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ORIGINAL PAPER

Sufficiency in multiobjective subset programming involving generalized type-I functions

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Abstract In this paper, sufficient optimality conditions for a multiobjective subset programming problem are established under generalized $(\mathcal{F}, \alpha, \rho, d)$-type-I functions.

Keywords Efficient solutions · Generalized n-set convex functions · Sufficient conditions

1 Introduction

The concept of optimizing n-set functions was initially developed by Morris [24], whose results are confined only to set functions of a single set. Such type of programming problems have various interesting applications in fluid flow [4], electrical insulator design [7], regional design (districting, facility location, warehouse layout, urban planning, etc.)[10,11], statistics [12,25] and optimal plasma confinement [28]. Corley [9] generalized the results of Morris [24] to n-set functions and discussed optimality conditions and Lagrangian duality. Several authors have shown interest in optimization involving differentiable n-set functions. For details, the readers are advised to consult [1–3,5,6,8,16,18,19,22,26,29,30].

Hanson and Mond [15] defined two new classes of functions, called type-I and type-II functions. Hachimi and Aghezzaf [14] introduced generalized $(\mathcal{F}, \alpha, \rho, d)$-type-I for vector-valued functions by combining the concepts of $(\mathcal{F}, \alpha, \rho, d)$-convex function [20,21] and type-I function [15,17]. Zalmai [31] discussed a fairly large number of sufficient efficiency conditions and duality results for multiobjective fractional subset programming problems under generalized $(\mathcal{F}, \alpha, \rho, \theta)$-V-convexity. Recently, Mishra

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[23] generalized the duality results in Ref. [31] involving generalized $(\mathcal{F}, \rho, \sigma, \theta)$-V-type-I functions but sufficient conditions were not discussed in his treatment.

We consider the following nonlinear multiobjective programming problem:

(P) Minimize $F(S) = [F_1(S), F_2(S), \ldots, F_k(S)]$
subject to $G_j(S) \leqq 0,\ j \in M,\ S = (S_1, S_2, \ldots, S_n) \in \mathcal{A}^n$,

where $\mathcal{A}^n$ is the n-fold product of σ-algebra $\mathcal{A}$ of subsets of a given set X, $F_i,\ i \in K = \{1, 2, \ldots, k\}$ and $G_j,\ j \in M = \{1, 2, \ldots, m\}$ are real-valued functions defined on $\mathcal{A}^n$. Let $X_\circ = \{S \in \mathcal{A}^n : G_j(S) \leqq 0,\ j \in M\}$ be the set of all feasible solutions of (P).

In this paper, motivated by Liang et al. [20,21], Hanson and Mond [15] and Preda et al. [27], we introduce a new class of generalized $(\mathcal{F}, \alpha, \rho, d)$-type-I for n-set functions. Based upon these functions, sufficient optimality conditions are discussed for properly efficient, efficient and weakly efficient solutions of (P).

2 Notations and definitions

The following conventions for vectors in R^n will be followed throughout this paper: $x \geqq y \Leftrightarrow x_p \geqq y_p,\ p = 1, 2, \ldots, n$; $x \geq y \Leftrightarrow x \geqq y$, and $x \neq y$; $x > y \Leftrightarrow x_p > y_p,\ p = 1, 2, \ldots, n$.

Let $(X, \mathcal{A}, \mu)$ be a finite atomless measure space with $L_1(X, \mathcal{A}, \mu)$ separable and let d be the pseudometric on $\mathcal{A}^n$ defined by

$$d(S, T) = \left[\sum_{p=1}^{n} \mu^2(S_p \triangle T_p)\right]^{1/2}, \quad S = (S_1, S_2, \ldots, S_n), \quad T = (T_1, T_2, \ldots, T_n) \in \mathcal{A}^n,$$

where $\triangle$ denotes symmetric difference; thus, $(\mathcal{A}^n, d)$ is a pseudometric space. For $h \in L_1(X, \mathcal{A}, \mu)$ and $Z \in \mathcal{A}$ with characteristic function $\chi_Z \in L_\infty(X, \mathcal{A}, \mu)$, the integral $\int_Z h\, \mathrm{d}\mu$ will be denoted by $\langle h, \chi_Z \rangle$.

We next define the notions of differentiability for n-set functions. This was originally introduced by Morris [24] for set functions, and subsequently extended by Corley [9] to n-set functions.

