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Mixed type duality for multiobjective variational problems with generalized (F, ρ)-convexity

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Abstract

A mixed type dual for multiobjective variational problems is formulated. Several duality theorems are established relating properly efficient solutions of the primal and dual variational problems under generalized (F, ρ)-convexity. Static mixed type dual multiobjective problems are particular cases of these problems.

Keywords: Multiobjective variational programming; Mixed type duality; Generalized (F, ρ)-convexity; Properly efficient solutions

1. Introduction

The relationship between mathematical programming and classical calculus of variation was explored and extended by Hanson [5]. Thereafter variational programming problems have attracted some attention in literature. Optimality conditions and duality results were obtained for scalar valued variational problems by Mond and Hanson [8] under convexity.

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Mathematical programs involving several conflicting objectives have been the subject of extensive study in the recent literature. By defining a restricted form of efficiency called proper efficiency, Geoffrion [3] established an equivalence between a convex multiobjective nonlinear program and a related parametric single objective program. Using parametric equivalence, Bector and Husain [2] formulated Wolfe and Mond–Weir type dual variational problems and established various duality results to relate properly efficient solutions of the primal and dual problems. The problems of [2] serve as the multiobjective version of the problems in [1,8].

Preda [11] introduced generalized (F,ρ)-convexity, an extension of F-convexity defined by Hanson and Mond [4] and generalized ρ-convexity defined by Vial [12]. In [10], Mukherjee and Rao have used the concept of efficiency to discuss duality results for multiobjective variational problems involving generalized ρ-convex functions.

In this paper, a mixed type dual is considered for a multiobjective variational problem and a number of duality results are established by relating properly efficient solutions between the primal and mixed dual problems under generalized (F,ρ)-convexity assumptions. Mainly these are generalizations of the results of Xu [14] for multiobjective variational problems.

2. Notations and preliminary results

Let $I=[a,b]$ be a real interval and let $P=\{1,2,\ldots,p\}$ and $M=\{1,2,\ldots,m\}$. In this paper, we assume $x(t)$ is an n-dimensional piecewise smooth function of t, and $\dot{x}(t)$ is the derivative of $x(t)$ with respect to t in $[a,b]$.

For notational simplicity, we write $x(t)$ and $\dot{x}(t)$ as x and $\dot{x}$, respectively. We denote the partial derivatives of f^1 with respect to t, x and $\dot{x}$ respectively by f_t^1, f_x^1 and $f_{\dot{x}}^1$ such that $f_x^1=(\frac{\partial f^1}{\partial x_1},\frac{\partial f^1}{\partial x_2},\ldots,\frac{\partial f^1}{\partial x_n})$ and $f_{\dot{x}}^1=(\frac{\partial f^1}{\partial \dot{x}_1},\frac{\partial f^1}{\partial \dot{x}_2},\ldots,\frac{\partial f^1}{\partial \dot{x}_n})$. Similarly, the partial derivatives of the vector function g can be written, using matrices with m rows instead of one. Let S denotes the space of n-dimensional piecewise smooth functions x with $\|x\|=\|x\|_\infty+\|Dx\|_\infty$, where the differentiation operator D is $u=Dx \Leftrightarrow x(t)=a_\circ+\int_a^t u(s)\,ds$, where $a_\circ$ is a given boundary value. Therefore $\frac{d}{dt}\equiv D$ except at discontinuities. No notational distinction is made between row and column vectors. Subscripts denote partial derivatives and superscripts denote vector components. Unless otherwise specified, for any index set $M=\{1,2,\ldots,m\}$, $\sum_M$ means the sum over all $i\in M$.

We consider the following multiobjective variational programming problem (MP) studied by Bector and Husain [2]:

$$\text{(MP)}\quad \text{Minimize} \int_a^b f(t,x,\dot{x})\,dt$$

$$\text{subject to}\quad g(t,x,\dot{x}) \leqslant 0,\quad t\in I,$$

$$x(a)=a_\circ,\qquad x(b)=b_\circ,$$

where $f=(f^1,f^2,\ldots,f^p): I\times R^n\times R^n\to R^p$, each component function is a continuously differentiable real scalar function, and $g=(g^1,g^2,\ldots,g^m): I\times R^n\times R^n\to R^m$ is an m-dimensional continuously differentiable vector function.

Let X denote the set of all feasible solutions of (MP), i.e.,

$$x \in X = \{x \in S\colon\ g(t,x,\dot{x}) \leqslant 0,\ t \in I,\ x(a) = a_\circ,\ x(b) = b_\circ\}.$$

Definition 1 (Geoffrion [3]). A point $u \in X$ is said to be efficient solution of (MP) if for all $x \in X$,

$$\int_a^b f^i(t,u,\dot{u})\,dt \geqslant \int_a^b f^i(t,x,\dot{x})\,dt \quad \text{for all } i \in P$$

$$\Rightarrow \quad \int_a^b f^i(t,u,\dot{u})\,dt = \int_a^b f^i(t,x,\dot{x})\,dt \quad \text{for all } i \in P.$$

