x},u,\dot{u})\mathrm{d}t$ is $(F,\alpha^*,\rho^*,\theta)$-V-quasiconvex at $(\bar{x},\bar{u})$ with $\tilde{\rho}+\hat{\rho}+\rho^*\geq 0$, then $(\bar{x},\bar{u})$ is a weakly efficient solution of (VP).*

Proof. Suppose contrary that $(\bar{x}, \bar{u})$ is not a weakly efficient solution of (VP), then there exists a feasible solution (x, u) such that

$$\int_a^b f_i(t, x, \dot{x}, u, \dot{u})\mathrm{d}t < \int_a^b f_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t, \quad \forall\, i \in P,$$

which by $\bar{\lambda} \geq 0$, $\sum_{i\in P} \bar{\lambda}_i = 1$, and $\tilde{\boldsymbol{\alpha}}_{\mathbf{i}} > 0$, $i \in P$ implies

$$\int_a^b \sum_{i\in P} \tilde{\boldsymbol{\alpha}}_{\mathbf{i}}\bar{\lambda}_i f_i(t, x, \dot{x}, u, \dot{u})\mathrm{d}t < \int_a^b \sum_{i\in P} \tilde{\boldsymbol{\alpha}}_{\mathbf{i}}\bar{\lambda}_i f_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t. \tag{9}$$

Also, we have

$$\int_a^b \sum_{k\in N} \boldsymbol{\alpha}^*_{\mathbf{k}}\bar{\nu}_k h_k(t, x, \dot{x}, u, \dot{u})\mathrm{d}t = \int_a^b \sum_{k\in N} \boldsymbol{\alpha}^*_{\mathbf{k}}\bar{\nu}_k h_k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\mathrm{d}t. \tag{10}$$

The $(F, \tilde{\alpha}, \tilde{\rho}, \theta)$-V-pseudoconvexity of $\int_a^b f^{\bar{\lambda}}(t, x, \dot{x}, u, \dot{u})\mathrm{d}t$ at $(\bar{x}, \bar{u})$ and (9) give

$$\begin{aligned}&\int_a^b F\Bigg(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}; \sum_{i\in P} \bar{\lambda}_i f_{ix}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \bar{\lambda}_i f_{iu}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\\ &\quad - D\Bigg(\sum_{i\in P} \bar{\lambda}_i f_{i\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \bar{\lambda}_i f_{i\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\Bigg)\Bigg)\mathrm{d}t\\ &< -\int_a^b \tilde{\rho}\theta^2(t, x, \bar{x}, u, \bar{u})\mathrm{d}t.\end{aligned} \tag{11}$$

By the $(F, \alpha^*, \rho^*, \theta)$-V-quasiconvexity of $\int_a^b h^{\bar{\nu}}(t, x, \dot{x}, u, \dot{u})\mathrm{d}t$ at $(\bar{x}, \bar{u})$ and (10), we obtain

$$\begin{aligned}&\int_a^b F\Bigg(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}; \sum_{k\in N} \bar{\nu}_k h_{kx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k h_{ku}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\\ &\quad - D\Bigg(\sum_{k\in N} \bar{\nu}_k h_{k\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k h_{k\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\Bigg)\Bigg)\mathrm{d}t\\ &\leq -\int_a^b \rho^*\theta^2(t, x, \bar{x}, u, \bar{u})\mathrm{d}t.\end{aligned} \tag{12}$$

Summing (11) and (12) and using the sublinearity of F, we get

$$\begin{aligned}&\int_a^b F\Bigg(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}; \sum_{i\in P} \bar{\lambda}_i f_{ix}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \bar{\lambda}_i f_{iu}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\\ &\quad - D\Bigg(\sum_{i\in P} \bar{\lambda}_i f_{i\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \bar{\lambda}_i f_{i\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\Bigg) + \sum_{k\in N} \bar{\nu}_k h_{kx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\\ &\quad + \sum_{k\in N} \bar{\nu}_k h_{ku}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - D\Bigg(\sum_{k\in N} \bar{\nu}_k h_{k\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k h_{k\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\Bigg)\Bigg)\mathrm{d}t\\ &< -\int_a^b (\tilde{\rho} + \rho^*)\theta^2(t, x, \bar{x}, u, \bar{u})\mathrm{d}t.\end{aligned}$$