A function $\phi : \mathcal{A} \to R$ is said to be differentiable at $S^* \in \mathcal{A}$ if there exist $D\phi(S^*) \in L_1(X, \mathcal{A}, \mu)$, called the derivative of ϕ at S^* and $\psi : \mathcal{A} \times \mathcal{A} \to R$ such that for each $S \in \mathcal{A}$,

$$\phi(S) = \phi(S^*) + \langle D\phi(S^*),\ I_S - I_{S^*} \rangle + \psi(S, S^*),$$

where $\psi(S, S^*)$ is $o(d(S, S^*))$, that is, $\lim_{d(S,S^*) \to 0} \frac{\psi(S,S^*)}{d(S,S^*)} = 0$.

A function $F : \mathcal{A}^n \to R$ is said to have a partial derivative at $S^* = (S_1^*, S_2^*, \cdots, S_n^*)$ with respect to its pth argument if the function

$$\phi(S_p) = F(S_1^*, \ldots, S_{p-1}^*, S_p, S_{p+1}^*, \ldots, S_n^*)$$

has derivative $D\phi(S_p^*)$ and we define $D_pF(S^*) = D\phi(S_p^*)$. If $D_pF(S^*),\ p = 1, 2, \ldots, n$, all exist, then we put $DF(S^*) = (D_1F(S^*), D_2F(S^*), \ldots, D_nF(S^*))$.

A function $F : \mathcal{A}^n \to R$ is said to be differentiable at S^* if there exist $DF(S^*)$ and $\psi : \mathcal{A}^n \times \mathcal{A}^n \to R$ such that

$$F(S) = F(S^*) + \sum_{p=1}^{n} \langle D_pF(S^*),\ I_{S_p} - I_{S_p^*} \rangle + \psi(S, S^*),$$

where $\psi(S, S^*)$ is $o(d(S, S^*))$ for all $S \in \mathcal{A}^n$.

Definition 2.1 A feasible solution S^* of (P) is said to be an efficient solution of (P), if there exists no other feasible S of (P) such that

$$F(S) \leq F(S^*).$$

Definition 2.2 A feasible solution S^* of (P) is said to be a weakly efficient solution of (P), if there exists no other feasible $S(S \neq S^*)$ of (P) such that

$$F(S) < F(S^*).$$

Definition 2.3 An efficient solution S^* of (P) is said to be a properly efficient solution of (P), if there exists a scalar $N > 0$ such that for each r and feasible S satisfying $F_r(S) < F_r(S^*)$, we have

$$F_r(S^*) - F_r(S) \leqq N[F_j(S) - F_j(S^*)]$$

for at least one j satisfying $F_j(S^*) < F_j(S)$.

Definition 2.4 A function $\mathcal{F}(S, S^*; .) : L_1^n(X, \mathcal{A}, \mu) \to R$ is said to be sublinear if it is subadditive and positively homogeneous, that is, if for fixed $S, S^* \in \mathcal{A}^n$ and for every $f, g \in L_1^n(X, \mathcal{A}, \mu)$ and $a \in R_+ \equiv [0, \infty)$,

$$\mathcal{F}(S, S^*; f + g) \leqq \mathcal{F}(S, S^*; f) + \mathcal{F}(S, S^*; g)$$

and

$$\mathcal{F}(S, S^*; af) = a\,\mathcal{F}(S, S^*; f).$$

Let $\mathcal{F}(S, S^*; .) : L_1^n(X, \mathcal{A}, \mu) \to R$ be a sublinear function and $\theta : \mathcal{A}^n \times \mathcal{A}^n \to \mathcal{A}^n \times \mathcal{A}^n$ be a function such that $S \neq S^* \Rightarrow \theta(S, S^*) \neq (0, 0)$. Let $\alpha = (\alpha^1, \alpha^2) : \mathcal{A}^n \times \mathcal{A}^n \to R_+ \setminus \{0\}$ and $\rho = (\rho^1, \rho^2)$ such that $\rho^1 = (\rho_1^1, \rho_2^1, \ldots, \rho_k^1) \in R^k$, $\rho^2 = (\rho_{k+1}^2, \rho_{k+2}^2, \ldots, \rho_{k+m}^2) \in R^m$, i.e., ρ^1 has k components corresponding to k components of F and ρ^2 has m components corresponding to m components of G. The number of components in ρ^1 and ρ^2 may vary depending upon the way of the objective and constraint functions are involved in various hypothesis, e.g., the hypothesis may be on F, G, λF, and uG, etc. For $S^* \in X_\circ$, $J(S^*) = \{j \in M : G_j(S^*) = 0\}$ and G_J will denote the vector of active constraints at S^*. The functions $F : \mathcal{A}^n \to R^k$, with components F_i, $i \in K$, and $G : \mathcal{A}^n \to R^m$ with components G_j, $j \in M$, be differentiable at $S^* \in \mathcal{A}^n$.

