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Second-order duality in nondifferentiable minmax programming involving type-I functions

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Abstract

Two types of second-order dual models are formulated for a nondifferentiable minmax programming problem and usual duality results are established involving generalized type-I functions.

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Keywords: Nondifferentiable programming; Minmax programming; Second-order duality; Generalized convexity

1. Introduction

In this paper, we consider the following nondifferentiable minmax programming problem:

$$\text{(P)} \quad \begin{array}{ll} \text{Minimize} & \psi(x) = \sup_{y \in Y} f(x, y) + (x^{\mathrm{T}} B x)^{1/2} \\ \text{subject to} & g(x) \leqslant 0, \quad x \in X, \end{array}$$

where Y is a compact subset of R^l, X is an open subset of R^n; $f(\cdot, \cdot) : X \times Y \to R$, and $g(\cdot) : X \to R^m$ are twice differentiable functions at $x \in X$, and B is an $n \times n$ positive semidefinite symmetric matrix. If $B = 0$, then (P) is a usual minmax programming problem which was frequently studied in [5,6,14,21–23].

Yadav and Mukherjee [24] employed the optimality conditions in [21] to construct two dual problems for a differentiable fractional minmax programming problem and derived duality results. Chandra and Kumar [7] pointed out certain omissions and inconsistencies in the dual formulation of Yadav and Mukherjee [24]; they constructed two modified dual problems for fractional minmax programming problem and proved duality theorems. Many other authors have shown their interest in developing optimality conditions and duality results for differentiable minmax fractional programming problems [1,15,25] and nondifferentiable minmax fractional programming problems [3,4,9–11,19].

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Mangasarian [16] first formulated the second-order dual for a nonlinear programming problem and established duality results under somewhat involved assumptions. Mond [20] reproved second-order duality results involving simpler assumptions than those previously given by Mangasarian [16], and showed that the second-order dual has computational advantages over the first-order dual.

In order to generalize the notion of convexity to second and higher order, and to extend the validity of results to larger classes of optimization problems, various attempts have been made. More precisely, Liang et al. [12,13] introduced the concept of (F, α, ρ, d)-convex functions, which was further extended to second-order (F, α, ρ, d)-convex functions by Ahmad and Husain [2] and to second-order (F, α, ρ, d)-type I functions by Hachimi and Aghezzaf [8].

Liu [14] discussed second-order duality results for differentiable minmax programming problems. Recently, Mishra and Rueda [17] proved second-order duality theorems for a general Mond–Weir type dual associated with nondifferentiable minmax programming problem using the concept of generalized second-order Type-I functions. Motivated by these authors, we discuss duality theorems under second-order (F, α, ρ, d)-type-I assumptions for second-order Wolfe and general Mond–Weir type duals to (P). The present work generalizes the results in [6,14,17].

2. Notations and preliminaries

Let $S = \{x \in X : g(x) \leqslant 0\}$ denote the set of all feasible solutions of (P). Any point $x \in S$ is called the feasible point of (P). The index set is $M = \{1, 2, \ldots, m\}$. For each $(x, y) \in S \times Y$, we define

$$J(x) = \{j \in M : g_j(x) = 0\},$$

$$Y(x) = \left\{ y \in Y : f(x, y) + (x^{\mathrm{T}}Bx)^{1/2} = \sup_{z \in Y} f(x, z) + (x^{\mathrm{T}}Bx)^{1/2} \right\},$$

and

$$K(x) = \left\{ (s, t, \tilde{y}) \in \mathbb{N} \times R^s_+ \times R^{ls} : 1 \leqslant s \leqslant n+1, t = (t_1, t_2, \ldots, t_s) \in R^s_+ \right.$$
$$\left. \text{with } \sum_{i=1}^{s} t_i = 1, \tilde{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_s) \text{ with } \bar{y}_i \in Y(x), i = 1, 2, \ldots, s \right\}.$$

Definition 2.1. A functional $F : X \times X \times R^n \to R$ is said to be sublinear in its third argument, if $\forall x, \bar{x} \in X$,

(i) $F(x, \bar{x}; a_1 + a_2) \leqslant F(x, \bar{x}; a_1) + F(x, \bar{x}; a_2)$ $\forall a_1, a_2 \in R^n$,
(ii) $F(x, \bar{x}; \alpha a) = \alpha F(x, \bar{x}; a)$ $\forall \alpha \in R_+, a \in R^n$.

By (ii), it is clear that $F(x, \bar{x}; 0\,a) = 0$.

We now rewrite the definitions of generalized second-order (F, α, ρ, d)-type-I functions [8] in the following form:

Let F be a sublinear functional. Let $\alpha = (\alpha^1, \alpha^2) : X \times X \to R_+ \backslash \{0\}$, and let $\rho = (\rho^1, \rho^2)$, where $\rho^1 = (\rho^1_1, \rho^1_2, \ldots, \rho^1_s) \in R^s$ and $\rho^2 = (\rho^2_1, \rho^2_2, \ldots, \rho^2_m) \in R^m$. Let $d(\cdot, \cdot) : X \times X \to R$. Let $\phi(\cdot, \cdot) : X \times Y \to R$ be twice differentiable at $\bar{x} \in X$. In what follows, ∇ stands for the gradient vector with respect to x throughout the paper.