As $\tilde{\rho} + \hat{\rho} + \rho^* \geq 0$, it follows that

$$\begin{aligned}&\int_a^b F\Bigg(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}; \sum_{i\in P} \bar{\lambda}_i f_{ix}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \bar{\lambda}_i f_{iu}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\\ &\quad - D\Bigg(\sum_{i\in P} \bar{\lambda}_i f_{i\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \bar{\lambda}_i f_{i\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\Bigg) + \sum_{k\in N} \bar{\nu}_k h_{kx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\end{aligned}$$

$$+\sum_{k\in N}\bar{\nu}_k h_{ku}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-D\left(\sum_{k\in N}\bar{\nu}_k h_{k\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{k\in N}\bar{\nu}_k h_{k\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\Bigg)\mathrm{d}t$$

$$<\int_a^b \hat{\rho}\theta^2(t,x,\bar{x},u,\bar{u})\mathrm{d}t. \tag{13}$$

The inequality (13) along with (7) gives

$$\int_a^b F\Bigg(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}};\sum_{j\in M}\bar{\mu}_j g_{jx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\bar{\mu}_j g_{ju}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$
$$-D\left(\sum_{j\in M}\bar{\mu}_j g_{j\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\bar{\mu}_j g_{j\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\Bigg)\mathrm{d}t+\int_a^b \hat{\rho}\theta^2(t,x,\bar{x},u,\bar{u})\mathrm{d}t>0,$$

which by $(F,\hat{\alpha},\hat{\rho},\theta)$-V-quasiconvexity of $\int_a^b g^{\bar{\mu}}(t,x,\dot{x},u,\dot{u})\mathrm{d}t$ at $(\bar{x},\bar{u})$ yields

$$\int_a^b \sum_{j\in M}\hat{\boldsymbol{\alpha}}_{\mathbf{j}}\bar{\mu}_j g_j(t,x,\dot{x},u,\dot{u})\mathrm{d}t>\int_a^b\sum_{j\in M}\hat{\boldsymbol{\alpha}}_{\mathbf{j}}\bar{\mu}_j g_j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t.$$

A contradiction to

$$\int_a^b \sum_{j\in M}\bar{\mu}_j g_j(t,x,\dot{x},u,\dot{u})\mathrm{d}t\le\int_a^b\sum_{j\in M}\bar{\mu}_j g_j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t$$

and $\hat{\boldsymbol{\alpha}}_{\mathbf{j}}>0, j\in M$. □

Theorem 4. *Let $(\bar{x},\bar{u})$ be a feasible solution of (VP). Then, there exist $\bar{\lambda}:I\mapsto R^p$, $\bar{\mu}:I\mapsto R^m$ and $\bar{\nu}:I\mapsto R^n$ satisfying* (i) *to* (v). *If $\int_a^b f^{\bar{\lambda}}(t,x,\dot{x},u,\dot{u})\mathrm{d}t$ is $(F,\hat{\alpha},\hat{\rho},\theta)$-V-strictly quasiconvex at $(\bar{x},\bar{u})$, $\int_a^b g^{\bar{\mu}}(t,x,\dot{x},u,\dot{u})\mathrm{d}t$ is $(F,\tilde{\alpha},\tilde{\rho},\theta)$-V-quasiconvex at $(\bar{x},\bar{u})$, and $\int_a^b h^{\bar{\nu}}(t,x,\dot{x},u,\dot{u})\mathrm{d}t$ is $(F,\alpha^*,\rho^*,\theta)$-V-quasiconvex at $(\bar{x},\bar{u})$ with $\hat{\rho}+\tilde{\rho}+\rho^*\ge 0$, then $(\bar{x},\bar{u})$ is an efficient solution of (VP).*

Proof. Suppose contrary that $(\bar{x},\bar{u})$ is not an efficient solution of (VP), then there exists a feasible solution (x,u) such that

$$\int_a^b f_i(t,x,\dot{x},u,\dot{u})\mathrm{d}t\le\int_a^b f_i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t,\quad\forall\, i\in P$$

and

$$\int_a^b f_l(t,x,\dot{x},u,\dot{u})\mathrm{d}t<\int_a^b f_l(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t,\quad\forall\, l\in P_i,$$

which by $\bar{\lambda}\ge 0$, $\sum_{i\in P}\bar{\lambda}_i=1$ and $\hat{\boldsymbol{\alpha}}_{\mathbf{i}}>0, i\in P$ give

$$\int_a^b \sum_{i\in P}\hat{\boldsymbol{\alpha}}_{\mathbf{i}}\bar{\lambda}_i f_i(t,x,\dot{x},u,\dot{u})\mathrm{d}t\le\int_a^b\sum_{i\in P}\hat{\boldsymbol{\alpha}}_{\mathbf{i}}\bar{\lambda}_i f_i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t. \tag{14}$$