An efficient solution u is said to be a properly efficient solution of (MP), if there exists a scalar $N > 0$ such that, for all $i \in P$,

$$\int_a^b f^i(t,u,\dot{u})\,dt - \int_a^b f^i(t,x,\dot{x})\,dt \leqslant N\left(\int_a^b f^j(t,x,\dot{x})\,dt - \int_a^b f^j(t,u,\dot{u})\,dt\right)$$

for some j, such that

$$\int_a^b f^j(t,x,\dot{x})\,dt > \int_a^b f^j(t,u,\dot{u})\,dt$$

whenever $x \in X$, and

$$\int_a^b f^i(t,x,\dot{x})\,dt < \int_a^b f^i(t,u,\dot{u})\,dt.$$

An efficient solution that is not properly efficient is said to be improperly efficient.

Definition 2. A point $u \in X$ is said to be weak minimum for (MP) if there exists no $x \in X$ for which

$$\int_a^b f(t,u,\dot{u})\,dt > \int_a^b f(t,x,\dot{x})\,dt.$$

It follows that if $u \in X$ is efficient for (MP), then it is also a weak minimum for (MP).

Definition 3. A functional $F : I \times R^n \times R^n \times R^n \times R^n \times R^n \to R$ is sublinear, if for any $x, \dot{x}, u, \dot{u} \in R^n$,

$$F(t,x,\dot{x},u,\dot{u};\xi_1+\xi_2) \leqslant F(t,x,\dot{x},u,\dot{u};\xi_1) + F(t,x,\dot{x},u,\dot{u};\xi_2) \tag{A}$$

for any $\xi_1, \xi_2 \in R^n$ and

$$F(t,x,\dot{x},u,\dot{u};\lambda\xi) = \lambda F(t,x,\dot{x},u,\dot{u};\xi) \tag{B}$$

for any $\lambda \in R$, $\lambda \geqslant 0$ and $\xi \in R^n$. From (B), $F(t, x, \dot{x}, u, \dot{u}; 0) = 0$, follows by substituting $\lambda = 0$.

Let $\Phi(x): S \to R$, denoted by $\Phi(x) = \int_a^b h(t, x, \dot{x})\, dt$ be Fréchet differentiable. Let $d(t, ., .)$ be a pseudometric on R^n and $\rho \in R$. For convenience and following [6,7], $\{d(t, x, u)\}^2$ has been written as $d^2(t, x, u)$ in the following definitions.

Definition 4. The functional $\Phi(x)$ is said to be (F, ρ)-convex at $u \in S$, if for all $x \in S$,

$$\Phi(x) - \Phi(u) \geqslant \int_a^b F\big(t, x, \dot{x}, u, \dot{u}; h_u(t, u, \dot{u}) - D\big(h_{\dot{u}}(t, u, \dot{u})\big)\big)\, dt + \rho \int_a^b d^2(t, x, u)\, dt.$$

Definition 5. The functional $\Phi(x)$ is said to be (F, ρ)-pseudoconvex at $u \in S$, if for all $x \in S$,

$$\int_a^b F\big(t, x, \dot{x}, u, \dot{u}; h_u(t, u, \dot{u}) - D\big(h_{\dot{u}}(t, u, \dot{u})\big)\big)\, dt \geqslant -\rho \int_a^b d^2(t, x, u)\, dt \quad \Rightarrow \quad \Phi(x) \geqslant \Phi(u),$$

or equivalently, if

$$\Phi(x) < \Phi(u) \quad \Rightarrow \quad \int_a^b F\big(t, x, \dot{x}, u, \dot{u}; h_u(t, u, \dot{u}) - D\big(h_{\dot{u}}(t, u, \dot{u})\big)\big)\, dt < -\rho \int_a^b d^2(t, x, u)\, dt.$$

Definition 6. The functional $\Phi(x)$ is said to be strictly (F, ρ)-pseudoconvex at $u \in S$, if for all $x \in S$, $x \neq u$,

$$\int_a^b F\big(t, x, \dot{x}, u, \dot{u}; h_u(t, u, \dot{u}) - D\big(h_{\dot{u}}(t, u, \dot{u})\big)\big)\, dt \geqslant -\rho \int_a^b d^2(t, x, u)\, dt \quad \Rightarrow \quad \Phi(x) > \Phi(u),$$

or equivalently, if

$$\Phi(x) \leqslant \Phi(u) \quad \Rightarrow \quad \int_a^b F\big(t, x, \dot{x}, u, \dot{u}; h_u(t, u, \dot{u}) - D\big(h_{\dot{u}}(t, u, \dot{u})\big)\big)\, dt < -\rho \int_a^b d^2(t, x, u)\, dt.$$

