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Sufficiency and Duality in Multiobjective Programming under Generalized Type I Functions

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Abstract In this paper, new classes of generalized (F, α, ρ, d)-V-type I functions are introduced for differentiable multiobjective programming problems. Based upon these generalized convex functions, sufficient optimality conditions are established. Weak, strong and strict converse duality theorems are also derived for Wolfe and Mond-Weir type multiobjective dual programs.

Keywords Multiobjective programming · Generalized type I functions · Weak Efficiency · Sufficiency · Duality

1 Introduction

In 1981, Hanson [1] introduced the concept of invexity and established Karush-Kuhn-Tucker type sufficient optimality conditions for a nonlinear programming problem. Later, Hanson and Mond [2] defined two new classes of functions, called type I and type II functions in nonlinear programming, which were further generalized to pseudo-type I and quasi-type I functions by Rueda and Hanson [3]. Both classes are related to, but more general than, invex functions. In [4], Kaul et al. considered a differentiable multiobjective programming problem involving generalized type I functions. They investigated Karush-Kuhn-Tucker type necessary and sufficient conditions and obtained duality results under generalized type I functions. Bector et al. [5] introduced the concept of univex functions as a generalization of B-vex functions

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introduced by Bector and Singh [6]. Combining the concepts of type I and univex functions, Rueda et al. [7] gave optimality conditions and duality results for several mathematical programming problems. Aghezzaf and Hachimi [8] introduced classes of generalized type I functions for a differentiable multiobjective programming problem and derived some Mond-Weir type duality results under the above generalized type I assumptions.

The concept of (F, ρ)-convexity was introduced by Preda [9] as an extension of F-convexity [10] and ρ-convexity [11]. Gulati and Islam [12] obtained sufficient optimality conditions and duality theorems for multiobjective programming problems under generalized F-convexity. In [13], Ahmad derived several sufficient optimality conditions and duality results involving generalized (F, ρ)-convex functions. Recently, combining the concepts of (F, α, ρ, d)-convexity [14] and generalized type I functions [2], Hachimi and Aghezzaf [15] gave sufficient optimality conditions and mixed type duality results for multiobjective programming problems. Other classes of generalized type I functions have been discussed in [16, 17].

In this paper, we will introduce new generalized classes of type I functions, called (F, α, ρ, d)-V-type I functions by combining the concepts of generalized (F, α, ρ, d)-type I functions [15] and generalized V-invex functions [18]. Sufficiency optimality conditions will be obtained for multiobjective programming problems involving generalized (F, α, ρ, d)-V-type I functions. Duality results will also be obtained by associating Wolfe and Mond-Weir type duals with the multiobjective programming problem.

2 Notations and Preliminaries

The following convention of vectors in R^n will be followed throughout this paper: $x \geqq y \Leftrightarrow x_i \geqq y_i,\ i = 1, 2, \dots, n;\ x \geq y \Leftrightarrow x \geqq y,\ x \neq y;\ x > y \Leftrightarrow x_i > y_i,\ i = 1, 2, \dots, n.$

We consider the following nonlinear multiobjective programming problem:

$$\text{(MP)} \qquad \begin{aligned} &\text{Min} \quad f(x) = (f_1(x), f_2(x), \dots, f_k(x)), \\ &\text{s.t.} \quad x \in X_\circ = \{x \in X : g(x) \leqq 0\}, \end{aligned}$$

where $X \subseteq R^n$ is an open set and the functions $f = (f_1, f_2, \dots, f_k) : X \mapsto R^k$ and $g = (g_1, g_2, \dots, g_m) : X \mapsto R^m$ are differentiable on X.

Since the objectives in multiobjective programming problems generally conflict with one another, an optimal solution is chosen from the set of efficient/weakly efficient solutions in the following sense [19].

Definition 2.1 A point $\bar{x} \in X_\circ$ is said to be an efficient solution of (MP) if there exists no $x \in X_\circ$ such that

$$f(x) \leq f(\bar{x}).$$

Definition 2.2 A point $\bar{x} \in X_\circ$ is said to be a weakly efficient solution of (MP), if there exists no $x \in X_\circ$ such that

$$f(x) < f(\bar{x}).$$

Let F be a sublinear functional and let $d : X \times X \mapsto R$ be a metric. We introduce the index sets $K = \{1, 2, \dots, k\}$ and $M = \{1, 2, \dots, m\}$. Also, for $\bar{x} \in X$, $J(\bar{x}) = \{j : g_j(\bar{x}) = 0\}$ and g_J denotes the vector of active constraints at $\bar{x}$.

Along the lines of Hachimi and Aghezzaf [15] and Jeyakumar and Mond [18], we now define the following classes of generalized convex functions.

Definition 2.3 (f, g) is said to be (F, α, ρ, d)-V-type I at $\bar{x} \in X$ if there exist vectors $\alpha = (\alpha_1^1, \dots, \alpha_k^1, \alpha_1^2, \dots, \alpha_m^2)$ and $\rho = (\rho_1^1, \dots, \rho_k^1, \rho_1^2, \dots, \rho_m^2)$, with $\alpha_i^1, \alpha_j^2 : X \times X \mapsto R_+ \setminus \{0\}$ and $\rho_i^1, \rho_j^2 \in R$ for $i \in K$, $j \in M$, such that, for each $x \in X_\circ$ and for all $i \in K$, $j \in M$,

$$f_i(x) - f_i(\bar{x}) \geqq F(x, \bar{x}; \alpha_i^1(x, \bar{x}) \nabla f_i(\bar{x})) + \rho_i^1 d^2(x, \bar{x}),$$
$$-g_j(\bar{x}) \geqq F(x, \bar{x}; \alpha_j^2(x, \bar{x}) \nabla g_j(\bar{x})) + \rho_j^2 d^2(x, \bar{x}).$$

If in the above definition the first inequality is satisfied as

$$f_i(x) - f_i(\bar{x}) > F(x, \bar{x}; \alpha_i^1(x, \bar{x}) \nabla f_i(\bar{x})) + \rho_i^1 d^2(x, \bar{x}),$$

then we say that (f, g) is semistrictly (F, α, ρ, d)-V-type I at $\bar{x}$.

Remark 2.1

(i) If $\alpha_i^1(x, \bar{x}) = \alpha^1(x, \bar{x})$ and $\alpha_j^2(x, \bar{x}) = \alpha^2(x, \bar{x})$ for $i \in K$, $j \in M$, the above definition become that of (F, α, ρ, d)-type I function introduced by Hachimi and Aghezzaf [15].

(ii) If $\rho_i^1 = \rho_j^2 = 0$ for $i \in K$, $j \in M$ and $F(x, \bar{x}; a) = a^T \eta(x, \bar{x})$, with $\eta : X \times X \mapsto R^n$, the inequalities become those of V-type I functions introduced by Hanson et al. [16].

(iii) If $\alpha_i^1(x, \bar{x}) = \alpha_j^2(x, \bar{x}) = 1$, $\rho_i^1 = \rho_j^2 = 0$ for $i \in K$, $j \in M$ and $F(x, \bar{x}; a) = a^T \eta(x, \bar{x})$, we obtain the definition of type I function given by Hanson and Mond [2].