The $(F,\hat{\alpha},\hat{\rho},\theta)$-V- strict quasiconvexity of $\int_a^b f^{\bar{\lambda}}(t,x,\dot{x},u,\dot{u})\mathrm{d}t$ at $(\bar{x},\bar{u})$ and (14) yields

$$\int_a^b F\Bigg(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}};\sum_{i\in P}\bar{\lambda}_i f_{ix}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{i\in P}\bar{\lambda}_i f_{iu}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$
$$-D\left(\sum_{i\in P}\bar{\lambda}_i f_{i\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{i\in P}\bar{\lambda}_i f_{i\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\Bigg)\mathrm{d}t$$
$$<-\int_a^b \hat{\rho}\theta^2(t,x,\bar{x},u,\bar{u})\mathrm{d}t,$$

which is precisely (11). Therefore, the remaining proof follows on the similar lines of Theorem 3. □

4. Wolfe type vector duality

This section deals with the following Wolfe type dual program (WD) of (VP) and related weak and strong duality theorems.

$$\text{(WD)}\quad \text{Maximize}\left(\int_a^b (f_1(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\mu^T g(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\mathrm{d}t,\ldots,\int_a^b (f_p(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\mu^T g(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\mathrm{d}t\right)$$

subject to

$$\bar{x}(a)=\alpha,\qquad \bar{x}(b)=\beta,$$

$$\sum_{i\in P}\lambda_i f_{ix}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\mu_j g_{jx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{k\in N}\nu_k h_{kx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$=D\left[\sum_{i\in P}\lambda_i f_{i\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\mu_j g_{j\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{k\in N}\nu_k h_{k\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right], \tag{15}$$

$$\sum_{i\in P}\lambda_i f_{iu}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\mu_j g_{ju}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{k\in N}\nu_k h_{ku}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$=D\left[\sum_{i\in P}\lambda_i f_{i\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\mu_j g_{j\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{k\in N}\nu_k h_{k\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right], \tag{16}$$

$$\int_a^b \sum_{k\in N}\nu_k h_k(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t=0, \tag{17}$$

$$\lambda\geq 0,\quad \sum_{i\in P}\lambda_i=1,\quad \mu\geq 0,\quad \lambda\in R^p,\quad \mu\in R^m,\quad \nu\in R^n.$$

Theorem 5 (*Weak Duality*). *Let* (x,u) *and* $(\bar{x},\bar{u},\lambda,\mu,\nu)$ *be the feasible solutions of (VP) and (WD), respectively. If*

(i) $(\int_a^b f_1,\int_a^b f_2,\ldots,\int_a^b f_p)(t,x,\dot{x},u,\dot{u})\mathrm{d}t$ *is* $(F,\hat{\alpha},\hat{\rho},\theta)$*-V-convex at* $(\bar{x},\bar{u})$*;*
(ii) $(\int_a^b g_1,\int_a^b g_2,\ldots,\int_a^b g_m)(t,x,\dot{x},u,\dot{u})\mathrm{d}t$ *is* $(F,\tilde{\alpha},\tilde{\rho},\theta)$*-V-convex at* $(\bar{x},\bar{u})$*;*
(iii) $(\int_a^b h_1,\int_a^b h_2,\ldots,\int_a^b h_n)(t,x,\dot{x},u,\dot{u})\mathrm{d}t$ *is* $(F,\alpha^*,\rho^*,\theta)$*-V-convex at* $(\bar{x},\bar{u})$*;*
(iv) $\hat{\alpha}_1=\hat{\alpha}_2=\cdots=\hat{\alpha}_p=\tilde{\alpha}_1=\tilde{\alpha}_2=\cdots=\tilde{\alpha}_m=\alpha_1^*=\alpha_2^*=\cdots=\alpha_k^*=\gamma>0$;
(v) *either* (a) $\lambda>0$, $\sum_{i\in P}\lambda_i\hat{\rho}_i+\sum_{j\in M}\mu_j\tilde{\rho}_j+\sum_{k\in N}\nu_k\rho_k^*\geq 0$, *or*
(b) $\sum_{i\in P}\lambda_i\hat{\rho}_i+\sum_{j\in M}\mu_j\tilde{\rho}_j+\sum_{k\in N}\nu_k\rho_k^*>0$,

then

$$\int_a^b f_i(t,x,\dot{x},u,\dot{u})\mathrm{d}t\leq\int_a^b\left(f_i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\mu_j g_j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\mathrm{d}t,\quad \forall\, i\in P, \tag{18}$$

and

$$\int_a^b f_l(t,x,\dot{x},u,\dot{u})\mathrm{d}t<\int_a^b\left(f_l(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\mu_j g_j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\mathrm{d}t,\quad \forall\, l\in P_i \tag{19}$$

cannot hold.