We now define a new class of $(\mathcal{F}, \alpha, \rho, d)$-type-I for n-set functions. This class of functions may be viewed as an n-set version of a combination of two classes of point-functions: $(\mathcal{F}, \alpha, \rho, d)$ functions and type-I functions, which were introduced by Liang et al. [21] and Mond and Hanson [15], respectively.

Definition 2.5 (F, G) is said to be $(\mathcal{F}, \alpha, \rho, d)$-type-I at $S^* \in \mathcal{A}^n$, if for each $S \in X_\circ$,

$$F_i(S) - F_i(S^*) \geqq \mathcal{F}(S, S^*; \alpha^1(S, S^*) DF_i(S^*)) + \rho_i^1 d^2(\theta(S, S^*)), \quad i \in K,$$
$$-G_j(S^*) \geqq \mathcal{F}(S, S^*; \alpha^2(S, S^*) DG_j(S^*)) + \rho_j^2 d^2(\theta(S, S^*)), \quad j \in M.$$

Remark 2.1 If $\mathcal{F}(S, S^*; \alpha^1(S, S^*) DF_i(S^*)) = \alpha_i(S, S^*) \sum_{k=1}^{n} \langle D_k F_i(S^*), I_{S_k} - I_{S_k^*} \rangle$, $i \in K$ and $\mathcal{F}(S, S^*; \alpha^2(S, S^*) DG_j(S^*)) = \beta_j(S, S^*) \sum_{k=1}^{n} \langle D_k G_j(S^*), I_{S_k} - I_{S_k^*} \rangle$, $j \in M$, the above definition becomes that of $(\bar{\rho}, \bar{\rho}', d)$-type-I function introduced by Preda et al. [27].

An analogous terminology can be applied to various generalizations of $(\mathcal{F}, \alpha, \rho, d)$-type-I n-set functions given below:

Definition 2.6 (F, G) is said to be $(\mathcal{F}, \alpha, \rho, d)$-pseudo type-I at $S^* \in \mathcal{A}^n$ if for each $S \in X_\circ$,

$$\mathcal{F}(S, S^*; \alpha^1(S, S^*)DF_i(S^*)) + \rho_i^1 d^2(\theta(S, S^*)) \geqq 0 \Rightarrow F_i(S) \geqq F_i(S^*), \quad i \in K,$$
$$\mathcal{F}(S, S^*; \alpha^2(S, S^*)DG_j(S^*)) + \rho_j^2 d^2(\theta(S, S^*)) \geqq 0 \Rightarrow -G_j(S^*) \geqq 0, \quad j \in M.$$

Definition 2.7 (F, G) is said to be $(\mathcal{F}, \alpha, \rho, d)$-quasi type-I at $S^* \in \mathcal{A}^n$, if for each $S \in X_\circ$,

$$F_i(S) \leqq F_i(S^*) \Rightarrow \mathcal{F}(S, S^*; \alpha^1(S, S^*)DF_i(S^*)) + \rho_i^1 d^2(\theta(S, S^*)) \leqq 0, \quad i \in K,$$
$$-G_j(S^*) \leqq 0 \Rightarrow \mathcal{F}(S, S^*; \alpha^2(S, S^*)DG_j(S^*)) + \rho_j^2 d^2(\theta(S, S^*)) \leqq 0, \quad j \in M.$$

Definition 2.8 (F, G) is said to be $(\mathcal{F}, \alpha, \rho, d)$-pseudoquasi type-I at $S^* \in \mathcal{A}^n$, if for each $S \in X_\circ$,

$$F_i(S) < F_i(S^*) \Rightarrow \mathcal{F}(S, S^*; \alpha^1(S, S^*)DF_i(S^*)) + \rho_i^1 d^2(\theta(S, S^*)) < 0, \quad i \in K,$$
$$-G_j(S^*) \leqq 0 \Rightarrow \mathcal{F}(S, S^*; \alpha^2(S, S^*)DG_j(S^*)) + \rho_j^2 d^2(\theta(S, S^*)) \leqq 0, \quad j \in M.$$

3 Sufficient conditions

In this section, we present the sufficiency for (P) under generalized $(\mathcal{F}, \alpha, \rho, d)$-type-I functions.