Example 2.1 Consider the following multiobjective programming problem:

$$\begin{aligned} &\text{Min} \quad f(x_1, x_2) = (x_2(\pi - x_2)e^{\cos x_1}, \sin^2 x_1, x_1 + \cos x_2), \\ &\text{s.t.} \quad (x_1, x_2) \in X, \quad g_1 = \pi - 4\, x_1 \leqq 0, \quad g_2 = -\cos x_2 \leqq 0, \end{aligned}$$

where $X = \{(x_1, x_2) : 0 < x_1 < \frac{3\pi}{2}, 0 < x_2 < \frac{3\pi}{2}\}$, $f = (f_1, f_2, f_3) : X \mapsto R^3$, $g = (g_1, g_2) : X \mapsto R^2$.
The feasible region is $X_\circ = \{(x_1, x_2) : \frac{\pi}{4} \leqq x_1 < \frac{3\pi}{2}, \ 0 < x_2 \leqq \frac{\pi}{2}\}$.

It can be seen that (f, g) is (F, α, ρ, d)-V-type I at $\bar{x} = (\frac{\pi}{4}, \frac{\pi}{2}) \in X_\circ$ for $F(x, \bar{x}; a) = a^T(x - \bar{x}), d(x, \bar{x}) = \sqrt{(x_1 - \frac{\pi}{4})^2 + (x_2 - \frac{\pi}{2})^2}$, $\alpha = (\alpha_1^1, \alpha_2^1, \alpha_3^1, \alpha_1^2, \alpha_2^2)$, $\rho = (\rho_1^1, \rho_2^1, \rho_3^1, \rho_1^2, \rho_2^2)$, where $\alpha_1^1 = 1, \alpha_2^1 = \frac{1}{x_1 + \frac{\pi}{4}}, \alpha_3^1 = \frac{3}{2\pi}, \alpha_1^2 = 1, \alpha_2^2 = 1, \rho_1^1 = -\frac{5}{2}, \rho_2^1 = -\frac{1}{3}, \rho_3^1 = \frac{1}{15}, \rho_1^2 = 0, \rho_2^2 = 0$. But (f, g) is not type I and V-type I function introduced by Hanson and Mond [2] and Hanson et al. [16] respectively as can be verified by taking the feasible point $(\frac{\pi}{4}, \frac{\pi}{4})$ or $(\pi, \frac{\pi}{2})$. Also (f, g) is not (F, α, ρ, d)-type I function introduced by Hachimi and Aghezzaf [15]. For this, if $\alpha_1^1 = \alpha_2^1 = \alpha_3^1 = 1$, then the inequalities for $i = 2, 3$ are not satisfied at $(\frac{2\pi}{3}, \frac{\pi}{2}) \in X_\circ$. Also, if $\alpha_1^1 = \alpha_2^1 = \alpha_3^1 = \frac{1}{(x_1 + \frac{\pi}{4})}$ (or $\frac{3}{2\pi}$), then the inequality for $i = 3$ (or 2) is not satisfied at $(\frac{\pi}{4}, \frac{\pi}{30})$ (or $(\frac{5\pi}{6}, \frac{\pi}{2})$)$\in X_\circ$.

Definition 2.4 (f, g) is said to be quasi $(F, \bar{\alpha}, \bar{\rho}, d)$-$V$-type I at $\bar{x} \in X$ if there exist $\bar{\alpha} = (\bar{\alpha}_1^1, \ldots, \bar{\alpha}_k^1, \bar{\alpha}_1^2, \ldots, \bar{\alpha}_m^2)$ and $\bar{\rho} = (\bar{\rho}^1, \bar{\rho}^2) \in R^2$, with $\bar{\alpha}_i^1, \bar{\alpha}_j^2 : X \times X \mapsto R_+ \setminus \{0\}$ for $i \in K$, $j \in M$, such that, for each $x \in X_\circ$,

$$\sum_{i=1}^{k} \bar{\alpha}_i^1 f_i(x) \leqq \sum_{i=1}^{k} \bar{\alpha}_i^1 f_i(\bar{x}) \quad \Rightarrow \quad F\left(x, \bar{x}; \sum_{i=1}^{k} \nabla f_i(\bar{x})\right) \leqq -\bar{\rho}^1 d^2(x, \bar{x}),$$

$$-\sum_{j=1}^{m} \bar{\alpha}_j^2 g_j(\bar{x}) \leqq 0 \quad \Rightarrow \quad F\left(x, \bar{x}; \sum_{j=1}^{m} \nabla g_j(\bar{x})\right) \leqq -\bar{\rho}^2 d^2(x, \bar{x}).$$

Definition 2.5 (f, g) is said to be pseudo $(F, \overset{\star}{\alpha}, \overset{\star}{\rho}, d)$-$V$-type I at $\bar{x} \in X$ if there exist $\overset{\star}{\alpha} = (\overset{\star 1}{\alpha}_1, \ldots, \overset{\star 1}{\alpha}_k, \overset{\star 2}{\alpha}_1, \ldots, \overset{\star 2}{\alpha}_m)$ and $\overset{\star}{\rho} = (\overset{\star 1}{\rho}, \overset{\star 2}{\rho}) \in R^2$, with $\overset{\star 1}{\alpha}_i, \overset{\star 2}{\alpha}_j : X \times X \mapsto R_+ \setminus \{0\}$ for $i \in K$, $j \in M$, such that, for each $x \in X_\circ$,

$$F\left(x, \bar{x}; \sum_{i=1}^{k} \nabla f_i(\bar{x})\right) \geqq -\overset{\star 1}{\rho} d^2(x, \bar{x}) \quad \Rightarrow \quad \sum_{i=1}^{k} \overset{\star 1}{\alpha}_i f_i(x) \geqq \sum_{i=1}^{k} \overset{\star 1}{\alpha}_i f_i(\bar{x}),$$

$$F\left(x, \bar{x}; \sum_{j=1}^{m} \nabla g_j(\bar{x})\right) \geqq -\overset{\star 2}{\rho} d^2(x, \bar{x}) \quad \Rightarrow \quad -\sum_{j=1}^{m} \overset{\star 2}{\alpha}_j g_j(\bar{x}) \geqq 0.$$

Definition 2.6 (f, g) is said to be pseudoquasi $(F, \tilde{\alpha}, \tilde{\rho}, d)$-$V$-type I at $\bar{x} \in X$ if there exist $\tilde{\alpha} = (\tilde{\alpha}_1^1, \ldots, \tilde{\alpha}_k^1, \tilde{\alpha}_1^2, \ldots, \tilde{\alpha}_m^2)$ and $\tilde{\rho} = (\tilde{\rho}^1, \tilde{\rho}^2) \in R^2$, with $\tilde{\alpha}_i^1, \tilde{\alpha}_j^2 : X \times X \mapsto R_+ \setminus \{0\}$ for $i \in K$, $j \in M$, such that, for each $x \in X_\circ$,

$$\sum_{i=1}^{k} \tilde{\alpha}_i^1 f_i(x) < \sum_{i=1}^{k} \tilde{\alpha}_i^1 f_i(\bar{x}) \quad \Rightarrow \quad F\left(x, \bar{x}; \sum_{i=1}^{k} \nabla f_i(\bar{x})\right) < -\tilde{\rho}^1 d^2(x, \bar{x}),$$

$$-\sum_{j=1}^{m} \tilde{\alpha}_j^2 g_j(\bar{x}) \leqq 0 \quad \Rightarrow \quad F\left(x, \bar{x}; \sum_{j=1}^{m} \nabla g_j(\bar{x})\right) \leqq -\tilde{\rho}^2 d^2(x, \bar{x}).$$

