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Higher-Order Duality in Nondifferentiable Minimax Programming with Generalized Type I Functions

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Abstract A unified higher-order dual for a nondifferentiable minimax programming problem is formulated. Weak, strong and strict converse duality theorems are discussed involving generalized higher-order (F, α, ρ, d)-Type I functions.

Keywords Nondifferentiable programming · Minimax programming · Higher-order duality

1 Introduction

Several authors have shown their interest in developing optimality conditions and duality results for minimax programming problems. Schmitendorf (Ref. [1]) considered the following minimax programming problem:

$$\text{(P)} \quad \operatorname{Min} \sup_{y \in Y} f(x, y),$$
$$\text{s.t.} \quad g(x) \leq 0, \quad x \in R^n,$$

where Y is a compact subset of R^l, the functions $f(\cdot,\cdot) : R^n \times R^l \to R$, and $g(\cdot) : R^n \to R^m$ are in C^1.

In Ref. [1], Schmitendorf established necessary and sufficient optimality conditions for (P) under convexity. Tanimoto (Ref. [2]) applied these optimality conditions

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to define a dual problem and derived duality theorems, which were extended for fractional analogue of (P) by several authors (Refs. [3–15]). Liu (Ref. [16]) discussed the second-order duality theorems for (P) using the concepts of second-order B-invex and related functions. A comprehensive view of the minimax problem can be seen in Ref. [17].

In this paper, we consider the following nondifferentiable minimax programming problem:

$$\text{(NP)} \quad \operatorname{Min} \sup_{y \in Y} f(x,y) + (x^T Bx)^{1/2},$$
$$\text{s.t.} \quad g(x) \leq 0, \quad x \in R^n,$$

where Y is a compact subset of R^l, $f(\cdot,\cdot): R^n \times R^l \to R$, $g(\cdot): R^n \to R^m$ are continuously differentiable functions at $x \in R^n$, and B is an $n \times n$ positive semidefinite symmetric matrix. Recently, Mishra and Rueda (Ref. [18]) discussed duality results for a general Mond-Weir type second-order dual of (NP) involving generalized Type-I functions.

In this paper, we formulate a unified higher-order dual of (NP) and establish weak, strong and strict converse duality theorems under higher-order (F, α, ρ, d)-Type I assumptions. More precisely, this paper is an extension of the second order duality results of Mishra and Rueda (Ref. [18]) to a class of higher-order duality, and hence, presents an answer of a question raised therein. Many of the previously published results in this class of optimization appear as special cases of our results.

2 Notations and Preliminary Results

Let R^n be the n-dimensional Euclidean space, R^n_+ be its nonnegative orthant and let X be an open set in R^n. In what follows ∇ stands for the gradient vector with respect to x throughout the paper.

Let $S = \{x \in X : g(x) \leq 0\}$ denote the set of all feasible solutions of (NP). Any point $x \in S$ is called the feasible point of (NP). Let $J = \{1, 2, \dots, m\}$ be an index set. For each $(x, y) \in S \times Y$, we define

$$J(x) = \{j \in J : g_j(x) = 0\},$$
$$Y(x) = \left\{y \in Y : f(x,y) + (x^T Bx)^{1/2} = \sup_{z \in Y} f(x,z) + (x^T Bx)^{1/2}\right\},$$

and

$$K = \Bigg\{(s, t, \tilde{y}) \in \mathbb{N} \times R^s_+ \times R^{ls} : 1 \leq s \leq n+1,\ t = (t_1, t_2, \dots, t_s) \in R^s_+$$
$$\text{with } \sum_{i=1}^{s} t_i = 1, \tilde{y} = (\bar{y}_1, \bar{y}_2, \dots, \bar{y}_s) \text{ with } \bar{y}_i \in Y(x), i = 1, 2, \dots, s\Bigg\}.$$

Definition 2.1 A functional $F : X \times X \times R^n \to R$ is said to be sublinear in its third argument if, $\forall x, \bar{x} \in X$,

(i) $F(x,\bar{x};a_1+a_2)\leq F(x,\bar{x};a_1)+F(x,\bar{x};a_2),\ \forall a_1,a_2\in R^n$,
(ii) $F(x,\bar{x};\alpha a)=\alpha F(x,\bar{x};a),\ \forall \alpha\in R_+, a\in R^n$.