Definition 7. The functional $\Phi(x)$ is said to be (F,ρ)-quasiconvex at $u \in S$, if for all $x \in S$,

$$\Phi(x) \leqslant \Phi(u) \quad \Rightarrow \quad \int_a^b F\big(t,x,\dot{x},u,\dot{u}; h_u(t,u,\dot{u}) - D\big(h_{\dot{u}}(t,u,\dot{u})\big)\big)\,dt \leqslant -\rho \int_a^b d^2(t,x,u)\,dt,$$

or equivalently, if

$$\int_a^b F\big(t,x,\dot{x},u,\dot{u}; h_u(t,u,\dot{u}) - D\big(h_{\dot{u}}(t,u,\dot{u})\big)\big)\,dt > -\rho \int_a^b d^2(t,x,u)\,dt \quad \Rightarrow \quad \Phi(x) > \Phi(u).$$

3. Mixed type duality

Let J be a subset of M and $K = M/J$ such that $J \cup K = M$, and let

$$\beta_J(t)g^J(t,x,\dot{x}) = \sum_J \beta_i(t)g^i(t,x,\dot{x})$$

and

$$\beta_K(t)g^K(t,x,\dot{x}) = \sum_K \beta_i(t)g^i(t,x,\dot{x}).$$

Now we present the following mixed type multiobjective variational dual problem for (MP):

$$\text{(MD)} \quad \text{Maximize} \int_a^b \big\{f(t,u,\dot{u}) + \beta_J(t)g^J(t,u,\dot{u})e\big\}\,dt$$

$$\text{subject to} \quad \big[\alpha f_u(t,u,\dot{u}) + \beta(t)g_u(t,u,\dot{u})\big] = D\big[\alpha f_{\dot{u}}(t,u,\dot{u}) + \beta(t)g_{\dot{u}}(t,u,\dot{u})\big], \tag{1}$$

$$\int_a^b \beta_K(t)g^K(t,u,\dot{u})\,dt \geqslant 0, \tag{2}$$

$$\beta(t) \geqslant 0, \quad \alpha \geqslant 0, \quad \alpha e = 1, \tag{3}$$

$$x(a) = a_\circ, \qquad x(b) = b_\circ,$$

where $e = (1,1,\ldots,1)$ is a p-dimensional vector. It may be noted here that the above dual constraints are written using the Karush–Kuhn–Tucker necessary conditions for the problem (MP).

Remark 1. Let $K = \phi$. Then the dual (MD) reduces to the well-known Wolfe dual. If $J = \phi$, then (MD) becomes Mond–Weir type dual [9].

Let Y be the set of all feasible solutions of the dual problem (MD).

Theorem 1. *Let $x \in X$ and $(u, \alpha, \beta(t)) \in Y$. If $f^i(t,.,.)$, $i = 1, 2, \ldots, p$, are (F, ρ_i)-convex, $g^j(t,.,.)$, $j = 1, 2, \ldots, m$, are (F, σ_j)-convex and either*

(a) $\alpha > 0$ *and* $\sum_P \alpha_i \rho_i + \sum_M \beta_j(t)\sigma_j \geqslant 0$, *or*
(b) $\sum_P \alpha_i \rho_i + \sum_M \beta_j(t)\sigma_j > 0$,

then the following cannot hold:

$$\int_a^b f^i(t, x, \dot{x})\, dt \leqslant \int_a^b \left\{ f^i(t, u, \dot{u}) + \beta_J(t) g^J(t, u, \dot{u}) \right\} dt \quad \text{for all } i \in P \tag{4}$$

and

$$\int_a^b f^i(t, x, \dot{x})\, dt < \int_a^b \left\{ f^i(t, u, \dot{u}) + \beta_J(t) g^J(t, u, \dot{u}) \right\} dt \quad \text{for some } i \in P. \tag{5}$$

Proof. Suppose to the contrary that (4) and (5) hold. Then in view of the feasibility of x for (MP) and $\beta(t) \geqslant 0$, the inequalities (4) and (5) imply that

$$\int_a^b \left\{ f^i(t, x, \dot{x}) + \beta_J(t) g^J(t, x, \dot{x}) \right\} dt \leqslant \int_a^b \left\{ f^i(t, u, \dot{u}) + \beta_J(t) g^J(t, u, \dot{u}) \right\} dt$$

for all $i \in P$ and for some $i \in P$,

$$\int_a^b \left\{ f^i(t, x, \dot{x}) + \beta_J(t) g^J(t, x, \dot{x}) \right\} dt < \int_a^b \left\{ f^i(t, u, \dot{u}) + \beta_J(t) g^J(t, u, \dot{u}) \right\} dt.$$

Since $\alpha_i > 0$, for all $i \in P$ and $\alpha e = 1$, the above inequalities give

$$\int_a^b \left\{ \alpha f(t, x, \dot{x}) + \beta_J(t) g^J(t, x, \dot{x}) \right\} dt < \int_a^b \left\{ \alpha f(t, u, \dot{u}) + \beta_J(t) g^J(t, u, \dot{u} \right\} dt. \tag{6}$$