If in the above definition the first inequality is satisfied as

$$\sum_{i=1}^{k} \tilde{\alpha}_i^1 f_i(x) \leqq \sum_{i=1}^{k} \tilde{\alpha}_i^1 f_i(\bar{x}) \quad \Rightarrow \quad F\left(x, \bar{x}; \sum_{i=1}^{k} \nabla f_i(\bar{x})\right) < -\tilde{\rho}^1 d^2(x, \bar{x}),$$

then we say that (f, g) is strictly pseudoquasi $(F, \tilde{\alpha}, \tilde{\rho}, d)$-$V$-type I at $\bar{x}$.

Definition 2.7 (f, g) is said to be quasipseudo $(F, \hat{\alpha}, \hat{\rho}, d)$-$V$-type I at $\bar{x} \in X$ if there exist $\hat{\alpha} = (\hat{\alpha}_1^1, \dots, \hat{\alpha}_k^1, \hat{\alpha}_1^2, \dots, \hat{\alpha}_m^2)$ and $\hat{\rho} = (\hat{\rho}^1, \hat{\rho}^2) \in R^2$, with $\hat{\alpha}_i^1, \hat{\alpha}_j^2 : X \times X \mapsto R_+ \setminus \{0\}$ for $i \in K$, $j \in M$, such that, for each $x \in X_\circ$,

$$F\left(x, \bar{x}; \sum_{i=1}^{k} \nabla f_i(\bar{x})\right) > -\hat{\rho}^1 d^2(x, \bar{x}) \quad \Rightarrow \quad \sum_{i=1}^{k} \hat{\alpha}_i^1 f_i(x) > \sum_{i=1}^{k} \hat{\alpha}_i^1 f_i(\bar{x}),$$

$$-\sum_{j=1}^{m} \hat{\alpha}_j^2 g_j(\bar{x}) < 0 \quad \Rightarrow \quad F\left(x, \bar{x}; \sum_{j=1}^{m} \nabla g_j(\bar{x})\right) < -\hat{\rho}^2 d^2(x, \bar{x}).$$

If in the above definition the second inequality is satisfied as

$$-\sum_{j=1}^{m} \hat{\alpha}_j^2 g_j(\bar{x}) \leqq 0 \quad \Rightarrow \quad F\left(x, \bar{x}; \sum_{j=1}^{m} \nabla g_j(\bar{x})\right) < -\hat{\rho}^2 d^2(x, \bar{x}),$$

then we say that (f, g) is quasistrictly pseudo $(F, \hat{\alpha}, \hat{\rho}, d)$-$V$-type I at $\bar{x}$.

Using the sublinearity of $F(x, \bar{x}; .)$, one can see easily from the above definitions that an (F, α, ρ, d)-V-type I function is both quasi (F, α, ρ, d)-V-type I (with $\bar{\alpha}_i^1 = \frac{1}{\alpha_i^1}, i \in K, \bar{\alpha}_j^2 = \frac{1}{\alpha_j^2}, j \in M$ and $\bar{\rho}^1 = \inf_{x \in X_\circ}\{\sum_{i=1}^{k} (\frac{1}{\alpha_i^1})\ \rho_i^1\}$, $\bar{\rho}^2 = \inf_{x \in X_\circ}\{\sum_{j=1}^{m}\ (\frac{1}{\alpha_j^2})\ \rho_j^2\}$) and pseudo (F, α, ρ, d)-V-type I (with $\overset{\star}{\alpha}{}_i^1 = \frac{1}{\alpha_i^1}, i \in K, \overset{\star}{\alpha}{}_j^2 = \frac{1}{\alpha_j^2}, j \in M$ and $\overset{\star}{\rho}{}^1 = \inf_{x \in X_\circ}\{\sum_{i=1}^{k} (\frac{1}{\alpha_i^1}) \rho_i^1\}$, $\overset{\star}{\rho}{}^2 = \inf_{x \in X_\circ}\{\sum_{j=1}^{m} (\frac{1}{\alpha_j^2})\ \rho_j^2\}$).

3 Sufficiency Conditions

We now establish several sufficiency theorems which enable a feasible solution of (MP) to be a weakly efficient solution of (MP). In this section and in Sect. 5, $f^{\bar{\lambda}}$ denotes the vector $(\bar{\lambda}_1 f_1, \bar{\lambda}_2 f_2, \dots, \bar{\lambda}_k f_k)$ and $g_J^{\bar{\mu}}$ denotes the vector whose components are $\bar{\mu}_j g_j$, $j \in J$.

Theorem 3.1 *Suppose that there exists a feasible solution* $\bar{x}$ *of* (MP) *and scalars* $\bar{\lambda}_i \geqq 0, i = 1, 2, \dots, k, \sum_{i=1}^{k} \bar{\lambda}_i = 1, \bar{\mu}_j \geqq 0, j \in J(\bar{x})$ *satisfying*

(i) $\sum_{i=1}^{k} \bar{\lambda}_i \nabla f_i(\bar{x}) + \sum_{j \in J(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x}) = 0,$

(ii) (f, g_J) *is* (F, α, ρ, d)-V*-type I at* $\bar{x}$,

(iii) $\sum_{i=1}^{k} \frac{\bar{\lambda}_i \rho_i^1}{\alpha_i^1(x,\bar{x})} + \sum_{j \in J(\bar{x})} \frac{\bar{\mu}_j \rho_j^2}{\alpha_j^2(x,\bar{x})} \geqq 0,$

then $\bar{x}$ *is a weakly efficient solution of* (MP).