By (ii), it is clear that $F(x,\bar{x};0a)=0$.

Liang, Huang and Pardalos (Refs. [19, 20]) introduced the concept of (F,α,ρ,d)-convex function, which was further extended to generalized second-order (F,α,ρ,d)-convex functions and generalized second-order (F,α,ρ,d)-Type I functions by Ahmad and Husain (Ref. [21]) and Hachimi and Aghezzaf (Ref. [22]), respectively. Here, we rewrite the recently introduced definitions of higher-order (F,α,ρ,d)-Type I functions (Ref. [23]) for a minimax programming problem (P) as follows:

Let F be sublinear and let $d(\cdot,\cdot):X\times X\to R$. Let $\rho=(\rho^1,\rho^2)$, where $\rho^1=(\rho_1^1,\rho_2^1,\dots,\rho_s^1)\in R^s$ and $\rho^2=(\rho_1^2,\rho_2^2,\dots,\rho_m^2)\in R^m$, and let $\alpha=(\alpha^1,\alpha^2)$, where $\alpha^1,\alpha^2:X\times X\to R_+\setminus\{0\}$. Let $f(\cdot,\cdot):X\times Y(x)\to R$ and $g(\cdot):X\to R^m$ be differentiable functions at $\bar{x}\in X$.

Definition 2.2 For each $j\in J$, (f,g_j) is said to be higher-order (F,α,ρ,d)-Type I at $\bar{x}\in X$ with respect to $p\in R^n$ if, for all $x\in S$ and $\bar{y}_i\in Y(x)$,

$$\begin{aligned}
&f(x,\bar{y}_i)\geq f(\bar{x},\bar{y}_i)+h(\bar{x},\bar{y}_i,p)-p^T\nabla_p h(\bar{x},\bar{y}_i,p)\\
&\qquad\qquad +F(x,\bar{x};\alpha^1(x,\bar{x})(\nabla_p h(\bar{x},\bar{y}_i,p)))+\rho_i^1 d^2(x,\bar{x}),\quad i=1,2,\dots,s,\\
&-[g_j(\bar{x})+k_j(\bar{x},p)-p^T\nabla_p k_j(\bar{x},p)]\\
&\quad\geq F(x,\bar{x};\alpha^2(x,\bar{x})(\nabla_p k_j(\bar{x},p)))+\rho_j^2 d^2(x,\bar{x}).
\end{aligned}$$

Definition 2.3 For each $j\in J$, (f,g_j) is said to be higher-order (F,α,ρ,d)-pseudoquasi-Type I at $\bar{x}\in X$ with respect to $p\in R^n$ if, for all $x\in S$ and $\bar{y}_i\in Y(x)$,

$$\begin{aligned}
&f(x,\bar{y}_i)<f(\bar{x},\bar{y}_i)+h(\bar{x},\bar{y}_i,p)-p^T\nabla_p h(\bar{x},\bar{y}_i,p)\\
&\quad\Rightarrow\quad F(x,\bar{x};\alpha^1(x,\bar{x})(\nabla_p h(\bar{x},\bar{y}_i,p)))<-\rho_i^1 d^2(x,\bar{x}),\quad i=1,2,\dots,s,\\
&-[g_j(\bar{x})+k_j(\bar{x},p)-p^T\nabla_p k_j(\bar{x},p)]\leq 0\\
&\quad\Rightarrow\quad F(x,\bar{x};\alpha^2(x,\bar{x})(\nabla_p k_j(\bar{x},p)))\leq-\rho_j^2 d^2(x,\bar{x}).
\end{aligned}$$

In the above definition, if

$$\begin{aligned}
&F(x,\bar{x};\alpha^1(x,\bar{x})(\nabla_p h(\bar{x},\bar{y}_i,p)))\geq-\rho_i^1 d^2(x,\bar{x})\\
&\quad\Rightarrow\quad f(x,\bar{y}_i)>f(\bar{x},\bar{y}_i)+h(\bar{x},\bar{y}_i,p)-p^T\nabla_p h(\bar{x},\bar{y}_i,p),\quad i=1,2,\dots,s,
\end{aligned}$$

then we say that (f,g_j) is higher-order (F,α,ρ,d)-strictlypseudoquasi Type I at $\bar{x}$.