Definition 2.2. For each $j \in M$, (ϕ, g_j) is said to be second-order (F, α, ρ, d)-type-I at $\bar{x} \in X$, if for all $x \in S$ and $y_i \in Y(x)$, we have

$$\phi(x, y_i) - \phi(\bar{x}, y_i) + \tfrac{1}{2} p^{\mathrm{T}} \nabla^2 \phi(\bar{x}, y_i) p$$
$$\geqslant F(x, \bar{x}; \alpha^1(x, \bar{x})\{\nabla \phi(\bar{x}, y_i) + \nabla^2 \phi(\bar{x}, y_i) p\}) + \rho^1_i d^2(x, \bar{x}), \quad i = 1, 2, \ldots, s,$$

$$-\left[g_j(\bar{x}) - \tfrac{1}{2} p^{\mathrm{T}} \nabla^2 g_j(\bar{x}) p \right] \geqslant F(x, \bar{x}; \alpha^2(x, \bar{x})\{\nabla g_j(\bar{x}) + \nabla^2 g_j(\bar{x}) p\}) + \rho^2_j d^2(x, \bar{x}), \quad j = 1, 2, \ldots, m.$$

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In the above definition, if the inequalities appear as strict inequalities, then we say that for each $j \in M$, (ϕ, g_j) is second-order strictly (F, α, ρ, d)-type-I at $\bar{x} \in X$.

Remark 2.1. If $\alpha^1(x, \bar{x}) = \alpha^2(x, \bar{x}) = 1$, $F(x, \bar{x}; a) = \eta^{\mathrm{T}}(x, \bar{x})a$, for a certain mapping $\eta : S \times X \to R^n$, and $\rho_i^1 = 0$, $i = 1, 2, \ldots, s$, $\rho_j^2 = 0$, $j = 1, 2, \ldots, m$, then the above definition reduces to that of second-order Type I function introduced by Mishra and Rueda [17].

Definition 2.3. For each $j \in M$, (ϕ, g_j) is said to be second-order (F, α, ρ, d)-pseudoquasi-type-I at $\bar{x} \in X$, if for all $x \in S$ and $y_i \in Y(x)$, we have

$$\begin{aligned}
&\phi(x, y_i) < \phi(\bar{x}, y_i) - \tfrac{1}{2} p^{\mathrm{T}} \nabla^2 \phi(\bar{x}, y_i) p \\
&\quad \Rightarrow F(x, \bar{x}; \alpha^1(x, \bar{x})\{\nabla \phi(\bar{x}, y_i) + \nabla^2 \phi(\bar{x}, y_i) p\}) < -\rho_i^1 d^2(x, \bar{x}), i = 1, 2, \ldots, s, \\
&\quad\quad -\left[g_j(\bar{x}) - \tfrac{1}{2} p^{\mathrm{T}} \nabla^2 g_j(\bar{x}) p\right] \leqslant 0 \Rightarrow F(x, \bar{x}; \alpha^2(x, \bar{x})\{\nabla g_j(\bar{x}) + \nabla^2 g_j(\bar{x}) p\}) \\
&\quad \leqslant -\rho_j^2 d^2(x, \bar{x}), \quad j = 1, 2, \ldots, m.
\end{aligned}$$

In the above definition, if

$$\begin{aligned}
&F(x, \bar{x}; \alpha^1(x, \bar{x})\{\nabla \phi(\bar{x}, y_i) + \nabla^2 \phi(\bar{x}, y_i) p\}) \geqslant -\rho_i^1 d^2(x, \bar{x}) \\
&\Rightarrow \phi(x, y_i) > \phi(\bar{x}, y_i) - \tfrac{1}{2} p^{\mathrm{T}} \nabla^2 \phi(\bar{x}, y_i) p, \quad i = 1, 2, \ldots, s,
\end{aligned}$$

then we say that for each $j \in M$, (ϕ, g_j) is second-order (F, α, ρ, d)-strictly pseudoquasi-type-I at $\bar{x} \in X$.