Proof By hypothesis (ii), we have

$$f_i(x) - f_i(\bar{x}) \geqq F(x, \bar{x}; \alpha_i^1(x, \bar{x}) \nabla f_i(\bar{x})) + \rho_i^1 d^2(x, \bar{x}), \quad i \in K,$$

$$0 = -g_j(\bar{x}) \geqq F(x, \bar{x}; \alpha_j^2(x, \bar{x}) \nabla g_j(\bar{x})) + \rho_j^2 d^2(x, \bar{x}), \quad j \in J(\bar{x}).$$

As $\alpha_i^1(x, \bar{x}) > 0,\ i \in K$ and $\alpha_j^2(x, \bar{x}) > 0,\ j \in J(\bar{x})$, the above inequalities along with the sublinearity of F give

$$\frac{f_i(x)}{\alpha_i^1(x, \bar{x})} - \frac{f_i(\bar{x})}{\alpha_i^1(x, \bar{x})} - \frac{\rho_i^1 d^2(x, \bar{x})}{\alpha_i^1(x, \bar{x})} \geqq F(x, \bar{x}; \nabla f_i(\bar{x})), \quad i \in K,$$

$$\frac{-\rho_j^2 d^2(x, \bar{x})}{\alpha_j^2(x, \bar{x})} \geqq F(x, \bar{x}; \nabla g_j(\bar{x})), \quad j \in J(\bar{x}).$$

Since $\bar{\lambda}_i \geqq 0, i = 1, 2, \ldots, k, \sum_{i=1}^{k} \bar{\lambda}_i = 1, \bar{\mu}_j \geqq 0,\ j \in J(\bar{x})$, using hypothesis (i) and the sublinearity of F, we get

$$\sum_{i=1}^{k} \frac{\bar{\lambda}_i f_i(x)}{\alpha_i^1(x, \bar{x})} - \sum_{i=1}^{k} \frac{\bar{\lambda}_i f_i(\bar{x})}{\alpha_i^1(x, \bar{x})} - \sum_{i=1}^{k} \frac{\bar{\lambda}_i \rho_i^1 d^2(x, \bar{x})}{\alpha_i^1(x, \bar{x})} - \sum_{j \in J(\bar{x})} \frac{\bar{\mu}_j \rho_j^2 d^2(x, \bar{x})}{\alpha_j^2(x, \bar{x})}$$

$$\geqq F\left(x, \bar{x}; \sum_{i=1}^{k} \bar{\lambda}_i \nabla f_i(\bar{x})\right) + F\left(x, \bar{x}; \sum_{j \in J(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x})\right) \geqq 0.$$

Thus,

$$\sum_{i=1}^{k} \frac{\bar{\lambda}_i f_i(x)}{\alpha_i^1(x, \bar{x})} - \sum_{i=1}^{k} \frac{\bar{\lambda}_i f_i(\bar{x})}{\alpha_i^1(x, \bar{x})} \geqq \left(\sum_{i=1}^{k} \frac{\bar{\lambda}_i \rho_i^1}{\alpha_i^1(x, \bar{x})} + \sum_{j \in J(\bar{x})} \frac{\bar{\mu}_j \rho_j^2}{\alpha_j^2(x, \bar{x})}\right) d^2(x, \bar{x}).$$

Now, using hypothesis (iii), the above inequality becomes

$$\sum_{i=1}^{k} \frac{\bar{\lambda}_i f_i(x)}{\alpha_i^1(x, \bar{x})} \geqq \sum_{i=1}^{k} \frac{\bar{\lambda}_i f_i(\bar{x})}{\alpha_i^1(x, \bar{x})}. \tag{1}$$

If $\bar{x}$ is not a weakly efficient solution of (MP), then there exists a feasible solution x $(x \neq \bar{x})$ of (MP) such that

$$f_i(x) < f_i(\bar{x}), \quad i \in K.$$

Since $\bar{\lambda}_i \geqq 0$, $\sum_{i=1}^{k} \bar{\lambda}_i = 1$ and $\alpha_i^1(x,\bar{x}) > 0$, we have

$$\sum_{i=1}^{k} \frac{\bar{\lambda}_i f_i(x)}{\alpha_i^1(x,\bar{x})} < \sum_{i=1}^{k} \frac{\bar{\lambda}_i f_i(\bar{x})}{\alpha_i^1(x,\bar{x})},$$

which contradicts (1). Hence, $\bar{x}$ is a weakly efficient solution of (MP). □

Remark 3.1 For the multiobjective programming problem considered in Example 2.1 it has been shown that (f, g_J) is (F,α,ρ,d)-V-type I at $\bar{x} = (\frac{\pi}{4}, \frac{\pi}{2}) \in X_\circ$. It is easy to see that the hypotheses (i) and (iii) of Theorem 3.1 are also satisfied at this point for $\lambda_1 = 0, \lambda_2 = \frac{1}{18}, \lambda_3 = \frac{17}{18}, \mu_1 = \frac{1}{4}$ and $\mu_2 = \frac{17}{18}$. Hence, $(\frac{\pi}{4}, \frac{\pi}{2}) \in X_\circ$ is a weakly efficient solution.

Theorem 3.2 *Suppose that there exists a feasible solution $\bar{x}$ of* (MP) *and scalars $\bar{\lambda}_i \geqq 0, i = 1,2,\ldots,k$, $\sum_{i=1}^{k} \bar{\lambda}_i = 1$, $\bar{\mu}_j \geqq 0$, $j \in J(\bar{x})$ satisfying*

(i) $\sum_{i=1}^{k} \bar{\lambda}_i \nabla f_i(\bar{x}) + \sum_{j\in J(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x}) = 0$,
(ii) $(f^{\bar{\lambda}}, g_J^{\bar{\mu}})$ *is pseudoquasi* $(F,\tilde{\alpha},\tilde{\rho},d)$-$V$*-type I at* $\bar{x}$,
(iii) $\tilde{\rho}^1 + \tilde{\rho}^2 \geqq 0$.

Then, $\bar{x}$ *is a weakly efficient solution of* (MP).

Proof Suppose that $\bar{x}$ is not a weakly efficient solution of (MP). Then, there exists a feasible solution x $(x \neq \bar{x})$ of (MP) such that

$$f_i(x) < f_i(\bar{x}), \quad i \in K.$$

Using $\bar{\lambda}_i \geqq 0$, $\sum_{i=1}^{k} \bar{\lambda}_i = 1$ and $\tilde{\alpha}_i^1(x,\bar{x}) > 0$, for $i \in K$, we get

$$\sum_{i=1}^{k} \tilde{\alpha}_i^1(x,\bar{x})\bar{\lambda}_i f_i(x) < \sum_{i=1}^{k} \tilde{\alpha}_i^1(x,\bar{x})\bar{\lambda}_i f_i(\bar{x}). \tag{2}$$

Also, $g_j(\bar{x}) = 0$, $j \in J(\bar{x})$ yield

$$\sum_{j\in J(\bar{x})} \tilde{\alpha}_j^2(x,\bar{x})\bar{\mu}_j g_j(\bar{x}) = 0. \tag{3}$$

By hypothesis (ii), inequalities (2) and (3) imply

$$F\left(x,\bar{x}; \sum_{i=1}^{k} \bar{\lambda}_i \nabla f_i(\bar{x})\right) < -\tilde{\rho}^1 d^2(x,\bar{x}),$$

$$F\left(x, \bar{x}; \sum_{j \in J(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x})\right) \leqq -\tilde{\rho}^2 d^2(x, \bar{x}).$$

Summing the above two inequalities and using the sublinearity of F, we obtain

$$\begin{aligned} &F\left(x, \bar{x}; \sum_{i=1}^{k} \bar{\lambda}_i \nabla f_i(\bar{x}) + \sum_{j \in J(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x})\right) \\ &\leqq F\left(x, \bar{x}; \sum_{i=1}^{k} \bar{\lambda}_i \nabla f_i(\bar{x})\right) + F\left(x, \bar{x}; \sum_{j \in J(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x})\right) \\ &< -(\tilde{\rho}^1 + \tilde{\rho}^2)\, d^2(x, \bar{x}). \end{aligned}$$

Using hypothesis (iii), we have

$$F\left(x, \bar{x}; \sum_{i=1}^{k} \bar{\lambda}_i \nabla f_i(\bar{x}) + \sum_{j \in J(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x})\right) < 0,$$

which in view of hypothesis (i) contradicts $F(x, \bar{x}; 0) = 0$. Hence, $\bar{x}$ is a weakly efficient solution of (MP). □

Remark 3.2 If the pseudoquasi-type I assumption in the above theorem is replaced by the strictly pseudoquasi-type I assumption, we get the stronger conclusion that $\bar{x}$ is an efficient solution of (MP). This result is stated below. The proof follows on the same lines.