Remark 2.1 If $\alpha^1(x,\bar{x})=\alpha^2(x,\bar{x})=1$, $h(\bar{x},\bar{y}_i,p)=p^T\nabla f(\bar{x},\bar{y}_i)+\frac{1}{2}p^T\nabla^2 f(\bar{x},\bar{y}_i)p$, $\rho_i^1=0, i=1,2,\dots,s$, $k_j(\bar{x},p)=p^T\nabla g_j(\bar{x})+\frac{1}{2}p^T\nabla^2 g_j(\bar{x})p$, $\rho_j^2=0$, $j\in J$ and $F(x,\bar{x};a)=\eta^T(x,\bar{x})a$, where $\eta:S\times X\to R^n$, then the above definitions reduce to that of second-order Type I/second-order pseudoquasi Type I functions given by Mishra and Rueda (Ref. [18]).

Lemma 2.1 (Generalized Schwartz Inequality) *Let B be a positive semidefinite symmetric matrix of order n. Then, for all $x, u \in R^n$,*

$$x^T Bu \leq (x^T Bx)^{1/2}(u^T Bu)^{1/2}.$$

We observe that equality holds if $Bx = \lambda Bu$, for some $\lambda \geq 0$. Evidently, if $u^T Bu \leq 1$, we have

$$x^T Bu \leq (x^T Bx)^{1/2}.$$

The following theorem is a particular case of Theorem 3.1 in Ref. [6] and will be needed in the present analysis.

Theorem 2.1 (Necessary Conditions) *If x^* is a (local or global) solution of (NP) satisfying $x^{*T} Bx^* > 0$, and if $\nabla g_j(x^*)$, $j \in J(x^*)$ are linearly independent, then there exist $(s^*, t^*, \tilde{y}^*) \in K$, $u^* \in R^n$, and $\mu^* \in R^m_+$ such that*

$$\sum_{i=1}^{s^*} t_i^* \nabla f(x^*, \bar{y}_i^*) + Bu^* + \sum_{j=1}^{m} \nabla \mu_j^* g_j(x^*) = 0,$$

$$\sum_{j=1}^{m} \mu_j^* g_j(x^*) = 0,$$

$$t_i^* \geq 0, \quad i = 1, 2, \dots, s^*, \sum_{i=1}^{s^*} t_i^* = 1,$$

$$u^{*T} Bu^* \leq 1, \quad (x^{*T} Bx^*)^{1/2} = x^{*T} Bu^*.$$

3 Unified Higher-Order Duality

In this section, we formulate the following unified higher-order dual of (NP) and derive duality results:

$$\text{(GD)} \quad \max_{(s,t,\tilde{y}) \in K} \sup_{(z,u,\mu,p) \in H(s,t,\tilde{y})} \sum_{i=1}^{s} t_i \{ f(z, \bar{y}_i) + h(z, \bar{y}_i, p) - p^T \nabla_p h(z, \bar{y}_i, p) \}$$
$$+ z^T Bu + \sum_{j \in J_\circ} \{ \mu_j g_j(z) + \mu_j k_j(z, p) - p^T \nabla_p (\mu_j k_j(z, p)) \},$$

where $H(s, t, \tilde{y})$ denotes the set of all $(z, u, \mu, p) \in R^n \times R^n \times R^m_+ \times R^n$ satisfying

$$\sum_{i=1}^{s} t_i \nabla_p h(z, \bar{y}_i, p) + Bu + \sum_{j=1}^{m} \nabla_p (\mu_j k_j(z, p)) = 0, \tag{1}$$

$$\sum_{j \in J_\beta} \{ \mu_j g_j(z) + \mu_j k_j(z, p) - p^T \nabla_p (\mu_j k_j(z, p)) \} \geq 0, \quad \beta = 1, 2, \dots, r, \tag{2}$$

$$u^T Bu \leq 1, \tag{3}$$

where $J_\beta \subseteq J = \{1, 2, \dots, m\}$, $\beta = 0, 1, 2, \dots, r$, with $J_\beta \cap J_\gamma = \emptyset$ if $\beta \neq \gamma$, $\bigcup_{\beta=0}^r J_\beta = J$. If, for a triplet $(s, t, \tilde{y}) \in K$, the set $H(s, t, \tilde{y}) = \emptyset$, then we define the supremum over it to be $-\infty$.