Now by the definition of (F, ρ_i)-convexity of $f^i(t,.,.)$, $i \in P$, and (F, σ_j)-convexity of $g^j(t,.,.)$, $j \in M$, we have

$$\begin{aligned} &\int_a^b \left\{ f^i(t, x, \dot{x}) - f^i(t, u, \dot{u}) \right\} dt \\ &\quad \geqslant \int_a^b F\left(t, x, \dot{x}, u, \dot{u};\, f_u^i(t, u, \dot{u}) - D\left(f_{\dot{u}}^i(t, u, \dot{u})\right)\right) dt + \rho_i \int_a^b d^2(t, x, u)\, dt \\ &\quad \text{for all } i \in P \end{aligned} \tag{7}$$

and

$$\int_a^b \{g^j(t,x,\dot{x}) - g^j(t,u,\dot{u})\}\,dt$$
$$\geqslant \int_a^b F\big(t,x,\dot{x},u,\dot{u}; g_u^j(t,u,\dot{u}) - D\big(g_{\dot{u}}^j(t,u,\dot{u})\big)\big)\,dt + \sigma_j \int_a^b d^2(t,x,u)\,dt$$
$$\text{for all } j \in M. \tag{8}$$

On multiplying (7) by $\alpha_i > 0$, $i \in P$, and (8) by $\beta_j(t)$, $j \in M$, and adding the inequalities and by sublinearity of F, we have

$$\int_a^b \{\alpha f(t,x,\dot{x}) + \beta(t)\, g(t,x,\dot{x}) - \alpha f(t,u,\dot{u}) - \beta(t)\, g(t,u,\dot{u})\}\,dt$$
$$\geqslant \int_a^b \{F\big(t,x,\dot{x},u,\dot{u}; \alpha f_u(t,u,\dot{u}) - D\big(\alpha f_{\dot{u}}(t,u,\dot{u})\big)\big)$$
$$+ F\big(t,x,\dot{x},u,\dot{u}; \beta(t) g_u(t,u,\dot{u}) - D\big(\beta(t) g_{\dot{u}}(t,u,\dot{u})\big)\big)\}\,dt$$
$$+ \left(\sum_P \alpha_i \rho_i + \sum_M \beta_j(t)\sigma_j\right) \int_a^b d^2(t,x,u)\,dt$$
$$\geqslant \int_a^b F\big(t,x,\dot{x},u,\dot{u}; \alpha f_u(t,u,\dot{u}) - D\big(\alpha f_{\dot{u}}(t,u,\dot{u})\big) + \beta(t) g_u(t,u,\dot{u})$$
$$- D\big(\beta(t) g_{\dot{u}}(t,u,\dot{u})\big)\big)\,dt + \left(\sum_P \alpha_i \rho_i + \sum_M \beta_j(t)\sigma_j\right) \int_a^b d^2(t,x,u)\,dt$$
$$\geqslant \left(\sum_P \alpha_i \rho_i + \sum_M \beta_j(t)\sigma_j\right) \int_a^b d^2(t,x,u)\,dt \quad \big(\text{by (1)}\big)$$
$$\geqslant 0 \quad \big(\text{using hypothesis (a)}\big). \tag{9}$$

Since $M = JUK$,

$$\beta(t) g(t,.,.) = \beta_J(t) g^J(t,.,.) + \beta_K(t) g^K(t,.,.). \tag{10}$$

The inequalities (6), (9) and (10) imply

$$\int_a^b \{\beta_K(t) g^K(t,x,\dot{x}) - \beta_K(t) g^K(t,u,\dot{u})\}\,dt > 0. \tag{11}$$

Now, since $(u, \alpha, \beta(t)) \in Y$, from (2), $\int_a^b \beta_K(t) g^K(t, x, \dot{x})\, dt > 0$, which is a contradiction to the fact that x is feasible for (MP) and hence (4) and (5) cannot hold.

Under hypothesis (b), strict inequality (6) holds as inequality. Therefore inequality (9) also holds as strict inequality. Hence we obtain (11), again contradicting the fact that x is feasible for (MP) and, (4) and (5) cannot hold. □

The above theorem has a number of special cases which can be easily identified by the suitable sublinear and algebraic properties of the (F, ρ)-convex functions. We shall state two of these as corollaries.

Corollary 1. *Let $x \in X$ and $(u, \alpha, \beta(t)) \in Y$. If $f^i(t, ., .)$, $i = 1, 2, \ldots, p$, are (F, ρ_i)-convex and $\beta_j(t) g^j(t, ., .)$, $j = 1, 2, \ldots, m$, are (F, σ_j)-convex and either*

(a) $\alpha > 0$ *and* $\sum_P \alpha_i \rho_i + \sum_M \sigma_j \geqslant 0$, *or*
(b) $\sum_P \alpha_i \rho_i + \sum_M \sigma_j > 0$,

then (4) *and* (5) *cannot hold.*

Corollary 2. *Let $x \in X$ and $(u, \alpha, \beta(t)) \in Y$. If $f^i(t, ., .)$, $i = 1, 2, \ldots, p$, are (F, ρ_i)-convex and $\beta_J(t) g^J(t, ., .)$ is (F, σ)-convex and either*