Proof. Suppose contrary that (18) and (19) hold. Then in view of $\lambda>0$, we get

$$\int_a^b\sum_{i\in P}\lambda_i f_i(t,x,\dot{x},u,\dot{u})\mathrm{d}t<\int_a^b\sum_{i\in P}\lambda_i\left(f_i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\mu_j g_j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\mathrm{d}t,$$

which along with $\mu\geq 0, g(t,x,\dot{x},u,\dot{u})\leq 0$ and $\sum_{i\in P}\lambda_i=1$ implies

$$\int_a^b\left(\sum_{i\in P}\lambda_i f_i(t,x,\dot{x},u,\dot{u})+\sum_{j\in M}\mu_j g_j(t,x,\dot{x},u,\dot{u})\right)\mathrm{d}t<\int_a^b\left(\sum_{i\in P}\lambda_i f_i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\mu_j g_j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\mathrm{d}t. \tag{20}$$

By hypothesis (i), (iv) and $\lambda > 0$, it follows that

$$\int_a^b \sum_{i\in P} \lambda_i f_i(t,x,\dot{x},u,\dot{u})\mathrm{d}t - \int_a^b \sum_{i\in P} \lambda_i f_i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t$$

$$\geq \int_a^b F\Bigg(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}};\gamma\Bigg(\sum_{i\in P}\lambda_i f_{ix}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{i\in P}\lambda_i f_{iu}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$-D\Bigg(\sum_{i\in P}\lambda_i f_{i\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{i\in P}\lambda_i f_{i\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\Bigg)\Bigg)\Bigg)\mathrm{d}t + \int_a^b \sum_{i\in P}\lambda_i\tilde{\rho}_i\theta^2(t,x,\bar{x},u,\bar{u})\mathrm{d}t. \tag{21}$$

From (2) and (17) and $\nu \neq 0$, we obtain

$$\int_a^b \sum_{k\in N}\nu_k h_k(t,x,\dot{x},u,\dot{u})\mathrm{d}t = 0 = \int_a^b \sum_{k\in N}\nu_k h_k(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t. \tag{22}$$

Now hypothesis (ii) and (iii) together with $\mu \geq 0$, $\nu \neq 0$ and (iv) imply that

$$\int_a^b \sum_{j\in M} \mu_j g_j(t,x,\dot{x},u,\dot{u})\mathrm{d}t - \int_a^b \sum_{j\in M} \mu_j g_j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t$$

$$\geq \int_a^b F\Bigg(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}};\gamma\Bigg(\sum_{j\in M}\mu_j g_{jx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{j\in M}\mu_j g_{ju}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$-D\Bigg(\sum_{j\in M}\mu_j g_{j\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{j\in M}\mu_j g_{j\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\Bigg)\Bigg)\Bigg)\mathrm{d}t + \int_a^b \sum_{j\in M}\mu_j\tilde{\rho}_j\theta^2(t,x,\bar{x},u,\bar{u})\mathrm{d}t, \tag{23}$$

and

$$\int_a^b \sum_{k\in N}\nu_k h_k(t,x,\dot{x},u,\dot{u})\mathrm{d}t - \int_a^b \sum_{k\in N}\nu_k h_k(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t$$

$$\geq \int_a^b F\Bigg(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}};\gamma\Bigg(\sum_{k\in N}\nu_k h_{kx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{k\in N}\nu_k h_{ku}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$-D\Bigg(\sum_{k\in N}\nu_k h_{k\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{k\in N}\nu_k h_{k\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\Bigg)\Bigg)\Bigg)\mathrm{d}t + \int_a^b \sum_{k\in N}\nu_k\rho_k^*\theta^2(t,x,\bar{x},u,\bar{u})\mathrm{d}t. \tag{24}$$

Using (22) in (24), we have

$$0 \geq \int_a^b F\Bigg(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}};\gamma\Bigg(\sum_{k\in N}\nu_k h_{kx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{k\in N}\nu_k h_{ku}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$-D\Bigg(\sum_{k\in N}\nu_k h_{k\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{k\in N}\nu_k h_{k\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\Bigg)\Bigg)\Bigg)\mathrm{d}t + \int_a^b \sum_{k\in N}\nu_k\rho_k^*\theta^2(t,x,\bar{x},u,\bar{u})\mathrm{d}t. \tag{25}$$