Lemma 2.1. (Generalized Schwartz inequality). *Let B be a positive semidefinite symmetric matrix of order n. Then, for all $x, w \in R^n$,*

$$x^{\mathrm{T}} B w \leqslant (x^{\mathrm{T}} B x)^{1/2} (w^{\mathrm{T}} B w)^{1/2}.$$

We observe that equality holds, if $Bx = \lambda B w$, for some $\lambda \geqslant 0$. Evidently, if $(w^{\mathrm{T}} B w)^{1/2} \leqslant 1$, we have

$$x^{\mathrm{T}} B w \leqslant (x^{\mathrm{T}} B x)^{1/2}.$$

Following theorem is a special case of [11, Theorem 3.1], and will be needed in the proofs of strong duality theorems:

Theorem 2.1. (Necessary conditions). *If x^* is a solution (local or global) of problem* (P) *satisfying ${x^*}^{\mathrm{T}} B x^* > 0$, and $\nabla g_j(x^*)$, $j \in J(x^*)$ are linearly independent, then there exist $(s, t, \bar{y}) \in K(x^*)$, $u \in R^n$, and $\mu \in R_+^m$ such that*

$$\nabla \sum_{i=1}^{s} t_i f(x^*, \bar{y}_i) + Bu + \nabla \sum_{j=1}^{m} \mu_j g_j(x^*) = 0,$$

$$\sum_{j=1}^{m} \mu_j g_j(x^*) = 0,$$

$$t_i \geqslant 0, \quad i = 1, 2, \ldots, s, \quad \sum_{i=1}^{s} t_i = 1,$$

$$({x^*}^{\mathrm{T}} B x^*)^{1/2} = {x^*}^{\mathrm{T}} B u,$$

$$u^{\mathrm{T}} B u \leqslant 1.$$

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3. First duality model

This section deals with the duality results for the following second-order dual to (P):

$$\text{(WD)} \quad \max_{(s,t,\bar{y})\in K(z)} \sup_{(z,u,\mu,p)\in H_1(s,t,\bar{y})} \sum_{i=1}^{s} t_i f(z,\bar{y}_i) + z^{\mathrm{T}}Bu + \sum_{j=1}^{m} \mu_j g_j(z) - \frac{1}{2} p^{\mathrm{T}} \nabla^2 \left[\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + \sum_{j=1}^{m} \mu_j g_j(z) \right] p,$$

where $H_1(s,t,\bar{y})$ denotes the set of all $(z,u,\mu,p) \in R^n \times R^n \times R^m_+ \times R^n$ satisfying

$$\nabla \sum_{i=1}^{s} t_i f(z,\bar{y}_i) + Bu + \nabla \sum_{j=1}^{m} \mu_j g_j(z) + \nabla^2 \sum_{i=1}^{s} t_i f(z,\bar{y}_i) p + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z) p = 0, \tag{3.1}$$

$$u^{\mathrm{T}}Bu \leqslant 1. \tag{3.2}$$

If, for a triplet $(s,t,\bar{y}) \in K(z)$, the set $H_1(s,t,\bar{y}) = \emptyset$, then we define the supremum over it to be $-\infty$.

Theorem 3.1. (Weak duality). *Let x and $(z,u,\mu,s,t,\bar{y},p)$ be the feasible solutions of* (P) *and* (WD), *respectively. Assume that $[f(\cdot,\bar{y}_i) + (\cdot)^{\mathrm{T}}Bu, i = 1,2,\ldots,s,\ g_j(\cdot), j = 1,2,\ldots,m]$ is second-order (F,α,ρ,d)-type-I at z with $\alpha^1(x,z) = \alpha^2(x,z)$, and $\sum_{i=1}^{s} t_i\rho_i^1 + \sum_{j=1}^{m}\mu_j\rho_j^2 \geqslant 0$. Then*

$$\sup_{y\in Y} f(x,y) + (x^{\mathrm{T}}Bx)^{\frac{1}{2}} \geqslant \sum_{i=1}^{s} t_i f(z,\bar{y}_i) + z^{\mathrm{T}}Bu + \sum_{j=1}^{m} \mu_j g_j(z) - \frac{1}{2} p^{\mathrm{T}} \nabla^2 \left[\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + \sum_{j=1}^{m} \mu_j g_j(z) \right] p.$$

Proof. Suppose to the contrary that

$$\sup_{y\in Y} f(x,y) + (x^{\mathrm{T}}Bx)^{1/2} < \sum_{i=1}^{s} t_i f(z,\bar{y}_i) + z^{\mathrm{T}}Bu + \sum_{j=1}^{m} \mu_j g_j(z) - \frac{1}{2} p^{\mathrm{T}} \nabla^2 \left[\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + \sum_{j=1}^{m} \mu_j g_j(z) \right] p.$$

Thus, we have

$$f(x,\bar{y}_i) + (x^{\mathrm{T}}Bx)^{1/2} < \sum_{i=1}^{s} t_i f(z,\bar{y}_i) + z^{\mathrm{T}}Bu + \sum_{j=1}^{m} \mu_j g_j(z) - \frac{1}{2} p^{\mathrm{T}} \nabla^2 \left[\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + \sum_{j=1}^{m} \mu_j g_j(z) \right] p,$$

for all $\bar{y}_i \in Y(x)$, $i = 1,2,\ldots,s$.