Theorem 3.3 *Suppose that there exists a feasible solution $\bar{x}$ of* (MP) *and scalars $\bar{\lambda}_i \geqq 0, i = 1, 2, \ldots, k, \sum_{i=1}^{k} \bar{\lambda}_i = 1, \bar{\mu}_j \geqq 0,\ j \in J(\bar{x})$ satisfying*

(i) $\sum_{i=1}^{k} \bar{\lambda}_i \nabla f_i(\bar{x}) + \sum_{j \in J(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x}) = 0,$
(ii) $(f^{\bar{\lambda}}, g_J^{\bar{\mu}})$ *is strictly pseudoquasi* $(F, \tilde{\alpha}, \tilde{\rho}, d)$*-V-type I at* $\bar{x}$,
(iii) $\tilde{\rho}^1 + \tilde{\rho}^2 \geqq 0.$

Then, $\bar{x}$ *is an efficient solution of* (MP).

Theorem 3.4 *Suppose that there exists a feasible solution $\bar{x}$ of* (MP) *and scalars $\bar{\lambda}_i \geqq 0, i = 1, 2, \ldots, k, \sum_{i=1}^{k} \bar{\lambda}_i = 1, \bar{\mu}_j \geqq 0,\ j \in J(\bar{x})$ satisfying*

(i) $\sum_{i=1}^{k} \bar{\lambda}_i \nabla f_i(\bar{x}) + \sum_{j \in J(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x}) = 0,$
(ii) $(f^{\bar{\lambda}}, g_J^{\bar{\mu}})$ *is quasistrictly pseudo* $(F, \hat{\alpha}, \hat{\rho}, d)$*-V-type I at* $\bar{x}$,
(iii) $\hat{\rho}^1 + \hat{\rho}^2 \geqq 0.$

Then, $\bar{x}$ *is a weakly efficient solution of* (MP).

Proof Suppose that $\bar{x}$ is not a weakly efficient solution of (MP). Then, following the proof of Theorem 3.2, we obtain

$$\sum_{i=1}^{k} \hat{\alpha}_i^1(x,\bar{x})\bar{\lambda}_i f_i(x) < \sum_{i=1}^{k} \hat{\alpha}_i^1(x,\bar{x})\bar{\lambda}_i f_i(\bar{x}), \tag{4}$$

$$\sum_{j\in J(\bar{x})} \hat{\alpha}_j^2(x,\bar{x})\bar{\mu}_j g_j(\bar{x}) = 0. \tag{5}$$

As hypothesis (ii) holds, we have

$$F\left(x,\bar{x};\sum_{i=1}^{k}\bar{\lambda}_i\nabla f_i(\bar{x})\right) > -\hat{\rho}^1 d^2(x,\bar{x})$$

$$\Rightarrow \quad \sum_{i=1}^{k} \hat{\alpha}_i^1(x,\bar{x})\bar{\lambda}_i f_i(x) > \sum_{i=1}^{k} \hat{\alpha}_i^1(x,\bar{x})\bar{\lambda}_i f_i(\bar{x}), \tag{6}$$

$$-\sum_{j\in J(\bar{x})} \hat{\alpha}_j^2(x,\bar{x})\bar{\mu}_j g_j(\bar{x}) \leqq 0$$

$$\Rightarrow \quad F\left(x,\bar{x};\sum_{j\in J(\bar{x})}\bar{\mu}_j\nabla g_j(\bar{x})\right) < -\hat{\rho}^2 d^2(x,\bar{x}). \tag{7}$$

In view of (7), (5) implies

$$F\left(x,\bar{x};\sum_{j\in J(\bar{x})}\bar{\mu}_j\nabla g_j(\bar{x})\right) < -\hat{\rho}^2 d^2(x,\bar{x}). \tag{8}$$

The hypothesis (i), along with the sublinearity of F, yields

$$\begin{aligned} 0 &= F\left(x,\bar{x};\sum_{i=1}^{k}\bar{\lambda}_i\nabla f_i(\bar{x}) + \sum_{j\in J(\bar{x})}\bar{\mu}_j\nabla g_j(\bar{x})\right) \\ &\leqq F\left(x,\bar{x};\sum_{i=1}^{k}\bar{\lambda}_i\nabla f_i(\bar{x})\right) + F\left(x,\bar{x};\sum_{j\in J(\bar{x})}\bar{\mu}_j\nabla g_j(\bar{x})\right). \end{aligned}$$

Hence

$$\begin{aligned} &F\left(x,\bar{x};\sum_{i=1}^{k}\bar{\lambda}_i\nabla f_i(\bar{x})\right) \\ &\quad \geqq -F\left(x,\bar{x};\sum_{j\in J(\bar{x})}\bar{\mu}_j\nabla g_j(\bar{x})\right) > \hat{\rho}^2 d^2(x,\bar{x}) \geqq -\hat{\rho}^1 d^2(x,\bar{x}), \end{aligned}$$

(using (8) and hypothesis (iii)). This together with (6) implies

$$\sum_{i=1}^{k}\hat{\alpha}_i^1(x,\bar{x})\bar{\lambda}_i f_i(x) > \sum_{i=1}^{k}\hat{\alpha}_i^1(x,\bar{x})\bar{\lambda}_i f_i(\bar{x}),$$

contradicting (4). Hence, $\bar{x}$ is a weakly efficient solution of (MP). □

4 Wolfe Type Duality

In this section, we consider the following Wolfe type dual for (MP) and establish weak, strong and strict converse duality theorems.

$$\text{(WD)} \qquad \text{Max} \quad f(y) + \sum_{j=1}^{m}\mu_j g_j(y)e,$$

$$\text{s.t.} \quad y \in X,$$

$$\sum_{i=1}^{k}\lambda_i \nabla f_i(y) + \sum_{j=1}^{m}\mu_j \nabla g_j(y) = 0, \tag{9}$$

$$\lambda_i \geqq 0, \quad i = 1, 2, \ldots, k, \tag{10}$$

$$\mu_j \geqq 0, \quad j = 1, 2, \ldots, m, \tag{11}$$

$$\sum_{i=1}^{k}\lambda_i = 1, \tag{12}$$

where e is a k-dimensional vector whose all components are all ones.