Theorem 3.1 *Suppose that there exists a feasible solution S^* of* (P) *and scalars $\lambda_i^* > 0$, $i \in K$, and $u_j^* \geqq 0$, $j \in J$, such that*

$$\mathcal{F}\left(S, S^*; \sum_{i=1}^{k} \lambda_i^* DF_i(S^*) + \sum_{j \in J} u_j^* DG_j(S^*)\right) \geqq 0. \tag{1}$$

If (F, G_J) is $(\mathcal{F}, \alpha, \rho, d)$-type-I at S^* and

$$\frac{\sum_{i=1}^{k} \lambda_i^* \rho_i^1}{\alpha^1(S, S^*)} + \frac{\sum_{j \in J} u_j^* \rho_j^2}{\alpha^2(S, S^*)} \geqq 0,$$

then S^* is a properly efficient solution of (P).

Proof Since (F, G_J) is $(\mathcal{F}, \alpha, \rho, d)$-type-I at S^*, we have for all $S \in X_\circ$

$$F_i(S) - F_i(S^*) \geqq \mathcal{F}(S, S^*; \alpha^1(S, S^*)DF_i(S^*)) + \rho_i^1 d^2(\theta(S, S^*)), \quad i \in K,$$
$$-G_j(S^*) \geqq \mathcal{F}(S, S^*; \alpha^2(S, S^*)DG_j(S^*)) + \rho_j^2 d^2(\theta(S, S^*)), \quad j \in J.$$

Using $\lambda_i^* > 0,\ i \in K,\ u_j^* \geqq 0,\ j \in J,\ \alpha^1(S,S^*) > 0, \alpha^2(S,S^*) > 0$ and the sublinearity of $\mathcal{F}$, we get

$$\frac{\sum_{i=1}^{k} \lambda_i^* F_i(S)}{\alpha^1(S,S^*)} - \frac{\sum_{i=1}^{k} \lambda_i^* F_i(S^*)}{\alpha^1(S,S^*)} \geqq \mathcal{F}\left(S,S^*;\sum_{i=1}^{k} \lambda_i^* DF_i(S^*)\right) + \frac{\sum_{i=1}^{k} \lambda_i^* \rho_i^1 d^2(\theta(S,S^*))}{\alpha^1(S,S^*)},$$

$$0 = -\sum_{j\in J} u_j^* G_j(S^*) \geqq \mathcal{F}\left(S,S^*;\sum_{j\in J} u_j^* DG_j(S^*)\right) + \frac{\sum_{j\in J} u_j^* \rho_j^2 d^2(\theta(S,S^*))}{\alpha^2(S,S^*)}.$$

By the sublinearity of $\mathcal{F}$, we summarize to get

$$\begin{aligned}
&\mathcal{F}\left(S,S^*;\sum_{i=1}^{k} \lambda_i^* DF_i(S^*) + \sum_{j\in J} u_j^* DG_j(S^*)\right) \\
&\leqq \mathcal{F}\left(S,S^*;\sum_{i=1}^{k} \lambda_i^* DF_i(S^*)\right) + \mathcal{F}\left(S,S^*;\sum_{j\in J} u_j^* DG_j(S^*)\right) \\
&\leqq \frac{\sum_{i=1}^{k} \lambda_i^* F_i(S)}{\alpha^1(S,S^*)} - \frac{\sum_{i=1}^{k} \lambda_i^* F_i(S^*)}{\alpha^1(S,S^*)} - \left(\frac{\sum_{i=1}^{k} \lambda_i^* \rho_i^1}{\alpha^1(S,S^*)} + \frac{\sum_{j\in J} u_j^* \rho_j^2}{\alpha^2(S,S^*)}\right) \\
&\quad \times d^2(\theta(S,S^*)).
\end{aligned} \tag{2}$$