Adding (21), (23) and (25), and by the sublinearity of F, we get

$$\int_a^b \left(\sum_{i\in P}\lambda_i f_i(t,x,\dot{x},u,\dot{u}) + \sum_{j\in M}\mu_j g_j(t,x,\dot{x},u,\dot{u})\right)\mathrm{d}t - \int_a^b \left(\sum_{i\in P}\lambda_i f_i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{j\in M}\mu_j g_j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\mathrm{d}t$$

$$\geq \int_a^b F\Bigg(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}};\gamma\Bigg(\sum_{i\in P}\lambda_i f_{ix}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{i\in P}\lambda_i f_{iu}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$-D\Bigg(\sum_{i\in P}\lambda_i f_{i\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{i\in P}\lambda_i f_{i\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\Bigg) + \sum_{k\in N}\nu_k h_{kx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$+\sum_{k\in N}\nu_k h_{ku}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-D\left(\sum_{k\in N}\nu_k h_{k\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{k\in N}\nu_k h_{k\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)$$

$$+\sum_{j\in M}\mu_j g_{jx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\mu_j g_{ju}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-D\left(\sum_{j\in M}\mu_j g_{j\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right.$$

$$\left.\left.\left.+\sum_{j\in M}\mu_j g_{j\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\right)\right)\mathrm{d}t+\left(\sum_{i\in P}\lambda_i\tilde{\rho}_i+\sum_{j\in M}\mu_j\hat{\rho}_j+\sum_{k\in N}\nu_k\rho_k^*\right)\int_a^b\theta^2(t,x,\bar{x},u,\bar{u})\mathrm{d}t,$$

$$\geq\int_a^b F\left(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}};\gamma\left(\sum_{i\in P}\lambda_i f_{ix}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{i\in P}\lambda_i f_{iu}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right.\right.$$

$$-D\left(\sum_{i\in P}\lambda_i f_{i\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{i\in P}\lambda_i f_{i\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)+\sum_{k\in N}\nu_k h_{kx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$+\sum_{k\in N}\nu_k h_{ku}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-D\left(\sum_{k\in N}\nu_k h_{k\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{k\in N}\nu_k h_{k\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)$$

$$+\sum_{j\in M}\mu_j g_{jx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\mu_j g_{ju}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-D\left(\sum_{j\in M}\mu_j g_{j\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right.$$

$$\left.\left.\left.+\sum_{j\in M}\mu_j g_{j\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\right)\right)\mathrm{d}t,\quad\text{(by hyp. (a))},\tag{26}$$

which in view of (15) and (16) and sublinearity of F implies that

$$\int_a^b\left(\sum_{i\in P}\lambda_i f_i(t,x,\dot{x},u,\dot{u})+\sum_{j\in M}\mu_j g_j(t,x,\dot{x},u,\dot{u})\right)\mathrm{d}t\geq\int_a^b\left(\sum_{i\in P}\lambda_i f_i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\mu_j g_j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\mathrm{d}t.$$

A contradiction to (20).

As assumption (b) holds, we get inequality instead of strict inequality in (20) and strict inequality instead of inequality in (26). Therefore, we again get a contradiction to (20). □

Theorem 6 (*Strong Duality*). *Let $(\bar{x},\bar{u})$ be an efficient solution for (VP) at which a constraint qualification is satisfied. Then, there exist $\bar{\lambda}\in R^p$ and piecewise smooth $\bar{\mu}: I\mapsto R^m$ and $\bar{\nu}: I\mapsto R^n$ such that $(\bar{x},\bar{u},\bar{\lambda},\bar{\mu},\bar{\nu})$ is feasible for (WD). Further, if weak duality (Theorem 5) holds between (VP) and (WD), then $(\bar{x},\bar{u},\bar{\lambda},\bar{\mu},\bar{\nu})$ is an efficient solution for (WD).*

Proof. Since $(\bar{x},\bar{u})$ is an efficient solution for (VP), then from Lemma 2, $(\bar{x},\bar{u})$ solves $(\mathrm{VP})_s$. As $(\bar{x},\bar{u})$ satisfies the constraint qualification for $(\mathrm{VP})_s$, it follows from Lemma 1 that there exist piecewise smooth $\bar{\lambda}^*\in R^{p-1}$, $\bar{\mu}^*: I\mapsto R^m$ and $\bar{\nu}^*: I\mapsto R^n$ such that for all $t\in I$,

$$f_{kx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{\substack{i=1\\i\neq k}}^{p}\bar{\lambda}_i^*f_{ix}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\bar{\mu}_j^*g_{jx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{k\in N}\bar{\nu}_k^*h_{kx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$=D\left[f_{k\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{\substack{i=1\\i\neq k}}^{p}\bar{\lambda}_i^*f_{i\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\bar{\mu}_j^*g_{j\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{k\in N}\bar{\nu}_k^*h_{k\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right],$$