Theorem 4.1 (Weak Duality) *Let x and (y,λ,μ) be feasible solutions of* (MP) *and* (WD) *respectively and let*

(i) *(f,g) be (F,α,ρ,d)-V-type I at y,*
(ii) *$\sum_{i=1}^{k}\frac{\lambda_i}{\alpha_i^1(x,y)} = 1$ and $\alpha_j^2(x,y) = 1$ for $j \in M$,*
(iii) *$\sum_{i=1}^{k}\frac{\lambda_i\rho_i^1}{\alpha_i^1(x,y)} + \sum_{j=1}^{m}\mu_j\rho_j^2 \geqq 0$.*

Then, the following inequality cannot hold:

$$f(x) < f(y) + \sum_{j=1}^{m}\mu_j g_j(y)e. \tag{13}$$

Proof Suppose that (13) holds. By (10), (12) and hypothesis (ii), we have

$$\sum_{i=1}^{k}\frac{\lambda_i f_i(x)}{\alpha_i^1(x,y)} < \sum_{i=1}^{k}\frac{\lambda_i f_i(y)}{\alpha_i^1(x,y)} + \sum_{j=1}^{m}\mu_j g_j(y). \tag{14}$$

Also, hypothesis (i) yields

$$f_i(x) - f_i(y) \geqq F(x, y; \alpha_i^1(x, y)\nabla f_i(y)) + \rho_i^1 d^2(x, y), \tag{15}$$

$$-g_j(y) \geqq F(x, y; \alpha_j^2(x, y)\nabla g_j(y)) + \rho_j^2 d^2(x, y). \tag{16}$$

Multiplying (15) by $\frac{\lambda_i}{\alpha_i^1(x,y)} \geqq 0$, $i \in K$, and (16) by $\mu_j \geqq 0$, $j \in M$, summing over all i and j and using $\alpha_j^2(x, y) = 1$ for $j \in M$, we obtain

$$\sum_{i=1}^k \frac{\lambda_i f_i(x)}{\alpha_i^1(x, y)} - \sum_{i=1}^k \frac{\lambda_i f_i(y)}{\alpha_i^1(x, y)} \geqq F\left(x, y; \sum_{i=1}^k \lambda_i \nabla f_i(y)\right) + \sum_{i=1}^k \frac{\lambda_i \rho_i^1 d^2(x, y)}{\alpha_i^1(x, y)},$$

$$-\sum_{j=1}^m \mu_j g_j(y) \geqq F\left(x, y; \sum_{j=1}^m \mu_j \nabla g_j(y)\right) + \sum_{j=1}^m \mu_j \rho_j^2 d^2(x, y).$$

Adding the above two inequalities and using the sublinearity of F along with (9), we get

$$\sum_{i=1}^k \frac{\lambda_i f_i(x)}{\alpha_i^1(x, y)} - \sum_{i=1}^k \frac{\lambda_i f_i(y)}{\alpha_i^1(x, y)} - \sum_{j=1}^m \mu_j g_j(y) \geqq \left(\sum_{i=1}^k \frac{\lambda_i \rho_i^1}{\alpha_i^1(x, y)} + \sum_{j=1}^m \mu_j \rho_j^2\right) d^2(x, y),$$

which by virtue of hypothesis (iii) implies

$$\sum_{i=1}^k \frac{\lambda_i f_i(x)}{\alpha_i^1(x, y)} \geqq \sum_{i=1}^k \frac{\lambda_i f_i(y)}{\alpha_i^1(x, y)} + \sum_{j=1}^m \mu_j g_j(y).$$

This inequality contradicts (14). Hence, (13) cannot hold. □

Theorem 4.2 (Strong Duality) *Suppose that $\bar{x}$ is a weakly efficient solution of* (MP) *for which the Kuhn-Tucker constraint qualification is satisfied. Then, there exist $\bar{\lambda} \in R^k$, $\bar{\mu} \in R^m$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for* (WD) *and the objective function values of* (MP) *and* (WD) *are equal.*

Furthermore, if the conditions of Theorem 4.1 hold for all feasible solutions of (MP) *and* (WD), *then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a weakly efficient solution of* (WD).

Proof Since $\bar{x}$ is a weakly efficient solution of (MP) for which the Kuhn-Tucker constraint qualification is satisfied, by the Karush-Kuhn-Tucker type necessary conditions ([19]. p. 40), there exist $\bar{\lambda} \in R^k$, $\bar{\mu} \in R^m$ satisfying

$$\sum_{i=1}^k \bar{\lambda}_i \nabla f_i(\bar{x}) + \sum_{j=1}^m \bar{\mu}_j \nabla g_j(\bar{x}) = 0,$$

$$\bar{\mu}_j g_j(\bar{x}) = 0, \quad j = 1, 2, \ldots, m,$$

$$\bar{\lambda}_i \geqq 0, \quad i = 1, 2, \ldots, k,$$

$$\bar{\mu}_j \geqq 0, \quad j = 1, 2, \ldots, m,$$
$$\sum_{i=1}^{k} \bar{\lambda}_i = 1.$$

Thus, $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for (WD) and the objective function values of (MP) and (WD) are equal.

Now, suppose that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is not a weakly efficient solution of (WD). Then, there exists a feasible solution $(y^\star, \lambda^\star, \mu^\star)$ of (WD) such that

$$f(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}) e < f_(y^\star) + \sum_{j=1}^{m} \mu_j^\star g_j(y^\star) e,$$

which in view of $\bar{\mu}_j \ g_j(\bar{x}) = 0$ becomes

$$f(\bar{x}) < f(y^\star) + \sum_{j=1}^{m} \mu_j^\star g_j(y^\star) e.$$

This contradicts Theorem 4.1. Hence, $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a weakly efficient solution of (WD). □

Theorem 4.3 (Strict Converse Duality) *Let $\bar{x}$ be a weakly efficient solution of* (MP), *let $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be a weakly efficient solution of* (WD) *and let*

(i) $\sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{x}) = \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{y})$,
(ii) (f, g) *be semistrictly* (F, α, ρ, d)-V*-type I at* $\bar{y}$ *with* $\alpha_i^1(\bar{x}, \bar{y}) = 1$, $i \in K$, $\alpha_j^2(\bar{x}, \bar{y}) = 1$, $j \in M$,
(iii) $\sum_{i=1}^{k} \bar{\lambda}_i \rho_i^1 + \sum_{j=1}^{m} \bar{\mu}_j \rho_j^2 \geqq 0$.

Then, $\bar{x} = \bar{y}$, *that is*, $\bar{y}$ *is a weakly efficient solution of* (MP).