It follows from $t_i \geqslant 0$, $i = 1, 2, \ldots, s$, that

$$t_i \left[(f(x, \bar{y}_i) + (x^{\mathrm{T}} Bx)^{1/2}) - \left(\sum_{i=1}^{s} t_i f(z, \bar{y}_i) + z^{\mathrm{T}} Bu + \sum_{j=1}^{m} \mu_j g_j(z) - \frac{1}{2} p^{\mathrm{T}} \nabla^2 \left[\sum_{i=1}^{s} t_i f(z, \bar{y}_i) + \sum_{j=1}^{m} \mu_j g_j(z) \right] p \right) \right] \leqslant 0, \quad i = 1, 2, \ldots, s,$$

with at least one strict inequality, since $t = (t_1, t_2, \ldots, t_s) \neq 0$. Taking summation over i and using $\sum_{i=1}^{s} t_i = 1$, we have

$$\sum_{i=1}^{s} t_i f(x, \bar{y}_i) + (x^{\mathrm{T}} Bx)^{1/2} < \sum_{i=1}^{s} t_i f(z, \bar{y}_i) + z^{\mathrm{T}} Bu + \sum_{j=1}^{m} \mu_j g_j(z) - \frac{1}{2} p^{\mathrm{T}} \nabla^2 \left[\sum_{i=1}^{s} t_i f(z, \bar{y}_i) + \sum_{j=1}^{m} \mu_j g_j(z) \right] p.$$

By (3.2) and Lemma 2.1, the above inequality implies

$$\sum_{i=1}^{s} t_i f(x, \bar{y}_i) + x^{\mathrm{T}} Bu < \sum_{i=1}^{s} t_i f(z, \bar{y}_i) + z^{\mathrm{T}} Bu + \sum_{j=1}^{m} \mu_j g_j(z) - \frac{1}{2} p^{\mathrm{T}} \nabla^2 \left[\sum_{i=1}^{s} t_i f(z, \bar{y}_i) + \sum_{j=1}^{m} \mu_j g_j(z) \right] p. \tag{3.3}$$

Now, the second-order (F, α, ρ, d)-type-I assumption on $[f(\cdot, \bar{y}_i) + (\cdot)^{\mathrm{T}} Bu,\ i = 1, 2, \ldots, s,\ g_j(\cdot),\ j = 1, 2, \ldots, m]$ at z gives

$$f(x, \bar{y}_i) + x^{\mathrm{T}} Bu - f(z, \bar{y}_i) - z^{\mathrm{T}} Bu + \tfrac{1}{2} p^{\mathrm{T}} \nabla^2 f(z, \bar{y}_i) p \geqslant F(x, z; \alpha^1(x, z)\{\nabla f(z, \bar{y}_i) + Bu + \nabla^2 f(z, \bar{y}_i) p\}) + \rho_i^1 d^2(x, z), \quad i = 1, 2, \ldots, s,$$

$$- g_j(z) + \tfrac{1}{2} p^{\mathrm{T}} \nabla^2 g_j(z) p \geqslant F(x, z; \alpha^2(x, z)\{\nabla g_j(z) + \nabla^2 g_j(z) p\}) + \rho_j^2 d^2(x, z), \quad j = 1, 2, \ldots, m.$$

On multiplying the first inequality by $t_i \geqslant 0$, $i = 1, 2, \ldots, s$, second by $\mu_j \geqslant 0$, $j = 1, 2, \ldots, m$, and on using the sublinearity of F with $\sum_{i=1}^{s} t_i = 1$, we get

$$\sum_{i=1}^{s} t_i f(x, \bar{y}_i) + x^{\mathrm{T}} Bu - \sum_{i=1}^{s} t_i f(z, \bar{y}_i) - z^{\mathrm{T}} Bu + \frac{1}{2} p^{\mathrm{T}} \nabla^2 \sum_{i=1}^{s} t_i f(z, \bar{y}_i) p \geqslant F\left(x, z; \alpha^1(x, z) \left\{ \nabla \sum_{i=1}^{s} t_i f(z, \bar{y}_i) + Bu + \nabla^2 \sum_{i=1}^{s} t_i f(z, \bar{y}_i) p \right\} \right) + \sum_{i=1}^{s} t_i \rho_i^1 d^2(x, z),$$

$$- \sum_{j=1}^{m} \mu_j g_j(z) + \frac{1}{2} p^{\mathrm{T}} \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z) p \geqslant F\left(x, z; \alpha^2(x, z) \left\{ \nabla \sum_{j=1}^{m} \mu_j g_j(z) + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z) p \right\} \right) + \sum_{j=1}^{m} \mu_j \rho_j^2 d^2(x, z).$$