$$f_{ku}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{\substack{i=1\\i\neq k}}^{p}\bar{\lambda}_i^*f_{iu}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\bar{\mu}_j^*g_{ju}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{k\in N}\bar{\nu}_k^*h_{ku}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$=D\left[f_{k\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{\substack{i=1\\i\neq k}}^{p}\bar{\lambda}_i^*f_{i\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{j\in M}\bar{\mu}_j^*g_{j\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sum_{k\in N}\bar{\nu}_k^*h_{k\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right],$$

$$\bar{\mu}^{*T}g(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t=0,$$

$$\bar{\mu}^* \geq 0.$$

Let $\frac{1}{\alpha} = 1 + \sum_{\substack{i=1\\ i\neq k}}^{p} \bar{\lambda}_i^*$. Then, we get

$$\sum_{i\in P} \bar{\lambda}_i f_{ix}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{j\in M} \bar{\mu}_j g_{jx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k h_{kx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$= D\left[\sum_{i\in P} \bar{\lambda}_i f_{i\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{j\in M} \bar{\mu}_j g_{j\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k h_{k\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right],$$

$$\sum_{i\in P} \bar{\lambda}_i f_{iu}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{j\in M} \bar{\mu}_j g_{ju}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k h_{ku}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$= D\left[\sum_{i\in P} \bar{\lambda}_i f_{i\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{j\in M} \bar{\mu}_j g_{j\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k h_{k\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right],$$

$$\bar{\mu}^T g(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t = 0,$$

$$\bar{\mu} \geq 0,$$

where, $\bar{\lambda}_k = \alpha > 0,\ \bar{\lambda}_i = \alpha\sum_{i\in P}\bar{\lambda}_i^*, i\neq k,\ \sum_{j\in M}\bar{\mu}_j = \alpha\sum_{j\in M}\bar{\mu}_j^*, \sum_{k\in N}\bar{\nu}_k = \alpha\sum_{k\in N}\bar{\nu}_k^*$.

Also, from (2), we have $\int_a^b \sum_{k\in N}\nu_k h_k(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) = 0$. Therefore, $(\bar{x},\bar{u},\bar{\lambda},\bar{\mu},\bar{\nu})$ is a feasible solution for (WD).

Assume that $(\bar{x},\bar{u},\bar{\lambda},\bar{\mu},\bar{\nu})$ is not an efficient solution for (WD). Then, there exists a feasible solution (x,u,λ,μ,ν) for (WD) such that

$$\int_a^b (f_i(t,x,\dot{x},u,\dot{u}) + \mu^T g(t,x,\dot{x},u,\dot{u}))\mathrm{d}t \geq \int_a^b (f_i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \bar{\mu}^T g(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\mathrm{d}t, \quad \forall i\in P,$$

$$\int_a^b (f_l(t,x,\dot{x},u,\dot{u}) + \mu^T g(t,x,\dot{x},u,\dot{u}))\mathrm{d}t > \int_a^b (f_l(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \bar{\mu}^T g(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\mathrm{d}t, \quad \forall l\in P_i.$$

Since $\bar{\mu}^T g(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) = 0$, we have

$$\int_a^b (f_i(t,x,\dot{x},u,\dot{u}) + \mu^T g(t,x,\dot{x},u,\dot{u}))\mathrm{d}t \geq \int_a^b f_i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t, \quad \forall i\in P,$$

$$\int_a^b (f_l(t,x,\dot{x},u,\dot{u}) + \mu^T g(t,x,\dot{x},u,\dot{u}))\mathrm{d}t > \int_a^b f_l(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t, \quad \forall l\in P_i,$$

which contradicts weak duality (Theorem 5). Hence, $(\bar{x},\bar{u},\bar{\lambda},\bar{\mu},\bar{\nu})$ is an efficient solution for (WD). □

5. Mond–Weir type vector duality

In this section, we present the following Mond–Weir type dual program (MD) for (VP) and prove weak and strong duality theorems.