Remark 3.1 Let $h(z, \bar{y}_i, p) = p^T \nabla f(z, \bar{y}_i) + \frac{1}{2} p^T \nabla^2 f(z, \bar{y}_i) p$, $i = 1, 2, \dots, s$, and $k_j(z, p) = p^T \nabla g_j(z) + \frac{1}{2} p^T \nabla^2 g_j(z) p$, $j \in J$. Then (GD) reduces to the second-order dual (Refs. [18, 24]). If, in addition, $B = 0$, then we get the dual formulated by Liu (Ref. [16]).

Theorem 3.1 (Weak Duality) *Let x and $(z, u, \mu, s, t, \tilde{y}, p)$ be the feasible solutions of (NP) and (GD), respectively. Suppose that*

$$\left[\sum_{i=1}^s t_i f(\cdot, \bar{y}_i) + (\cdot)^T Bu + \sum_{j \in J_\circ} \mu_j g_j(\cdot), \sum_{j \in J_\beta} \mu_j g_j(\cdot), \beta = 1, 2, \dots, r\right]$$

is higher-order (F, α, ρ, d)-pseudoquasi Type I at z and

$$\left(\frac{\rho_1^1}{\alpha^1(x, z)} + \frac{\sum_{\beta=1}^r \rho_\beta^2}{\alpha^2(x, z)}\right) \geq 0.$$

Then

$$\begin{aligned} &\sup_{y \in Y} f(x, y) + (x^T Bx)^{1/2} \\ &\quad \geq \sum_{i=1}^s t_i \{f(z, \bar{y}_i) + h(z, \bar{y}_i, p) - p^T \nabla_p h(z, \bar{y}_i, p)\} + z^T Bu \\ &\qquad + \sum_{j \in J_\circ} \{\mu_j g_j(z) + \mu_j k_j(z, p) - p^T \nabla_p (\mu_j k_j(z, p))\}. \end{aligned}$$

Proof Suppose, contrary to the result, that

$$\begin{aligned} &\sup_{y \in Y} f(x, y) + (x^T Bx)^{1/2} \\ &\quad < \sum_{i=1}^s t_i \{f(z, \bar{y}_i) + h(z, \bar{y}_i, p) - p^T \nabla_p h(z, \bar{y}_i, p)\} + z^T Bu \\ &\qquad + \sum_{j \in J_\circ} \{\mu_j g_j(z) + \mu_j k_j(z, p) - p^T \nabla_p (\mu_j k_j(z, p))\}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} &f(x, \bar{y}_i) + (x^T Bx)^{1/2} \\ &\quad < \sum_{i=1}^s t_i \{f(z, \bar{y}_i) + h(z, \bar{y}_i, p) - p^T \nabla_p h(z, \bar{y}_i, p)\} + z^T Bu \end{aligned}$$

$$+\sum_{j\in J_\circ}\{\mu_j g_j(z)+\mu_j k_j(z,p)-p^T\nabla_p(\mu_j k_j(z,p))\},$$

for all $\bar{y}_i \in Y(x), i = 1, 2, \dots, s$.

It follows from $t_i \geq 0, i = 1, 2, \dots, s$, that

$$t_i\Bigg[\left(f(x,\bar{y}_i)+(x^TBx)^{1/2}\right)-\Bigg(\sum_{i=1}^{s}t_i\{f(z,\bar{y}_i)+h(z,\bar{y}_i,p)-p^T\nabla_p h(z,\bar{y}_i,p)\}$$
$$+z^TBu+\sum_{j\in J_\circ}\{\mu_j g_j(z)+\mu_j k_j(z,p)-p^T\nabla_p(\mu_j k_j(z,p))\}\Bigg)\Bigg]\leq 0,$$
$$i=1,2,\dots,s,$$

with at least one strict inequality, since $t = (t_1, t_2, \dots, t_s) \neq 0$. Taking summation over i and using $\sum_{i=1}^{s} t_i = 1$, we have

$$\sum_{i=1}^{s}t_i f(x,\bar{y}_i)+(x^TBx)^{1/2}$$
$$<\sum_{i=1}^{s}t_i\{f(z,\bar{y}_i)+h(z,\bar{y}_i,p)-p^T\nabla_p h(z,\bar{y}_i,p)\}+z^TBu$$
$$+\sum_{j\in J_\circ}\{\mu_j g_j(z)+\mu_j k_j(z,p)-p^T\nabla_p(\mu_j k_j(z,p))\},$$

which, by the feasibility of x for (NP) and $\mu \in R_+^m$, gives

$$\sum_{i=1}^{s}t_i f(x,\bar{y}_i)+(x^TBx)^{1/2}+\sum_{j\in J_\circ}\mu_j g_j(x)$$
$$<\sum_{i=1}^{s}t_i\{f(z,\bar{y}_i)+h(z,\bar{y}_i,p)-p^T\nabla_p h(z,\bar{y}_i,p)\}$$
$$+z^TBu+\sum_{j\in J_\circ}\{\mu_j g_j(z)+\mu_j k_j(z,p)-p^T\nabla_p(\mu_j k_j(z,p))\}.$$