Adding the above inequalities along with $\alpha^1(x,z)=\alpha^2(x,z)$ and the sublinearity of F, to get

$$\sum_{i=1}^{s} t_i f(x,\bar{y}_i) + x^{\mathrm{T}}Bu - \sum_{i=1}^{s} t_i f(z,\bar{y}_i) - z^{\mathrm{T}}Bu + \frac{1}{2}p^{\mathrm{T}}\nabla^2 \sum_{i=1}^{s} t_i f(z,\bar{y}_i)p - \sum_{j=1}^{m}\mu_j g_j(z)$$
$$+\frac{1}{2}p^{\mathrm{T}}\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p \geqslant F\left(x,z;\alpha^1(x,z)\left\{\nabla\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + Bu + \nabla^2\sum_{i=1}^{s} t_i f(z,\bar{y}_i)p\right.\right.$$
$$\left.\left.+\nabla\sum_{j=1}^{m}\mu_j g_j(z) + \nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p\right\}\right) + \left(\sum_{i=1}^{s} t_i\rho_i^1 + \sum_{j=1}^{m}\mu_j\rho_j^2\right)d^2(x,z).$$

Since $(\sum_{i=1}^{s} t_i\rho_i^1 + \sum_{j=1}^{m}\mu_j\rho_j^2)\geqslant 0$, we have

$$\sum_{i=1}^{s} t_i f(x,\bar{y}_i) + x^{\mathrm{T}}Bu - \sum_{i=1}^{s} t_i f(z,\bar{y}_i) - z^{\mathrm{T}}Bu - \sum_{j=1}^{m}\mu_j g_j(z)$$
$$+\frac{1}{2}p^{\mathrm{T}}\nabla^2\left[\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + \sum_{j=1}^{m}\mu_j g_j(z)\right]p \geqslant F\left(x,z;\alpha^1(x,z)\left\{\nabla\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + Bu\right.\right.$$
$$\left.\left.+\nabla^2\sum_{i=1}^{s} t_i f(z,\bar{y}_i)p + \nabla\sum_{j=1}^{m}\mu_j g_j(z) + \nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p\right\}\right),$$

which along with (3.3) and $\alpha^1(x,z)>0$, implies

$$F\left(x,z;\nabla\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + Bu + \nabla\sum_{j=1}^{m}\mu_j g_j(z) + \nabla^2\sum_{i=1}^{s} t_i f(z,\bar{y}_i)p + \nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p\right) < 0,$$

a contradictuon to (3.1), since $F(x,z;0)=0$. □

Theorem 3.2 (*Strong duality*). *Assume that x^* is an optimal solution of* (P) *and $\nabla g_j(x^*)$, $j\in J(x^*)$ are linearly independent. Then there exist $(s^*,t^*,\bar{y}^*)\in K(x^*)$ and $(x^*,u^*,\mu^*,p^*{=}0)\in H_1(s^*,t^*,\bar{y}^*)$ such that $(x^*,u^*,\mu^*,s^*,t^*,\bar{y}^*,p^*=0)$ is a feasible solution of* (WD) *and the two objectives have the same values. Further, if the weak duality* (Theorem 3.1) *holds for all feasible solutions $(z,u,\mu,s,t,\bar{y},p)$ of* (WD), *then $(x^*,u^*,\mu^*,s^*,t^*,\bar{y}^*,p^*=0)$ is an optimal solution of* (WD).

Proof. Since x^* is an optimal solution of (P) and $\nabla g_j(x^*)$, $j\in J(x^*)$ are linearly independent, then by Theorem 2.1, there exist $(s^*,t^*,\bar{y}^*)\in K(x^*)$ and $(x^*,u^*,\mu^*,p^*=0)\in H_1(s^*,t^*,\bar{y}^*)$ such that $(x^*,u^*,\mu^*,s^*,t^*,\bar{y}^*,p^*=0)$ is a feasible solution of (WD) and the two objectives have the same values. Optimality of $(x^*,u^*,\mu^*,s^*,t^*,\bar{y}^*,p^*=0)$ for (WD) thus follows from the weak duality (Theorem 3.1).

Theorem 3.3 (*Strict converse duality*). *Let x^* and $(z^*,u^*,\mu^*,s^*,t^*,\bar{y}^*,p^*)$ be the optimal solutions of* (P) *and* (WD), *respectively. Suppose that $[f(\cdot,\bar{y}_i^*)+(\cdot)^{\mathrm{T}}Bu^*,\ i=1,2,\ldots,s^*,\ g_j(\cdot),\ j=1,2,\ldots,m]$ is second-order strictly (F,α,ρ,d)-type-*I *at z^* with $\alpha^1(x^*,z^*)=\alpha^2(x^*,z^*)$ and $\sum_{i=1}^{s^*}t_i^*\rho_i^1+\sum_{j=1}^{m}\mu_j^*\rho_j^2\geqslant 0$, and $\nabla g_j(x^*)$, $j\in J(x^*)$ are linearly independent. Then $z^*=x^*$, that is, z^* is an optimal solution of* (P).