$$\text{(MD)Maximize}\left(\int_a^b f_1(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t,\ldots,\int_a^b f_p(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t\right)$$

subject to

$$\bar{x}(a) = \alpha, \qquad \bar{x}(b) = \beta,$$

$$\sum_{i\in P} \lambda_i f_{ix}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{j\in M} \mu_j g_{jx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{k\in N} \nu_k h_{kx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$= D\left[\sum_{i\in P} \lambda_i f_{i\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{j\in M} \mu_j g_{j\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{k\in N} \nu_k h_{k\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right], \tag{27}$$

$$\sum_{i\in P} \lambda_i f_{iu}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{j\in M} \mu_j g_{ju}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{k\in N} \nu_k h_{ku}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$= D\left[\sum_{i\in P}\lambda_i f_{i\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{j\in M}\mu_j g_{j\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{k\in N}\nu_k h_{k\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right], \tag{28}$$

$$\int_a^b \sum_{j\in M}\mu_j g_j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t \geq 0, \tag{29}$$

$$\int_a^b \sum_{k\in N}\nu_k h_k(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t = 0, \tag{30}$$

$$\lambda \geq 0, \quad \sum_{i\in P}\lambda_i = 1, \quad \mu \geq 0, \quad \lambda\in R^p, \quad \mu \in R^m, \quad \nu\in R^n.$$

Theorem 7 (*Weak Duality*). *Let (x,u) and $(\bar{x},\bar{u},\lambda,\mu,\nu)$ be the feasible solutions of (VP) and (MD), respectively. If*

(i) $\int_a^b f^{\lambda}(t,x,\dot{x},u,\dot{u})\mathrm{d}t$ *is $(F,\tilde{\alpha},\tilde{\rho},\theta)$-V-pseudoconvex at $(\bar{x},\bar{u})$;*
(ii) $\int_a^b g^{\mu}(t,x,\dot{x},u,\dot{u})\mathrm{d}t$ *is $(F,\hat{\alpha},\hat{\rho},\theta)$-V-quasiconvex at $(\bar{x},\bar{u})$;*
(iii) $\int_a^b h^{\nu}(t,x,\dot{x},u,\dot{u})\mathrm{d}t$ *is $(F,\alpha^*,\rho^*,\theta)$-V-quasiconvex at $(\bar{x},\bar{u})$; and*
(iv) $\lambda > 0$, $\tilde{\rho}+\hat{\rho}+\rho^* \geq 0$,

then

$$\int_a^b f_i(t,x,\dot{x},u,\dot{u})\mathrm{d}t \leq \int_a^b f_i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t, \quad \forall\, i\in P, \tag{31}$$

and

$$\int_a^b f_l(t,x,\dot{x},u,\dot{u})\mathrm{d}t < \int_a^b f_l(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t, \quad \forall\, l\in P_i \tag{32}$$

cannot hold.

Proof. Suppose contrary that (31) and (32) hold. Then, by $\lambda > 0$ and $\tilde{\alpha}_i > 0$, $i\in P$, we get

$$\int_a^b \sum_{i\in P}\tilde{\alpha}_i\lambda_i f_i(t,x,\dot{x},u,\dot{u})\mathrm{d}t < \int_a^b \sum_{i\in P}\tilde{\alpha}_i\lambda_i f_i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t, \tag{33}$$

which along with hypothesis (i) gives

$$\int_a^b F\Bigg(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}};\sum_{i\in P}\lambda_i f_{ix}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{i\in P}\lambda_i f_{iu}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$
$$-D\left(\sum_{i\in P}\lambda_i f_{i\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{i\in P}\lambda_i f_{i\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\Bigg)\mathrm{d}t + \int_a^b \tilde{\rho}\theta^2(t,x,\bar{x},u,\bar{u})\mathrm{d}t < 0. \tag{34}$$

From (1) and (29), $\mu \geq 0$ and $\hat{\alpha}_j > 0$, $j\in M$, we have

$$\int_a^b \sum_{j\in M}\hat{\alpha}_j\mu_j g_j(t,x,\dot{x},u,\dot{u})\mathrm{d}t \leq \int_a^b \sum_{j\in M}\hat{\alpha}_j\mu_j g_j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t, \tag{35}$$

and from (2) and (30), $\nu \neq 0$ and $\alpha_k^* > 0$, $k\in N$, we get

$$\int_a^b \sum_{k\in N}\alpha_k^*\nu_k h_k(t,x,\dot{x},u,\dot{u})\mathrm{d}t = \int_a^b \sum_{k\in N}\alpha_k^*\nu_k h_k(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\mathrm{d}t. \tag{36}$$

The inequality (33) together with hypothesis (ii) implies

$$\int_a^b F\Bigg(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}};\sum_{j\in M}\mu_j g_{jx}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{j\in M}\mu_j g_{ju}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$
$$-D\left(\sum_{j\in M}\mu_j g_{j\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{j\in M}\mu_j g_{j\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\Bigg)\mathrm{d}t + \int_a^b \hat{\rho}\theta^2(t,x,\bar{x},u,\bar{u})\mathrm{d}t \leq 0. \tag{37}$$