Proof We suppose that $\bar{x} \neq \bar{y}$ and exhibit a contradiction. Using (10–12), hypothesis (ii) and summing over i and j, we obtain

$$\sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{x}) - \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) > F\left(\bar{x}, \bar{y}; \sum_{i=1}^{k} \bar{\lambda}_i \nabla f_i(\bar{y})\right) + \sum_{i=1}^{k} \bar{\lambda}_i \rho_i^1 d^2(\bar{x}, \bar{y}),$$
$$-\sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{y}) \geqq F\left(\bar{x}, \bar{y}; \sum_{j=1}^{m} \bar{\mu}_j \nabla g_j(\bar{y})\right) + \sum_{j=1}^{m} \bar{\mu}_j \rho_j^2 d^2(\bar{x}, \bar{y}).$$

Adding the above two inequalities and using the sublinearity of F along with (9), we get

$$\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{x}) - \sum_{i=1}^{k}\lambda_i f_i(\bar{y}) - \sum_{j=1}^{m}\bar{\mu}_j g_j(\bar{y})$$

$$> \left(\sum_{i=1}^{k}\bar{\lambda}_i\rho_i^1 + \sum_{j=1}^{m}\bar{\mu}_j\rho_j^2\right)d^2(\bar{x},\bar{y}),$$

which in view of hypothesis (iii) yields

$$\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{x}) > \sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{y}) + \sum_{j=1}^{m}\bar{\mu}_j g_j(\bar{y}).$$

This contradicts hypothesis (i). Hence, $\bar{x} = \bar{y}$. □

5 Mond-Weir Type Duality

In this section, we consider the following Mond-Weir type dual for (MP) and establish weak, strong and strict converse duality theorems:

(MD) Max $f(y)$,

s.t. $y \in X$,

$$\sum_{i=1}^{k}\lambda_i \nabla f_i(y) + \sum_{j=1}^{m}\mu_j \nabla g_j(y) = 0, \tag{17}$$

$$\sum_{j=1}^{m}\mu_j g_j(y) \geqq 0, \tag{18}$$

$$\lambda_i \geqq 0,\ i = 1,2,\ldots,k, \sum_{i=1}^{k}\lambda_i = 1, \tag{19}$$

$$\mu_j \geqq 0, \quad j = 1,2,\ldots,m. \tag{20}$$

Theorem 5.1 (Weak Duality) *Let x and (y,λ,μ) be feasible solutions of* (MP) *and* (MD) *respectively and let*

(i) $(f^\lambda, \mu^T g)$ *be pseudoquasi* $(F,\tilde{\alpha},\tilde{\rho},d)$-$V$-*type I at* y,
(ii) $\tilde{\rho}^1 + \tilde{\rho}^2 \geqq 0$.

Then, the following inequality cannot hold:

$$f(x) < f(y). \tag{21}$$

Proof Suppose that (21) holds. Using (19) and $\tilde{\alpha}_i^1(x,y) > 0, i \in K$, we have

$$\sum_{i=1}^{k} \tilde{\alpha}_i^1(x,y)\lambda_i f_i(x) < \sum_{i=1}^{k} \tilde{\alpha}_i^1(x,y)\lambda_i f_i(y).$$

Also, as $\tilde{\alpha}^2(x,y) > 0$, the dual constraint (18) yields

$$-\sum_{j=1}^{m} \tilde{\alpha}^2(x,y)\mu_j g_j(y) \leqq 0.$$

Therefore, hypothesis (i) implies

$$F\left(x,y;\sum_{i=1}^{k}\lambda_i \nabla f_i(y)\right) < -\tilde{\rho}^1 d^2(x,y),$$
$$F\left(x,y;\sum_{j=1}^{m}\mu_j \nabla g_j(y)\right) \leqq -\tilde{\rho}^2 d^2(x,y).$$

The above inequalities and the sublinearity of F give

$$F\left(x,y;\sum_{i=1}^{k}\bar{\lambda}_i \nabla f_i(y) + \sum_{j=1}^{m}\bar{\mu}_j \nabla g_j(y)\right)$$
$$\leqq F\left(x,y;\sum_{i=1}^{k}\bar{\lambda}_i \nabla f_i(y)\right) + F\left(x,y;\sum_{j=1}^{m}\bar{\mu}_j \nabla g_j(y)\right) < -(\tilde{\rho}^1+\tilde{\rho}^2)d^2(x,y) \leqq 0$$

(using hypothesis (ii)). This in view of (17) contradicts $F(x,y;0)=0$. Hence, (21) cannot hold. □

We now apply the above weak duality theorem on Example 2.1. Its Mond-Weir type dual is

$$\begin{aligned}
\text{Max}\quad & f(y_1,y_2) = (y_2(\pi-y_2)e^{\cos y_1}, \sin^2 y_1,\ y_1+\cos y_2),\\
\text{s.t.}\quad & y=(y_1,y_2)\in X,\\
& \lambda_1\{y_2(\pi-y_2)e^{\cos y_1}(-\sin y_1)\} + \lambda_2\{2\ \sin y_1 \cos y_1\} + \lambda_3 - 4\mu_1 = 0,\\
& \lambda_1\{(\pi-2y_2)e^{\cos y_1}\} - \lambda_3\{\sin y_2\} + \mu_2\{\sin y_2\} = 0,\\
& \mu_1(\pi-4y_1) + \mu_2(-\cos y_2) \geqq 0,\\
& \lambda_i \geqq 0,\quad \mu_j \geqq 0,\quad i=1,2,3,\quad j=1,2,\quad \sum_{i=1}^{3}\lambda_i = 1.
\end{aligned}$$

The point (y,λ,μ), where $y=(\frac{\pi}{2},\pi)\in X$, $\lambda=(0,0,1)$, $\mu=(\frac{1}{4},\frac{\pi}{4})$, is a dual feasible solution. Also, (i) it can be seen that (f,g) is (F,α,ρ,d)-V-type I at

$y=(\frac{\pi}{2},\pi)\in X$ for $F(x,y;a)=a^T(x-y)$, $d(x,y)=\sqrt{(x_1-\frac{\pi}{2})^2+(x_2-\pi)^2}$, $\alpha=(\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_1^2,\alpha_2^2)$, $\rho=(\rho_1^1,\rho_2^1,\rho_3^1,\rho_1^2,\rho_2^2)$, where $\alpha_1^1=\frac{x_2}{4\pi x_1},\alpha_2^1=1,\alpha_3^1=\frac{2}{x_2+2\pi},\alpha_1^2=1,\alpha_2^2=1,\rho_1^1=0,\rho_2^1=-\frac{1}{2},\rho_3^1=\frac{13}{100},\rho_1^2=0,\rho_2^2=-\frac{1}{2}$. Therefore using the sublinearity of F, it follows that $(f^\lambda,\mu^T g)$ is pseudoquasi $(F,\tilde{\alpha},\tilde{\rho},d)$-$V$-type I at $y=(\frac{\pi}{2},\pi)\in X$ for $F(x,y;a)$, $d(x,y)$ as defined above, $\tilde{\alpha}=(\tilde{\alpha}_1^1,\tilde{\alpha}_2^1,\tilde{\alpha}_3^1,\tilde{\alpha}^2)$ and $\tilde{\rho}=(\tilde{\rho}^1,\tilde{\rho}^2)$, where $\tilde{\alpha}_1^1=\frac{4\pi x_1}{x_2},\tilde{\alpha}_2^1=1,\tilde{\alpha}_3^1=\frac{x_2+2\pi}{2},\tilde{\alpha}^2=1,\tilde{\rho}^1=\frac{13\pi}{100},\tilde{\rho}^2=-\frac{\pi}{8}$, (ii) $\tilde{\rho}^1+\tilde{\rho}^2\geqq 0$.