It follows from Lemma 2.1 and (3) that

$$\sum_{i=1}^{s}t_i f(x,\bar{y}_i)+x^TBu+\sum_{j\in J_\circ}\mu_j g_j(x)$$
$$<\sum_{i=1}^{s}t_i\{f(z,\bar{y}_i)+h(z,\bar{y}_i,p)-p^T\nabla_p h(z,\bar{y}_i,p)\}+z^TBu$$
$$+\sum_{j\in J_\circ}\{\mu_j g_j(z)+\mu_j k_j(z,p)-p^T\nabla_p(\mu_j k_j(z,p))\}. \tag{4}$$

Also, from (2), we have

$$-\sum_{j\in J_\beta}\{\mu_j g_j(z)+\mu_j k_j(z,p)-p^T\nabla_p(\mu_j k_j(z,p))\}\leq 0,\quad \beta=1,2,\ldots,r. \tag{5}$$

The higher-order (F,α,ρ,d)-pseudoquasi Type I assumption on

$$\left[\sum_{i=1}^{s} t_i f(\cdot,\bar{y}_i)+(\cdot)^T Bu+\sum_{j\in J_\circ}\mu_j g_j(\cdot),\ \sum_{j\in J_\beta}\mu_j g_j(\cdot),\quad \beta=1,2,\ldots,r\right]$$

at z, with (4) and (5), implies

$$F\left(x,z;\alpha^1(x,z)\left\{\sum_{i=1}^{s} t_i\nabla_p h(z,\bar{y}_i,p)+Bu+\sum_{j\in J_\circ}\nabla_p(\mu_j k_j(z,p))\right\}\right)$$
$$<-\rho_1^1 d^2(x,z),$$
$$F\left(x,z;\alpha^2(x,z)\sum_{j\in J_\beta}\nabla_p(\mu_j k_j(z,p))\right)\leq -\rho_\beta^2 d^2(x,z),\quad \beta=1,2,\ldots,r.$$

By using $\alpha^1(x,z)>0$, $\alpha^2(x,z)>0$, and the sublinearity of F in the above inequalities, we summarize to get

$$F\left(x,z;\sum_{i=1}^{s} t_i\nabla_p h(z,\bar{y}_i,p)+Bu+\sum_{j=1}^{m}\nabla_p(\mu_j k_j(z,p))\right)$$
$$<-\left(\frac{\rho_1^1}{\alpha^1(x,z)}+\frac{\sum_{\beta=1}^{r}\rho_\beta^2}{\alpha^2(x,z)}\right)d^2(x,z). \tag{6}$$

Since

$$\left(\frac{\rho_1^1}{\alpha^1(x,z)}+\frac{\sum_{\beta=1}^{r}\rho_\beta^2}{\alpha^2(x,z)}\right)\geq 0,$$

inequality (6) yields

$$F\left(x,z;\sum_{i=1}^{s} t_i\nabla_p h(z,\bar{y}_i,p)+Bu+\sum_{j=1}^{m}\nabla_p(\mu_j k_j(z,p))\right)<0,$$

which contradicts (1), as $F(x,z;0)=0$. □

Theorem 3.2 (Strong Duality) *Let x^* be an optimal solution of* (NP) *and let $\nabla g_j(x^*)$, $j\in J(x^*)$ be linearly independent. Assume that*

$$\begin{aligned} &h(x^*,\bar{y}_i^*,0)=0, \qquad \nabla_p h(x^*,\bar{y}_i^*,0)=\nabla f(x^*,\bar{y}_i^*),\quad i=1,2,\ldots,s,\\ &k_j(x^*,0)=0, \qquad \nabla_p k_j(x^*,0)=\nabla g_j(x^*),\quad j\in J. \end{aligned} \tag{7}$$