Since $\frac{\sum_{i=1}^{k} \lambda_i^* \rho_i^1}{\alpha^1(S,S^*)} + \frac{\sum_{j\in J} u_j^* \rho_j^2}{\alpha^2(S,S^*)} \geqq 0$, inequality (2) gives

$$\begin{aligned}
\frac{\sum_{i=1}^{k} \lambda_i^* F_i(S)}{\alpha^1(S,S^*)} - \frac{\sum_{i=1}^{k} \lambda_i^* F_i(S^*)}{\alpha^1(S,S^*)} &\geqq \mathcal{F}\left(S,S^*;\sum_{i=1}^{k} \lambda_i^* DF_i(S^*) + \sum_{j\in J} u_j^* DG_j(S^*)\right) \\
&\geqq 0 \quad (\text{ by}(1))
\end{aligned}$$

or

$$\frac{\sum_{i=1}^{k} \lambda_i^* F_i(S)}{\alpha^1(S,S^*)} \geqq \frac{\sum_{i=1}^{k} \lambda_i^* F_i(S^*)}{\alpha^1(S,S^*)}.$$

As $\alpha^1(S,S^*) > 0$, it follows that

$$\sum_{i=1}^{k} \lambda_i^* F_i(S) \geqq \sum_{i=1}^{k} \lambda_i^* F_i(S^*).$$

Hence, by Theorem 1 in Ref. [13], S^* is a properly efficient solution of (P).

Theorem 3.2 *Suppose that there exists a feasible solution S^* of* (P) *and scalars $\lambda_i^* > 0,\ i \in K$, and $u_j^* \geqq 0,\ j \in J$, such that*

$$\mathcal{F}\left(S, S^*; \sum_{i=1}^{k} \lambda_i^* DF_i(S^*) + \sum_{j \in J} u_j^* DG_j(S^*)\right) \geqq 0. \tag{3}$$

If $(\lambda^* F, u_J^* G_J)$ is $(\mathcal{F}, \alpha, \rho, d)$-pseudoquasi-type-I at S^* and

$$\frac{\rho_1^1}{\alpha^1(S, S^*)} + \frac{\rho_2^2}{\alpha^2(S, S^*)} \geqq 0,$$

then S^* is an efficient solution of (P).

Proof Suppose that S^* is not an efficient solution of (P). Then there exists a feasible solution S such that

$$F(S) \leq F(S^*).$$

Since $\lambda_i^* > 0,\ i \in K$, we get

$$\sum_{i=1}^{k} \lambda_i^* F_i(S) < \sum_{i=1}^{k} \lambda_i^* F_i(S^*). \tag{4}$$

Also $G_j(S^*) = 0$ and $u_j^* \geqq 0,\ j \in J$, yield

$$\sum_{j \in J} u_j^* G_j(S^*) = 0. \tag{5}$$

By the $(\mathcal{F}, \alpha, \rho, d)$-pseudoquasi-type-I assumption on $(\lambda^* F, u_J^* G_J)$ at S^*, inequalities (4) and (5) imply

$$\mathcal{F}\left(S, S^*; \alpha^1(S, S^*) \sum_{i=1}^{k} \lambda_i^* DF_i(S^*)\right) + \rho_1^1 d^2(\theta(S, S^*)) < 0,$$

$$\mathcal{F}\left(S, S^*; \alpha^2(S, S^*) \sum_{j \in J} u_j^* DG_j(S^*)\right) + \rho_2^2 d^2(\theta(S, S^*)) \leqq 0.$$

Since $\alpha^1(S, S^*) > 0,\ \alpha^2(S, S^*) > 0$, and $\mathcal{F}$ is sublinear, we get

$$\mathcal{F}\left(S, S^*; \sum_{i=1}^{k} \lambda_i^* DF_i(S^*)\right) < -\frac{\rho_1^1 d^2(\theta(S, S^*))}{\alpha^1(S, S^*)},$$

$$\mathcal{F}\left(S, S^*; \sum_{j \in J} u_j^* DG_j(S^*)\right) \leqq -\frac{\rho_2^2 d^2(\theta(S, S^*))}{\alpha^2(S, S^*)}.$$