Proof. Suppose to the contrary that $z^*\neq x^*$, and exhibit a contradiction. Since x^* and $(z^*,u^*,\mu^*,s^*,t^*,\bar{y}^*,p^*)$ are the optimal solutions of (P) and (WD), respectively, and $\nabla g_j(x^*)$, $j\in J(x^*)$ are linearly independent, therefore from the strong duality (Theorem 3.2), we reach

$$\sup_{y\in Y} f(x^*, y^*) + (x^{*\mathrm{T}}Bx^*)^{1/2} = \sum_{i=1}^{s^*} t_i^* f(z^*, \bar{y}_i^*) + z^{*\mathrm{T}}Bu^* + \sum_{j=1}^{m} \mu_j^* g_j(z^*) - \frac{1}{2} p^{*\mathrm{T}} \nabla^2 \left[\sum_{i=1}^{s^*} t_i^* f(z^*, \bar{y}_i^*) + \sum_{j=1}^{m} \mu_j^* g_j(z^*) \right] p^*.$$

Thus, we have

$$f(x^*, \bar{y}_i^*) + (x^{*\mathrm{T}}Bx^*)^{1/2} \leqslant \sum_{i=1}^{s^*} t_i^* f(z^*, \bar{y}_i^*) + z^{*\mathrm{T}}Bu^* + \sum_{j=1}^{m} \mu_j^* g_j(z^*) - \frac{1}{2} p^{*\mathrm{T}} \nabla^2 \left[\sum_{i=1}^{s^*} t_i^* f(z^*, \bar{y}_i^*) + \sum_{j=1}^{m} \mu_j^* g_j(z^*) \right] p^*,$$

for all $\bar{y}_i^* \in Y(x^*)$, $i = 1, 2, \ldots, s^*$.

Now proceeding as in Theorem 3.1, we get

$$\sum_{i=1}^{s^*} t_i^* f(x^*, \bar{y}_i^*) + x^{*\mathrm{T}}Bu^* \leqslant \sum_{i=1}^{s^*} t_i^* f(z^*, \bar{y}_i^*) + z^{*\mathrm{T}}Bu^* + \sum_{j=1}^{m} \mu_j^* g_j(z^*) - \frac{1}{2} p^{*\mathrm{T}} \nabla^2 \left[\sum_{i=1}^{s^*} t_i^* f(z^*, \bar{y}_i^*) + \sum_{j=1}^{m} \mu_j^* g_j(z^*) \right] p^*. \quad (3.4)$$

The second-order strict (F, α, ρ, d)-type-I assumption on $[f(\cdot, \bar{y}_i^*)+(\cdot)^{\mathrm{T}}Bu^*,\ i=1, 2, \ldots, s^*,\ g_j(\cdot),\ j=1, 2, \ldots, m]$ at z^*, yields

$$f(x^*, \bar{y}_i^*) + x^{*\mathrm{T}}Bu^* - f(z^*, \bar{y}_i^*) - z^{*\mathrm{T}}Bu^* + \tfrac{1}{2} p^{*\mathrm{T}}\nabla^2 f(z^*, \bar{y}_i^*)p^* > F(x^*, z^*; \alpha^1(x^*, z^*)\{\nabla f(z^*, \bar{y}_i^*) + Bu^* + \nabla^2 f(z^*, \bar{y}_i^*)p^*\}) + \rho_i^1 d^2(x^*, z^*), \quad i = 1, 2, \ldots, s^*,$$

$$-g_j(z^*) + \tfrac{1}{2} p^{*\mathrm{T}}\nabla^2 g_j(z^*)p^* > F(x^*, z^*; \alpha^2(x^*, z^*)\{\nabla g_j(z^*) + \nabla^2 g_j(z^*)p^*\}) + \rho_j^2 d^2(x^*, z^*), \quad j = 1, 2, \ldots, m.$$

On multiplying the first inequality by $t_i^* \geqslant 0$, $i = 1, 2, \ldots, s^*$, second by $\mu_j^* \geqslant 0$, $j = 1, 2, \ldots, m$, respectively, and on using the sublinearity of F with $\sum_{i=1}^{s^*} t_i^* = 1$, we get

$$\sum_{i=1}^{s^*} t_i^* f(x^*, \bar{y}_i^*) + x^{*\mathrm{T}}Bu^* - \sum_{i=1}^{s^*} t_i^* f(z^*, \bar{y}_i^*) - z^{*\mathrm{T}}Bu^* + \frac{1}{2} p^{*\mathrm{T}}\nabla^2 \sum_{i=1}^{s^*} t_i^* f(z^*, \bar{y}_i^*)p^* > F\left(x^*, z^*; \alpha^1(x^*, z^*)\left\{\nabla \sum_{i=1}^{s^*} t_i^* f(z^*, \bar{y}_i^*) + Bu^* + \nabla^2 \sum_{i=1}^{s^*} t_i^* f(z^*, \bar{y}_i^*)p^*\right\}\right) + \sum_{i=1}^{s^*} t_i^* \rho_i^1 d^2(x^*, z^*),$$

$$-\sum_{j=1}^{m} \mu_j^* g_j(z^*) + \frac{1}{2} p^{*\mathrm{T}}\nabla^2 \sum_{j=1}^{m} \mu_j^* g_j(z^*)p^* \geqslant F\left(x^*, z^*; \alpha^2(x^*, z^*)\left\{\nabla \sum_{j=1}^{m} \mu_j^* g_j(z^*) + \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j(z^*)p^*\right\}\right) + \sum_{j=1}^{m} \mu_j^* \rho_j^2 d^2(x^*, z^*).$$