Then there exist $(s^*, t^*, \tilde{y}^*) \in K$ *and* $(x^*, u^*, \mu^*, p^*) \in H(s^*, t^*, \tilde{y}^*)$ *such that* $(x^*, u^*, \mu^*, s^*, t^*, \tilde{y}^*, p^* = 0)$ *is a feasible solution of* (GD) *and the two objectives have the same values. Furthermore, if the assumptions of weak duality* (*Theorem* 3.1) *hold for all feasible solutions of* (NP) *and* (GD), *then* $(x^*, u^*, \mu^*, s^*, t^*, \tilde{y}^*, p^* = 0)$ *is an optimal solution of* (GD).

Proof Since x^* is an optimal solution of (NP) and $\nabla g_j(x^*),\ j \in J(x^*)$ are linearly independent, by Theorem 2.1, there exist $(s^*, t^*, \tilde{y}^*) \in K$ and $(x^*, u^*, \mu^*, p^*) \in H(s^*, t^*, \tilde{y}^*)$ such that

$$\sum_{i=1}^{s^*} t_i^* \nabla f(x^*, \bar{y}_i^*) + Bu^* + \sum_{j=1}^{m} \nabla \mu_j^* g_j(x^*) = 0, \tag{8}$$

$$\sum_{j=1}^{m} \mu_j^* g_j(x^*) = 0, \tag{9}$$

$$t_i^* \geq 0, \quad i = 1, 2, \ldots, s^*, \quad \sum_{i=1}^{s^*} t_i^* = 1, \tag{10}$$

$$u^{*T} B u^* \leq 1, \tag{11}$$

$$(x^{*T} B x^*)^{1/2} = x^{*T} B u^*. \tag{12}$$

Thus the relations (8)–(11) along with (7) imply that $(x^*, u^*, \mu^*, s^*, t^*, \tilde{y}^*, p^* = 0)$ is a feasible solution of (GD). Also (7), (9) and (12) with $p^* = 0$ show the equality of objective values. Optimality of $(x^*, u^*, \mu^*, s^*, t^*, \tilde{y}^*, p^* = 0)$ thus follows from weak duality (Theorem 3.1). □

Theorem 3.3 (Strict Converse Duality) *Let* x^* *and* $(z^*, u^*, \mu^*, s^*, t^*, \tilde{y}^*, p^*)$ *be the optimal solutions of* (NP) *and* (GD), *respectively. Suppose that*

$$\left[\sum_{i=1}^{s^*} t_i^* f(\cdot, \bar{y}_i^*) + (\cdot)^T B u^* + \sum_{j \in J_\circ} \mu_j^* g_j(\cdot),\ \sum_{j \in J_\beta} \mu_j^* g_j(\cdot),\ \beta = 1, 2, \ldots, r \right]$$

is higher-order (F, α, ρ, d)*-strictly pseudoquasi Type I at* z^* *with*

$$\left(\frac{\rho_1^1}{\alpha^1(x^*, z^*)} + \frac{\sum_{\beta=1}^{r} \rho_\beta^2}{\alpha^2(x^*, z^*)} \right) \geq 0,$$

and that $\nabla g_j(x^*),\ j \in J(x^*)$, *are linearly independent. Then,* $z^* = x^*$; *that is,* z^* *is an optimal solution of* (NP).

Proof Suppose to the contrary that $z^* \neq x^*$ to exhibit a contradiction. Since x^* and $(z^*, u^*, \mu^*, s^*, t^*, \tilde{y}^*, p^*)$ are the optimal solutions of (NP) and (GD), respectively, and since $\nabla g_j(x^*)$, $j \in J(x^*)$ are linearly independent, therefore from the strong duality (Theorem 3.2), we obtain