By the sublinearity of $\mathcal{F}$, we summarize to get

$$\begin{aligned}
&\mathcal{F}(S, S^*; \sum_{i=1}^{k} \lambda_i^* DF_i(S^*) + \sum_{j\in J} u_j^* DG_j(S^*)) \\
&\leqq \mathcal{F}(S, S^*; \sum_{i=1}^{k} \lambda_i^* DF_i(S^*)) + \mathcal{F}(S, S^*; \sum_{j\in J} u_j^* DG_j(S^*)) \\
&< -\left(\frac{\rho_1^1}{\alpha^1(S, S^*)} + \frac{\rho_2^2}{\alpha^2(S, S^*)}\right) \times d^2(\theta(S, S^*)). \qquad (6)
\end{aligned}$$

As $\frac{\rho_1^1}{\alpha^1(S, S^*)} + \frac{\rho_2^2}{\alpha^2(S, S^*)} \geqq 0$, (6) reduces to

$$\mathcal{F}\left(S, S^*; \sum_{i=1}^{k} \lambda_i^* DF_i(S^*) + \sum_{j\in J} u_j^* DG_j(S^*)\right) < 0,$$

which contradicts (3). Hence S^* is an efficient solution of (P).

Remark 3.1 If $\lambda_i^* > 0,\ i \in K$ in the above theorems (Theorems 3.1 and 3.2) is replaced by $\lambda_i^* \geqq 0,\ i \in K, \sum_{i=1}^{k} \lambda_i^* = 1$, we get weaker conclusion that S^* is a weakly efficient solution of (P). These results are given below.

Theorem 3.3 *Suppose that there exists a feasible solution S^* of* (P) *and scalars $\lambda_i^* \geqq 0,\ i \in K, \sum_{i=1}^{k} \lambda_i^* = 1$ and $u_j^* \geqq 0,\ j \in J$, such that*

$$\mathcal{F}\left(S, S^*; \sum_{i=1}^{k} \lambda_i^* DF_i(S^*) + \sum_{j\in J} u_j^* DG_j(S^*)\right) \geqq 0.$$

If (F, G_J) is $(\mathcal{F}, \alpha, \rho, d)$-type-I at S^* and

$$\frac{\sum_{i=1}^{k} \lambda_i^* \rho_i^1}{\alpha^1(S, S^*)} + \frac{\sum_{j\in J} u_j^* \rho_j^2}{\alpha^2(S, S^*)} \geqq 0,$$

then S^* is a weakly efficient solution of (P).

Proof Following the proof of Theorem 3.1

$$\frac{\sum_{i=1}^{k} \lambda_i^* F_i(S)}{\alpha^1(S, S^*)} \geqq \frac{\sum_{i=1}^{k} \lambda_i^* F_i(S^*)}{\alpha^1(S, S^*)}. \qquad (7)$$

If S^* is not a weakly efficient solution of (P), then there exists a feasible solution S $(S \neq S^*)$ of (P) such that

$$F_i(S) < F_i(S^*), \quad i \in K.$$

Since $\lambda_i^* \geqq 0,\ i \in K, \sum_{i=1}^{k} \lambda_i^* = 1$ and $\alpha^1(S, S^*) > 0$, we have

$$\frac{\sum_{i=1}^{k} \lambda_i^* F_i(S)}{\alpha^1(S, S^*)} < \frac{\sum_{i=1}^{k} \lambda_i^* F_i(S^*)}{\alpha^1(S, S^*)},$$

which contradicts (7). Hence S^* is a weakly efficient solution of (P).

Theorem 3.4 *Suppose that there exists a feasible solution S^* of* (P) *and scalars $\lambda_i^* \geqq 0,\ i \in K, \sum_{i=1}^{k} \lambda_i^* = 1$ and $u_j^* \geqq 0,\ j \in J$, such that*

$$\mathcal{F}\left(S, S^*; \sum_{i=1}^{k} \lambda_i^* DF_i(S^*) + \sum_{j \in J} u_j^* DG_j(S^*)\right) \geqq 0.$$

If $(\lambda^* F, u_J^* G_J)$ is $(\mathcal{F}, \alpha, \rho, d)$-pseudoquasi-type-I at S^* and

$$\frac{\rho_1^1}{\alpha^1(S, S^*)} + \frac{\rho_2^2}{\alpha^2(S, S^*)} \geqq 0,$$

then S^* is a weakly efficient solution of (P).

Proof Its proof follows on the lines of Theorem 3.2.

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