Combining these inequalities together with $\alpha^1(x^*, z^*) = \alpha^2(x^*, z^*)$, and the sublinearity of F, to imply

$$\sum_{i=1}^{s^*} t_i^* f(x^*, \bar{y}_i^*) + x^{*\mathrm{T}} Bu^* - \sum_{i=1}^{s^*} t_i^* f(z^*, \bar{y}_i^*) - z^{*\mathrm{T}} Bu^* + \frac{1}{2} p^{*\mathrm{T}} \nabla^2 \sum_{i=1}^{s^*} t_i^* f(z^*, \bar{y}_i^*) p^* - \sum_{j=1}^{m} \mu_j^* g_j(z^*)$$
$$+ \frac{1}{2} p^{*\mathrm{T}} \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j(z^*) p^* > F\left(x^*, z^*; \alpha^1(x^*, z^*)\left\{\nabla \sum_{i=1}^{s^*} t_i^* f(z^*, \bar{y}_i^*) + Bu^* + \nabla^2 \sum_{i=1}^{s^*} t_i^* f(z^*, \bar{y}_i^*) p^*\right.\right.$$
$$\left.\left.+\nabla \sum_{j=1}^{m} \mu_j^* g_j(z^*) + \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j(z^*) p^*\right\}\right) + \left(\sum_{i=1}^{s^*} t_i^* \rho_i^1 + \sum_{j=1}^{m} \mu_j^* \rho_j^2\right) d^2(x^*, z^*).$$

Since $\sum_{i=1}^{s^*} t_i^* \rho_i^1 + \sum_{j=1}^{m} \mu_j^* \rho_j^2 \geqslant 0$, we obtain

$$\sum_{i=1}^{s^*} t_i^* f(x^*, \bar{y}_i^*) + x^{*\mathrm{T}} Bu^* - \sum_{i=1}^{s^*} t_i^* f(z^*, \bar{y}_i^*) - z^{*\mathrm{T}} Bu^* - \sum_{j=1}^{m} \mu_j^* g_j(z^*)$$
$$+ \frac{1}{2} p^{*\mathrm{T}} \nabla^2 \left[\sum_{i=1}^{s^*} t_i^* f(z^*, \bar{y}_i^*) + \sum_{j=1}^{m} \mu_j^* g_j(z^*)\right] p^* > F\left(x^*, z^*; \alpha^1(x^*, z^*)\left\{\nabla \sum_{i=1}^{s^*} t_i^* f(z^*, \bar{y}_i^*)\right.\right.$$
$$\left.\left.+Bu^* + \nabla^2 \sum_{i=1}^{s^*} t_i^* f(z^*, \bar{y}_i^*) p^* + \nabla \sum_{j=1}^{m} \mu_j^* g_j(z^*) + \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j(z^*) p^*\right\}\right).$$

The above inequality along with (3.1) and the sublinearity of F reduces to

$$\sum_{i=1}^{s^*} t_i^* f(x^*, \bar{y}_i^*) + x^{*\mathrm{T}} Bu^* > \sum_{i=1}^{s^*} t_i^* f(z^*, \bar{y}_i^*) + z^{*\mathrm{T}} Bu^* + \sum_{j=1}^{m} \mu_j^* g_j(z^*)$$
$$- \frac{1}{2} p^{*\mathrm{T}} \nabla^2 \left[\sum_{i=1}^{s^*} t_i^* f(z^*, \bar{y}_i^*) + \sum_{j=1}^{m} \mu_j^* g_j(z^*)\right] p^*,$$

which is a contradiction to (3.4). Hence $z^* = x^*$. □

4. Second duality model

In this section, we discuss usual duality results for the following second-order dual to (P):

$$\text{(MD)} \quad \max_{(s,t,\bar{y}) \in K(z)} \sup_{(z,u,\mu,p) \in H_2(s,t,\bar{y})} \sum_{i=1}^{s} t_i f(z, \bar{y}_i) + z^{\mathrm{T}} Bu + \sum_{j \in J_0} \mu_j g_j(z)$$
$$- \frac{1}{2} p^{\mathrm{T}} \nabla^2 \left(\sum_{i=1}^{s} t_i f(z, \bar{y}_i) + \sum_{j \in J_0} \mu_j g_j(z)\right) p,$$

where $H_2(s, t, \bar{y})$ denotes the set of all $(z, u, \mu, p) \in R^n \times R^n \times R_+^m \times R^n$ satisfying

$$\nabla \sum_{i=1}^{s} t_i f(z, \bar{y}_i) + Bu + \nabla \sum_{j=1}^{m} \mu_j g_j(z) + \nabla^2 \sum_{i=1}^{s} t_i f(z, \bar{y}_i) p + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z) p = 0, \tag{4.1}$$

$$\sum_{j \in J_\beta} \mu_j g_j(z) - \frac{1}{2} p^{\mathrm{T}} \nabla^2 \sum_{j \in J_\beta} \mu_j g_j(z) p \geqslant 0, \quad \beta = 1, 2, \ldots, r, \tag{4.2}$$

$$u^{\mathrm{T}} Bu \leqslant 1, \tag{4.3}$$

where $J_\beta \subseteq M$, $\beta = 0, 1, 2, \ldots, r$ with $\bigcup_{\beta=0}^{r} J_\beta = M$ and $J_\beta \cap J_\gamma = \emptyset$, if $\beta \neq \gamma$.