(a) $\alpha > 0$ *and* $\sum_P \alpha_i \rho_i + \sigma \geqslant 0$, *or*
(b) $\sum_P \alpha_i \rho_i + \sigma > 0$,

then (4) *and* (5) *cannot hold.*

Theorem 2. *Let $x \in X$ and $(u, \alpha, \beta(t)) \in Y$ and let*

(i) $\beta_K(t) g^K(t, ., .)$ *is (F, ρ)-quasiconvex. Also assume that one of the following three conditions holds*:
  (a) $\alpha_i > 0$ *for all $i \in P$, and $f^i(t, ., .) + \beta_J(t) g^J(t, ., .)$, $i \in P$, is both (F, σ_i)-quasiconvex and (F, σ_i)-pseudoconvex with $\rho + \sum_P \alpha_i \sigma_i \geqslant 0$;*
  (b) $\alpha_i > 0$ *for all $i \in P$, and $f^i(t, ., .) + \beta_J(t) g^J(t, ., .)$ is (F, σ_i)-quasiconvex and there exists some $k \in P$ such that it is strictly (F, σ_k)-pseudoconvex with $\sum_P \alpha_i \sigma_i + \rho \geqslant 0$;*
  (c) $\alpha_i > 0$ *for all $i \in P$, and $\alpha f(t, ., .) + \beta_J(t) g^J(t, ., .)$ is (F, σ)-pseudoconvex with $\rho + \sigma \geqslant 0$.*

Then (4) *and* (5) *cannot hold.*

Proof. Since $x \in X$ and $(u, \alpha, \beta(t)) \in Y$,

$$\int_a^b \beta_K(t) g^K(t, x, \dot{x})\, dt \leqslant 0 \leqslant \int_a^b \beta_K(t) g^K(t, u, \dot{u})\, dt. \tag{12}$$

Now (12) and hypothesis (i) imply

$$\int_a^b F\big(t,x,\dot{x},u,\dot{u};\beta_K(t)g_u^K(t,u,\dot{u})-D\big(\beta_K(t)g_{\dot{u}}^K(t,u,\dot{u})\big)\big)\,dt \leqslant -\rho\int_a^b d^2(t,x,u)\,dt. \tag{13}$$

The first constraint of the dual problem (MD), (10), (13) and sublinearity of F yield

$$\int_a^b F\big(t,x,\dot{x},u,\dot{u};\alpha f_u(t,u,\dot{u})+\beta_J(t)g_u^J(t,u,\dot{u}) - D\big[\alpha f_{\dot{u}}(t,u,\dot{u})+\beta_J(t)g_{\dot{u}}^J(t,u,\dot{u})\big]\big)\,dt \geqslant \rho\int_a^b d^2(t,x,u)\,dt. \tag{14}$$

By hypothesis (a), the sublinearity of F, $\alpha e = 1$ and (14) we have

$$\begin{aligned}\int_a^b \sum_P \alpha_i F\big(t,x,\dot{x},u,\dot{u};\, f_u(t,u,\dot{u})+\beta_J(t)g_u^J(t,u,\dot{u}) &- D\big[f_{\dot{u}}(t,u,\dot{u})+\beta_J(t)g_{\dot{u}}^J(t,u,\dot{u})\big]\big)\,dt \\ = \int_a^b F\big(t,x,\dot{x},u,\dot{u};\alpha f_u(t,u,\dot{u})+\beta_J(t)g_u^J(t,u,\dot{u}) &- D\big[\alpha f_{\dot{u}}(t,u,\dot{u})+\beta_J(t)g_{\dot{u}}^J(t,u,\dot{u})\big]\big)\,dt \\ \geqslant \rho\int_a^b d^2(t,x,u)\,dt \geqslant -\sum_P \alpha_i\sigma_i\int_a^b d^2(t,x,u)\,dt.\end{aligned} \tag{15}$$

Since $\alpha_i > 0,\ i \in P$, it follows from (15) that either

$$\int_a^b F\big(t,x,\dot{x},u,\dot{u};\, f_u^i(t,u,\dot{u})+\beta_J(t)g_u^J(t,u,\dot{u}) - D\big[\alpha f_{\dot{u}}^i(t,u,\dot{u})+\beta_J(t)g_{\dot{u}}^J(t,u,\dot{u})\big]\big)\,dt = -\sigma_i\int_a^b d^2(t,x,u)\,dt \quad \text{for all } i \in P, \tag{16}$$

or

$$\int_a^b F\big(t, x, \dot{x}, u, \dot{u};\ f_u^i(t, u, \dot{u}) + \beta_J(t) g_u^J(t, u, \dot{u}) \\ - D\big[\alpha f_{\dot{u}}^i(t, u, \dot{u}) + \beta_J(t) g_{\dot{u}}^J(t, u, \dot{u})\big]\big)\, dt \\ > -\sigma_i \int_a^b d^2(t, x, u)\, dt \quad \text{for some } i \in P. \tag{17}$$