$$\begin{aligned}
&\sup_{y^* \in Y} f(x^*, y^*) + (x^{*T} B x^*)^{1/2} \\
&\quad = \sum_{i=1}^{s^*} t_i^* \{ f(z^*, \bar{y}_i^*) + h(z^*, \bar{y}_i^*, p^*) - p^{*T} \nabla_p h(z^*, \bar{y}_i^*, p^*) \} \\
&\qquad + z^{*T} B u^* + \sum_{j \in J_\circ} \{ \mu_j^* g_j(z^*) + \mu_j^* k_j(z^*, p^*) - p^{*T} \nabla_p (\mu_j^* k_j(z^*, p^*)) \}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&f(x^*, \bar{y}_i^*) + (x^{*T} B x^*)^{1/2} \\
&\quad \le \sum_{i=1}^{s^*} t_i^* \{ f(z^*, \bar{y}_i^*) + h(z^*, \bar{y}_i^*, p^*) - p^{*T} \nabla_p h(z^*, \bar{y}_i^*, p^*) \} \\
&\qquad + z^{*T} B u^* + \sum_{j \in J_\circ} \{ \mu_j^* g_j(z^*) + \mu_j^* k_j(z^*, p^*) - p^{*T} \nabla_p (\mu_j^* k_j(z^*, p^*)) \},
\end{aligned}$$

for all $\bar{y}_i^* \in Y(x^*)$, $i = 1, 2, \ldots, s^*$.

Now, proceeding as in Theorem 3.1, we get

$$\begin{aligned}
&\sum_{i=1}^{s^*} t_i^* f(x^*, \bar{y}_i^*) + x^{*T} B u^* + \sum_{j \in J_\circ} \mu_j^* g_j(x^*) \\
&\quad < \sum_{i=1}^{s^*} t_i^* \{ f(z^*, \bar{y}_i^*) + h(z^*, \bar{y}_i^*, p^*) - p^{*T} \nabla_p h(z^*, \bar{y}_i^*, p^*) \} + z^{*T} B u^* \\
&\qquad + \sum_{j \in J_\circ} \{ \mu_j^* g_j(z^*) + \mu_j^* k_j(z^*, p^*) - p^{*T} \nabla_p (\mu_j^* k_j(z^*, p^*)) \}. \qquad (13)
\end{aligned}$$

From (2), we have

$$-\sum_{j \in J_\beta} \{ \mu_j^* g_j(z^*) + \mu_j^* k_j(z^*, p^*) - p^{*T} \nabla_p (\mu_j^* k_j(z^*, p^*)) \} \le 0, \quad \beta = 1, 2, \ldots, r,$$

which by the second part of higher-order (F, α, ρ, d)-strictlypseudoquasi Type I assumption on

$$\left[\sum_{i=1}^{s^*} t_i^* f(\cdot, \bar{y}_i^*) + (\cdot)^T B u^* + \sum_{j \in J_\circ} \mu_j^* g_j(\cdot),\ \sum_{j \in J_\beta} \mu_j^* g_j(\cdot),\ \beta = 1, 2, \ldots, r \right]$$

at z^* gives

$$F\left(x^*, z^*; \alpha^2(x^*, z^*) \sum_{j\in J_\beta} \nabla_p(\mu_j^* k_j(z^*, p^*))\right) \leq -\rho_\beta^2 d^2(x^*, z^*), \quad \beta = 1, 2, \ldots, r.$$

As $\alpha^2(x^*, z^*) > 0$ and as F is sublinear, it follows that

$$F\left(x^*, z^*; \sum_{j\in J_\beta} \nabla_p(\mu_j^* k_j(z^*, p^*))\right) \leq -\frac{\rho_\beta^2}{\alpha^2(x^*, z^*)} d^2(x^*, z^*), \quad \beta = 1, 2, \ldots, r. \tag{14}$$

From the first dual constraint, (14) and the sublinearity of F, we have

$$F\left(x^*, z^*; \sum_{i=1}^{s^*} t_i^* \nabla_p h(z^*, \bar{y}_i^*, p^*) + Bu^* + \sum_{j\in J_\circ} \nabla_p(\mu_j^* k_j(z^*, p^*))\right)$$
$$\geq \frac{\sum_{\beta=1}^r \rho_\beta^2}{\alpha^2(x^*, z^*)} d^2(x^*, z^*).$$

In view of

$$\left(\frac{\rho_1^1}{\alpha^1(x^*, z^*)} + \frac{\sum_{\beta=1}^r \rho_\beta^2}{\alpha^2(x^*, z^*)}\right) \geq 0,$$

$\alpha^1(x^*, z^*) > 0$ and the sublinearity of F, the above inequality becomes

$$F\left(x^*, z^*; \alpha^1(x^*, z^*)\left\{\sum_{i=1}^{s^*} t_i^* \nabla_p h(z^*, \bar{y}_i^*, p^*) + Bu^* + \sum_{j\in J_\circ} \nabla_p(\mu_j^* k_j(z^*, p^*))\right\}\right)$$
$$\geq -\rho_1^1 d^2(x^*, z^*).$$