Therefore, for any $x\in X_\circ$ weak duality holds, that is,

$$f(x)<f\left(\frac{\pi}{2},\pi\right)\quad\text{or}\quad\begin{pmatrix}x_2(\pi-x_2)e^{\cos x_1}\\ \sin^2 x_1\\ x_1+\cos x_2\end{pmatrix}<\begin{pmatrix}0\\ 1\\ \frac{\pi}{2}-1\end{pmatrix}$$

cannot hold. This is true as $x_2(\pi-x_2)e^{\cos x_1}>0$ for any $x\in X_\circ$.

The following strong duality theorem is stated without proof as it would run analogously to that of Theorem 4.2. However, here we invoke Theorem 5.1.

Theorem 5.2 (Strong Duality) *Suppose that $\bar{x}$ is a weakly efficient solution of* (MP) *for which the Kuhn-Tucker constraint qualification is satisfied. Then, there exist $\bar{\lambda}\in R^k$, $\bar{\mu}\in R^m$ such that $(\bar{x},\bar{\lambda},\bar{\mu})$ is feasible for* (MD) *and the objective function values of* (MP) *and* (MD) *are equal.*

Furthermore, if the conditions of Theorem 5.1 hold for all feasible solutions of (MP) *and* (MD), *then $(\bar{x},\bar{\lambda},\bar{\mu})$ is a weakly efficient solution of* (MD).

Theorem 5.3 (Strict Converse Duality) *Let $\bar{x}$ be a weakly efficient solution of* (MP), *let $(\bar{y},\bar{\lambda},\bar{\mu})$ be a weakly efficient solution of* (MD) *and let*

(i) $\sum_{i=1}^k\bar{\lambda}_i f_i(\bar{x})=\sum_{i=1}^k\bar{\lambda}_i f_i(\bar{y})$,

(ii) *$(f^{\bar{\lambda}},\bar{\mu}^T g)$ be strictly pseudoquasi $(F,\tilde{\alpha},\tilde{\rho},d)$-$V$-type I at $\bar{y}$ with $\tilde{\alpha}_i^1(\bar{x},\bar{y})=1$, $i\in K$,*

(iii) $\tilde{\rho}^1+\tilde{\rho}^2\geqq 0$.

Then, $\bar{x}=\bar{y}$, that is, $\bar{y}$ is a weakly efficient solution of (MP).

Proof We assume that $\bar{x}\neq\bar{y}$ and exhibit a contradiction. By hypothesis (ii), we have

$$F\left(\bar{x},\bar{y};\sum_{i=1}^k\bar{\lambda}_i\nabla f_i(\bar{y})\right)\geqq-\tilde{\rho}^1 d^2(\bar{x},\bar{y})\quad\Rightarrow\quad\sum_{i=1}^k\bar{\lambda}_i f_i(\bar{x})>\sum_{i=1}^k\bar{\lambda}_i f_i(\bar{y}),\tag{22}$$

$$-\sum_{j=1}^m\tilde{\alpha}^2(\bar{x},\bar{y})\bar{\mu}_j g_j(\bar{y})\leqq 0\quad\Rightarrow\quad F\left(\bar{x},\bar{y};\sum_{j=1}^m\bar{\mu}_j\nabla g_j(\bar{y})\right)\leqq-\tilde{\rho}^2 d^2(\bar{x},\bar{y}).\tag{23}$$

Since $(\bar{y}, \bar{\lambda}, \bar{\mu})$ is feasible for (MD) and $\tilde{\alpha}^2(\bar{x}, \bar{y}) > 0$, (18) implies

$$-\sum_{j=1}^{m} \tilde{\alpha}^2(\bar{x}, \bar{y})\bar{\mu}_j g_j(\bar{y}) \leqq 0,$$

which with (23) yields

$$F\left(\bar{x}, \bar{y}; \sum_{j=1}^{m} \bar{\mu}_j \nabla g_j(\bar{y})\right) \leqq -\tilde{\rho}^2 d^2(\bar{x}, \bar{y}). \tag{24}$$

The dual constraint (17) and the sublinearity of F give

$$\begin{aligned} 0 &= F\left(\bar{x}, \bar{y}; \sum_{i=1}^{k} \bar{\lambda}_i \nabla f_i(\bar{y}) + \sum_{j=1}^{m} \bar{\mu}_j \nabla g_j(\bar{y})\right) \\ &\leqq F\left(\bar{x}, \bar{y}; \sum_{i=1}^{k} \bar{\lambda}_i \nabla f_i(\bar{y})\right) + F\left(\bar{x}, \bar{y}; \sum_{j=1}^{m} \bar{\mu}_j \nabla g_j(\bar{y})\right). \end{aligned}$$

Hence,

$$\begin{aligned} &F\left(\bar{x}, \bar{y}; \sum_{i=1}^{k} \bar{\lambda}_i \nabla f_i(\bar{y})\right) \\ &\quad \geqq -F\left(\bar{x}, \bar{y}; \sum_{j=1}^{m} \bar{\mu}_j \nabla g_j(\bar{y})\right) \geqq \tilde{\rho}^2 d^2(\bar{x}, \bar{y}) \geqq -\tilde{\rho}^1 d^2(\bar{x}, \bar{y}) \end{aligned}$$

(using (24) and hypothesis (iii)). This with (22) implies

$$\sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{x}) > \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}),$$

contradicting hypothesis (i). Hence, $\bar{x} = \bar{y}$. □

Remark 5.1 It may be noted that, if in Theorem 4.3 hypothesis (i) is replaced by $\sum_{i=1}^{k} \frac{\bar{\lambda}_i f_i(\bar{x})}{\alpha_i^1(\bar{x},\bar{y})} = \sum_{i=1}^{k} \frac{\bar{\lambda}_i f_i(\bar{y})}{\alpha_i^1(\bar{x},\bar{y})} + \sum_{j=1}^{m} \frac{\bar{\mu}_j g_j(\bar{y})}{\alpha_j^2(\bar{x},\bar{y})}$, then the part $\alpha_i^1(\bar{x}, \bar{y}) = 1,\ i \in K$, $\alpha_j^2(\bar{x}, \bar{y}) = 1,\ j \in M$ of hypothesis (ii) is not required and hypothesis (iii) is to be replaced by $\sum_{i=1}^{k} \frac{\bar{\lambda}_i \rho_i^1}{\alpha_i^1(\bar{x},\bar{y})} + \sum_{j=1}^{m} \frac{\bar{\mu}_j \rho_j^2}{\alpha_j^2(\bar{x},\bar{y})} \geqq 0$. Similarly in Theorem 5.3, if hypothesis (i) is replaced by $\sum_{i=1}^{k} \tilde{\alpha}_i^1(\bar{x}, \bar{y})\bar{\lambda}_i f_i(\bar{x}) = \sum_{i=1}^{k} \tilde{\alpha}_i^1(\bar{x}, \bar{y})\bar{\lambda}_i f_i(\bar{y})$, then $\tilde{\alpha}_i^1(\bar{x}, \bar{y}) = 1,\ i \in K$ in hypothesis (ii) is not required.

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