If (16) and (17) hold, then by the both (F, σ_i)-pseudoconvexity and (F, σ_i)-quasiconvexity of $f^i(t, ., .) + \beta_J(t) g^J(t, ., .)$, $i \in P$, we yield

$$\int_a^b \big\{f^i(t, x, \dot{x}) + \beta_J(t) g^J(t, x, \dot{x})\big\}\, dt \geqslant \int_a^b \big\{f^i(t, u, \dot{u}) + \beta_J(t) g^J(t, u, \dot{u})\big\}\, dt \tag{18}$$

for all $i \in P$ and for some $i \in P$,

$$\int_a^b \big\{f^i(t, x, \dot{x}) + \beta_J(t) g^J(t, x, \dot{x})\big\}\, dt > \int_a^b \big\{f^i(t, u, \dot{u}) + \beta_J(t) g^J(t, u, \dot{u})\big\}\, dt. \tag{19}$$

Equations (18) and (19) along with the feasibility of x for (MP) yield

$$\int_a^b f^i(t, x, \dot{x})\, dt \geqslant \int_a^b \big\{f^i(t, u, \dot{u}) + \beta_J(t) g^J(t, u, \dot{u})\big\}\, dt \tag{20}$$

and

$$\int_a^b f^i(t, x, \dot{x})\, dt > \int_a^b \big\{f^i(t, u, \dot{u}) + \beta_J(t) g^J(t, u, \dot{u})\big\}\, dt. \tag{21}$$

Obviously (20) and (21) show that (4) and (5) cannot hold.

Under hypothesis (b) inequalities (18) and (19) hold as strict inequalities. Therefore (20) also holds as strict inequality. This means that (4) and (5) cannot hold.

As for hypothesis (c), inequality (14) along with $\rho + \sigma \geqslant 0$ gives

$$\int_a^b F\big(t, x, \dot{x}, u, \dot{u}; \alpha f_u(t, u, \dot{u}) + \beta_J(t) g_u^J(t, u, \dot{u}) \\ - D\big[\alpha f_{\dot{u}}(t, u, \dot{u}) + \beta_J(t) g_{\dot{u}}^J(t, u, \dot{u})\big]\big)\, dt \\ \geqslant -\sigma \int_a^b d^2(t, x, u)\, dt. \tag{22}$$

By the (F, σ)-pseudoconvexity assumption in (c),

$$\int_a^b \big\{\alpha f(t, x, \dot{x}) + \beta_J(t) g^J(t, x, \dot{x})\big\}\, dt \geqslant \int_a^b \big\{\alpha f(t, u, \dot{u}) + \beta_J(t) g^J(t, u, \dot{u})\big\}\, dt.$$

This inequality and the feasibility of x for (MP) yield

$$\int_a^b \alpha f(t,x,\dot{x})\,dt \geqslant \int_a^b \left\{\alpha f(t,u,\dot{u}) + \beta_J(t) g^J(t,u,\dot{u})\right\} dt,$$

which implies that (4) and (5) cannot hold, since $\alpha_i > 0$, $i \in P$. The proof is complete. □

In the proofs of the above theorems we first use the inequality constraint

$$\int_a^b \beta_K g^K(t,u,\dot{u})\,dt \geqslant 0$$

of (MD). The use of the equality constraint of (MD) first leads to the following theorem.

Theorem 3. *Let $x \in X$ and $(u,\alpha,\beta(t)) \in Y$. If any of the following holds*:

(a) $\alpha_i > 0$, *for all* $i \in P$ *and* $\alpha f^i(t,.,.) + \beta(t) g(t,.,.)$ *is* (F,ρ_i)*-pseudoconvex with* $\sum_P \alpha_i \rho_i \geqslant 0$;
(b) $\alpha f^i(t,.,0) + \beta(t) g(t,.,.)$ *is strictly* $(F,0)$*-pseudoconvex*;

then (4) *and* (5) *cannot hold.*

Proof. Using $F(t,x,\dot{x},u,\dot{u},0) = 0$ in Definition 3 and the first constraint of the dual problem (MD),

$$\int_a^b F\big(t,x,\dot{x},u,\dot{u};\alpha f_u(t,u,\dot{u}) + \beta(t) g_u(t,u,\dot{u}) - D\big[\alpha f_{\dot{u}}(t,u,\dot{u}) + \beta(t) g_{\dot{u}}(t,u,\dot{u})\big]\big)\,dt = 0. \tag{23}$$

Since $\alpha_i > 0$ for all $i \in P$ and from the condition $\alpha e = 1$, we get

$$\sum_P \alpha_i \int_a^b F\big(t,x,\dot{x},u,\dot{u};\, f_u^i(t,u,\dot{u}) + \beta(t) g_u(t,u,\dot{u}) - D\big[f_{\dot{u}}^i(t,u,\dot{u}) + \beta(t) g_{\dot{u}}(t,u,\dot{u})\big]\big)\,dt = 0. \tag{24}$$

Given that $\sum_P \alpha_i \rho_i \geqslant 0$ and $\int_a^b d^2(t,x,u)\,dt$ is always positive, therefore

$$\sum_P \alpha_i \int_a^b F\big(t,x,\dot{x},u,\dot{u};\, f_u^i(t,u,\dot{u}) + \beta(t) g_u(t,u,\dot{u}) - D\big[f_{\dot{u}}^i(t,u,\dot{u}) + \beta(t) g_{\dot{u}}(t,u,\dot{u})\big]\big)\,dt \geqslant -\sum_P \alpha_i \rho_i \int_a^b d^2(t,x,u)\,dt. \tag{25}$$

By hypothesis (a), we have

$$\sum_P \alpha_i \int_a^b \left\{ f^i(t,x,\dot{x}) + \beta(t) g(t,x,\dot{x}) \right\} dt$$
$$\geqslant \sum_P \alpha_i \int_a^b \left\{ f^i(t,u,\dot{u}) + \beta(t) g(t,u,\dot{u}) \right\} dt. \tag{26}$$

This inequality along with the feasibility of x for (MP) implies that (4) and (5) cannot hold.