On using the first part of the said assumption imposed on

$$\left[\sum_{i=1}^{s^*} t_i^* f(\cdot, \bar{y}_i^*) + (\cdot)^T Bu^* + \sum_{j\in J_\circ} \mu_j^* g_j(\cdot), \sum_{j\in J_\beta} \mu_j^* g_j(\cdot),\ \beta = 1, 2, \ldots, r\right]$$

at z^*, it follows that

$$\sum_{i=1}^{s^*} t_i^* f(x^*, \bar{y}_i^*) + x^{*T} Bu^* + \sum_{j\in J_\circ} \mu_j^* g_j(x^*)$$
$$> \sum_{i=1}^{s^*} t_i^* [f(z^*, \bar{y}_i^*) + h(z^*, \bar{y}_i^*, p^*) - p^{*T} \nabla_p h(z^*, \bar{y}_i^*, p^*)]$$
$$+ z^{*T} Bu^* + \sum_{j\in J_\circ} [\mu_j^* g_j(z^*) + \mu_j^* k_j(z^*, p^*) - p^{*T} \nabla_p(\mu_j^* k_j(z^*, p^*))],$$

which is a contradiction to (13). Hence, $z^* = x^*$. □

Remark 3.2

(i) If we take $h(z,\bar{y}_i,p)=p^T\nabla f(z,\bar{y}_i)+\frac{1}{2}p^T\nabla^2 f(z,\bar{y}_i)p,\ i=1,2,\ldots,s,$ and $k_j(z,p)=p^T\nabla g_j(z)+\frac{1}{2}p^T\nabla^2 g_j(z)p,\ j\in J$ in Theorems 3.1–3.3, then we get Theorems 4.1–4.3 discussed by Ahmad, Husain and Sharma (Ref. [24]). If, in addition, $\alpha^1(x,\bar{x})=1=\alpha^2(x,\bar{x}),\ \rho_1^1=0=\rho_\beta^2,\ \beta=1,2,\ldots,r$ and $F(x,z;a)=\eta^T(x,z)a$ for a certain mapping $\eta: S\times X\to R^n$, we obtain Theorems 3.1–3.3 in (Ref. [18]).

(ii) If we put $h(z,\bar{y}_i,p)=p^T\nabla f(z,\bar{y}_i)+\frac{1}{2}p^T\nabla^2 f(z,\bar{y}_i)p,\quad i=1,2,\ldots,s,$ $k_j(z,p)=p^T\nabla g_j(z)+\frac{1}{2}p^T\nabla^2 g_j(z)p,\ j\in J$, and $J_\beta=\phi,\ \beta=1,2,\ldots,r$ in Theorems 3.1–3.3, then we obtain Theorems 3.1–3.3 in Ref. [24].

4 Further Development

The results appeared in this paper can be further generalized to the following related classes of nondifferentiable minimax programming problems:

$$\text{(P1)}\qquad \operatorname{Min}\sup_{y\in Y}\frac{\phi(x,y)+(x^TBx)^{1/2}}{\psi(x,y)-(x^TDx)^{1/2}},$$

$$\text{s.t.}\quad g(x)\le 0,\quad x\in R^n,$$

$$\text{(P2)}\qquad \operatorname{Min}\sup_{\nu\in W}\frac{\operatorname{Re}[\phi(\xi,\nu)+(z^HBz)^{1/2}]}{\operatorname{Re}[\psi(\xi,\nu)-(z^HDz)^{1/2}]},$$

$$\text{s.t.}\quad -g(\xi)\in S^\circ,\quad \xi\in C^{2n},$$

where $\xi=(z,\bar{z}),\ \nu=(\omega,\bar{\omega})$ for $z\in C^n,\ \omega\in C^l.\phi(\cdot,\cdot): C^{2n}\times C^{2l}\to C$ and $\psi(\cdot,\cdot): C^{2n}\times C^{2l}\to C$ are analytic with respect to ξ, W is a specified compact subset in C^{2l}, S° is a polyhedral cone in C^m and $g: C^{2n}\to C^m$ is analytic. Also $B, D\in C^{n\times n}$ are positive semidefinite Hermitian matrices.

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