Also, from (36) and hypothesis (iii) yields

$$\int_a^b F\Bigg(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}; \sum_{k\in N} \nu_k h_{kx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \nu_k h_{ku}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})$$

$$-D\left(\sum_{k\in N} \nu_k h_{k\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \nu_k h_{k\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\right)\Bigg)\mathrm{d}t + \int_a^b \rho^* \theta^2(t, x, \bar{x}, u, \bar{u})\mathrm{d}t \leq 0. \tag{38}$$

Adding (34), (37) and (38) and using the sublinearity of F, we get

$$\int_a^b F\Bigg(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}; \sum_{i\in P} \lambda_i f_{ix}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \lambda_i f_{iu}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})$$

$$-D\left(\sum_{i\in P} \lambda_i f_{i\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \lambda_i f_{i\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\right) + \sum_{j\in M} \mu_j g_{jx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})$$

$$+ \sum_{j\in M} \mu_j g_{ju}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - D\left(\sum_{j\in M} \mu_j g_{j\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j\in M} \mu_j g_{j\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\right)$$

$$+ \sum_{k\in N} \nu_k h_{kx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \nu_k h_{ku}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - D\Bigg(\sum_{k\in N} \nu_k h_{k\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})$$

$$+ \sum_{k\in N} \nu_k h_{k\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\Bigg)\Bigg)\mathrm{d}t + (\tilde{\rho} + \hat{\rho} + \rho^*) \int_a^b \theta^2(t, x, \bar{x}, u, \bar{u})\mathrm{d}t < 0.$$

As hypothesis (iv) holds, we have

$$\int_a^b F\Bigg(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}; \sum_{i\in P} \lambda_i f_{ix}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \lambda_i f_{iu}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})$$

$$-D\Bigg(\sum_{i\in P} \lambda_i f_{i\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{i\in P} \lambda_i f_{i\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})) + \sum_{j\in M} \mu_j g_{jx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})$$

$$+ \sum_{j\in M} \mu_j g_{ju}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - D\left(\sum_{j\in M} \mu_j g_{j\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j\in M} \mu_j g_{j\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\right)$$

$$+ \sum_{k\in N} \nu_k h_{kx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \nu_k h_{ku}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - D\Bigg(\sum_{k\in N} \nu_k h_{k\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})$$

$$+ \sum_{k\in N} \nu_k h_{k\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\Bigg)\Bigg)\mathrm{d}t < 0,$$

a contradiction to (26) and (27) as $F(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}; 0) = 0$. □

Theorem 8 (*Weak Duality*). *Let* (x, u) *and* $(\bar{x}, \bar{u}, \lambda, \mu, \nu)$ *be the feasible solutions of (VP) and (MD), respectively. If*

(i) $\int_a^b f^{\lambda}(t, x, \dot{x}, u, \dot{u})\mathrm{d}t$ *is* $(F, \tilde{\alpha}, \tilde{\rho}, \theta)$*-V-strictly quasiconvex at* $(\bar{x}, \bar{u})$*;*
(ii) $\int_a^b g^{\mu}(t, x, \dot{x}, u, \dot{u})\mathrm{d}t$ *is* $(F, \hat{\alpha}, \hat{\rho}, \theta)$*-V-quasiconvex at* $(\bar{x}, \bar{u})$*;*
(iii) $\int_a^b h^{\nu}(t, x, \dot{x}, u, \dot{u})\mathrm{d}t$ *is* $(F, \alpha^*, \rho^*, \theta)$*-V-quasiconvex at* $(\bar{x}, \bar{u})$*; and*
(iv) $\tilde{\rho} + \hat{\rho} + \rho^* \geq 0$,

then (8) and (9) cannot hold.

Proof. The proof follows on the similar lines of Theorem 7. □

Theorem 9 (*Strong Duality*). *Let* $(\bar{x}, \bar{u})$ *be an efficient solution for (VP) at which a constraint qualification is satisfied. Then, there exist* $\bar{\lambda} \in R^p$ *and piecewise smooth* $\bar{\mu} : I \mapsto R^m$ *and* $\bar{\nu} : I \mapsto R^n$ *such that* $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}, \bar{\nu})$ *is feasible for (MD). Further, if any of the weak duality theorem holds between (VP) and (MD), then* $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}, \bar{\nu})$ *is an efficient solution for (MD).*

Proof. The proof follows along the lines of Theorem 6 of Section 4. □

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