For hypothesis (b), from (23) and the strict $(F,0)$-pseudoconvex assumption of $\alpha f(t,.,.) + \beta(t) g(t,.,.)$, we have

$$\int_a^b \left\{ \alpha f(t,x,\dot{x}) + \beta(t) g(t,x,\dot{x}) \right\} dt$$
$$> \int_a^b \left\{ \alpha f(t,u,\dot{u}) + \beta(t) g(t,u,\dot{u}) \right\} dt, \quad x \neq u. \tag{27}$$

Now the feasibility of x for (MP) and $(u, \alpha, \beta(t))$ for (MD), lead us to the desired conclusion that (4) and (5) cannot hold. The proof is complete. □

The condition $\alpha_i > 0$ for all $i \in P$ is very important, as we see in the previous Theorems 1–3. Of course, to get the desired results without this condition, other conditions should be enforced, which lead to the following theorem.

Theorem 4. *Let $x \in X$ and $(u, \alpha, \beta(t)) \in Y$. If any of the following holds*:

(a) $\alpha f(t,.,.) + \beta(t) g(t,.,.)$ *is strictly (F,σ)-convex with $\rho \geqslant 0$*;
(b) $\beta_K(t) g^K(t,.,.)$ *is (F,ρ)-quasiconvex, and for all $i \in P$, $f^i(t,.,.) + \beta_J(t) g^J(t,.,.)$ is strictly (F,σ_i)-pseudoconvex with $\rho + \sum_P \alpha_i \sigma_i \geqslant 0$*;
(c) $\beta_K(t) g^K(t,.,.)$ *is (F,ρ)-quasiconvex, and $\alpha f(t,.,.) + \beta_J(t) g^J(t,.,.)$ is strictly (F,σ)-pseudoconvex with $\rho + \sigma \geqslant 0$*;

then (4) *and* (5) *cannot hold.*

The proof follows on the lines of Theorems 1–3.

We now turn our attention to a discussion of strong duality theorem. The following proposition, the continuous version of Theorem 2.2 [13], is for that purpose.

Proposition 1. *Let $\bar{x}$ be a weak minimum for* (MP) *at which the Kuhn–Tucker constraint qualification is satisfied. Then there exist $\bar{\alpha} \in R^p$ and a piecewise smooth function $\bar{\beta}(.) : I \to R^m$ such that*

$$\left[\bar{\alpha} f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{\beta}(t) g_x(t, \bar{x}, \dot{\bar{x}})\right] = D\left[\bar{\alpha} f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{\beta}(t) g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}})\right], \tag{28}$$

$$\int_a^b \bar{\beta}(t) g(t, \bar{x}, \dot{\bar{x}})\, dt = 0, \tag{29}$$

$$\bar{\beta}(t) \geqslant 0, \quad \bar{\alpha} \geqslant 0, \quad \bar{\alpha} e = 1, \tag{30}$$

where $e = (1, 1, \ldots, 1)$ *is a* p*-dimensional vector.*

Theorem 5 (Strong duality). *Let* $\bar{x}$ *be a properly efficient solution for* (MP) *and assume that* $\bar{x}$ *satisfies the Kuhn–Tucker constraint qualification for* (MP). *Then there exist* $\bar{\alpha} \in R^p$ *and a piecewise smooth function* $\beta(t) : I \to R^m$ *such that* $(\bar{x}, \bar{\alpha}, \bar{\beta}(t))$ *is feasible for* (MD) *along with the condition* $\int_a^b \beta(t) g(t, \bar{x}, \dot{\bar{x}})\, dt = 0$. *Furthermore, if any weak duality* (*any of the Theorems* 1–4) *also holds between* (MP) *and* (MD), *then* $(\bar{x}, \bar{\alpha}, \bar{\beta}(t))$ *is a properly efficient solution of the problem* (MD).

Proof. Since $\bar{x}$ is a properly efficient solution of (MP), it is also efficient solution and every efficient solution for (MP) is also a weak minimum. Therefore by Proposition 1, there exists $\bar{\alpha} \in R^p$ and a piecewise smooth function $\bar{\beta}(t) : I \to R^m$ satisfying (28) to (30). Hence $(\bar{x}, \bar{\alpha}, \bar{\beta}(t)) \in Y$ and the two objective functionals have same values.

Now we claim that $(\bar{x}, \bar{\alpha}, \bar{\beta}(t))$ is an efficient solution of (MD). If not, then there exists $(x^\star, \alpha^\star, \beta^\star(t)) \in Y$ such that

$$\int_a^b \left\{f^r(t, x^\star, \dot{x}^\star) + \beta_J(t) g^J(t, x^\star, \dot{x}^\star)\right\} dt > \int_a^b f^r(t, \bar{x}, \dot{\bar{x}})\, dt$$
$$\text{for some } r \in \{1, 2, \ldots, p\}$$

and

$$\int_a^b \left\{f^i(t, x^\star, \dot{x}^\star) + \beta_J(t) g^J(t, x^\star, \dot{x}^\star)\right\} dt \geqslant \int_a^b f^i(t, \bar{x}, \dot{\bar{x}})\, dt$$
$$\text{for all } i \in \{1, 2, \ldots, p\}/\{r\}.$$

The right-hand side in the above inequalities contains only one term since

$$\int_a^b \bar{\beta}_J(t) g^J(t, \bar{x}, \dot{\bar{x}})\, dt = 0.$$

These inequalities contradict the conclusion of any weak duality (Theorems 1–4). Hence $(\bar{x}, \bar{\alpha}, \bar{\beta}(t))$ is an efficient solution of (MD).

Assume now that it is not a properly efficient solution of (MD). Then there exist $(x^\star, \alpha^\star, \beta^\star(t)) \in Y$ and $i \in \{1, 2, \ldots, p\}$ such that

$$\int_a^b \left\{f^i(t, x^\star, \dot{x}^\star) + \beta_J^\star(t) g^J(t, x^\star, \dot{x}^\star)\right\} dt > \int_a^b f^i(t, \bar{x}, \dot{\bar{x}})\, dt$$

and

$$\int_a^b \{f^i(t, x^\star, \dot{x}^\star) + \beta_J^\star(t) g^J(t, x^\star, \dot{x}^\star)\}\,dt - \int_a^b f^i(t, \bar{x}, \dot{\bar{x}})\,dt$$
$$> N\left[\int_a^b f^j(t, \bar{x}, \dot{\bar{x}})\,dt - \int_a^b \{f^j(t, x^\star, \dot{x}^\star) + \beta_J^\star(t) g^J(t, x^\star \dot{x}^\star)\}\,dt\right]$$

for all $N > 0$ and for some $j \in P$ satisfying

$$\int_a^b f^j(t, \bar{x}, \dot{\bar{x}})\,dt > \int_a^b \{f^j(t, x^\star, \dot{x}^\star) + \beta_J^\star(t) g^J(t, x^\star, \dot{x}^\star)\}\,dt.$$

This means that

$$\int_a^b \{f^i(t, x^\star, \dot{x}^\star) + \beta_J^\star(t) g^J(t, x^\star, \dot{x}^\star)\}\,dt - \int_a^b f^i(t, \bar{x}, \dot{\bar{x}})\,dt$$

can be made arbitrarily large, whereas

$$\int_a^b f^j(t, \bar{x}, \dot{\bar{x}})\,dt - \int_a^b \{f^j(t, x^\star, \dot{x}^\star) + \beta_J^\star(t) g^J(t, x^\star, \dot{x}^\star)\}\,dt$$

is finite for all $j \neq i$. Therefore,

$$\alpha^{\star i} \int_a^b \left[\{f^i(t, x^\star, \dot{x}^\star) + \beta_J^\star(t) g^J(t, x^\star, \dot{x}^\star)\} - f^i(t, \bar{x}, \dot{\bar{x}})\right] dt$$
$$> \sum_{j \neq i} \alpha^{\star j} \int_a^b \left[f^j(t, \bar{x}, \dot{\bar{x}}) - \{f^j(t, x^\star, \dot{x}^\star) + \beta_J^\star(t) g^J(t, x^\star, \dot{x}^\star)\}\right] dt,$$

or

$$\alpha^\star \int_a^b \left[f(t, x^\star, \dot{x}^\star) + \beta_J^\star g^J(t)(t, x^\star, \dot{x}^\star) e\right] dt > \alpha^\star \int_a^b f(t, \bar{x}, \dot{\bar{x}})\,dt.$$

This shows that inequalities (4) and (5) hold. Hence $(\bar{x}, \bar{\alpha}, \bar{\beta}(t))$ is a properly efficient solution for (MD). □

4. Multiobjective mathematical programming

If the time dependency of problems (MP) and (MD) is removed, then these problems essentially reduce to the following multiobjective nonlinear programs studied by Xu [14]:

$$\text{(NP)}\quad \begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & g(x) \leqslant 0; \end{array}$$

$$\text{(ND)}\quad \begin{array}{ll} \text{Maximize} & f(u) + \beta_J g^J(u)e \\ \text{subject to} & \alpha\nabla f(u) + \beta\nabla g(u) = 0, \\ & \beta_K g^K(u) \geqslant 0, \\ & \beta \geqslant 0, \quad \alpha \geqslant 0, \quad \alpha e = 1. \end{array}$$

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