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If, for a triplet $(s, t, \bar{y}) \in K(z)$, the set $H_2(s, t, \bar{y}) = \emptyset$, then we define the supremum over it to be $-\infty$.

Theorem 4.1 (*Weak duality*). *Let x and $(z, u, \mu, s, t, \bar{y}, p)$ be the feasible solutions of* (P) *and* (MD), *respectively. Assume that* $[\sum_{i=1}^{s} t_i f(\cdot, \bar{y}_i) + (\cdot)^{\mathrm{T}} Bu + \sum_{j\in J_0} \mu_j g_j(\cdot), \sum_{j\in J_\beta} \mu_j g_j(\cdot),\ \beta = 1, 2, \ldots, r]$ *is second-order* (F, α, ρ, d)-*pseudoquasi-type*-I *at z and*

$$\left(\frac{\rho_1^1}{\alpha^1(x,z)} + \frac{\sum_{\beta=1}^{r} \rho_\beta^2}{\alpha^2(x,z)}\right) \geqslant 0.$$

Then

$$\sup_{y\in Y} f(x,y) + (x^{\mathrm{T}}Bx)^{1/2} \geqslant \sum_{i=1}^{s} t_i f(z,\bar{y}_i) + z^{\mathrm{T}}Bu + \sum_{j\in J_0} \mu_j g_j(z) - \frac{1}{2} p^{\mathrm{T}} \nabla^2 \left(\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + \sum_{j\in J_0} \mu_j g_j(z)\right) p.$$

Proof. Suppose to the contrary that

$$\sup_{y\in Y} f(x,y) + (x^{\mathrm{T}}Bx)^{1/2} < \sum_{i=1}^{s} t_i f(z,\bar{y}_i) + z^{\mathrm{T}}Bu + \sum_{j\in J_0} \mu_j g_j(z) - \frac{1}{2} p^{\mathrm{T}} \nabla^2 \left(\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + \sum_{j\in J_0} \mu_j g_j(z)\right) p.$$

Thus, we obtain

$$f(x,\bar{y}_i) + (x^{\mathrm{T}}Bx)^{1/2} < \sum_{i=1}^{s} t_i f(z,\bar{y}_i) + z^{\mathrm{T}}Bu + \sum_{j\in J_0} \mu_j g_j(z) - \frac{1}{2} p^{\mathrm{T}} \nabla^2 \left(\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + \sum_{j\in J_0} \mu_j g_j(z)\right) p,$$

for all $\bar{y}_i \in Y(x)$, $i = 1, 2, \ldots, s$.

It follows from $t_i \geqslant 0$, $i = 1, 2, \ldots, s$, that

$$t_i \left[(f(x,\bar{y}_i) + (x^{\mathrm{T}}Bx)^{1/2}) - \left(\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + z^{\mathrm{T}}Bu + \sum_{j\in J_0} \mu_j g_j(z) - \frac{1}{2} p^{\mathrm{T}} \nabla^2 \left(\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + \sum_{j\in J_0} \mu_j g_j(z)\right) p \right) \right] \leqslant 0, \quad i = 1, 2, \ldots, s,$$

with at least one strict inequality, since $t = (t_1, t_2, \ldots, t_s) \neq 0$. Taking summation over i and using $\sum_{i=1}^{s} t_i = 1$, we have

$$\sum_{i=1}^{s} t_i f(x,\bar{y}_i) + (x^{\mathrm{T}}Bx)^{1/2} < \sum_{i=1}^{s} t_i f(z,\bar{y}_i) + z^{\mathrm{T}}Bu + \sum_{j\in J_0} \mu_j g_j(z) - \frac{1}{2} p^{\mathrm{T}} \nabla^2 \left(\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + \sum_{j\in J_0} \mu_j g_j(z)\right) p,$$

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which by (4.3) and Lemma 2.1, yields

$$\sum_{i=1}^{s} t_i f(x,\bar{y}_i) + x^{\mathrm{T}}Bu < \sum_{i=1}^{s} t_i f(z,\bar{y}_i) + z^{\mathrm{T}}Bu + \sum_{j\in J_0}\mu_j g_j(z) - \frac{1}{2}p^{\mathrm{T}}\nabla^2\left(\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + \sum_{j\in J_0}\mu_j g_j(z)\right)p. \tag{4.4}$$

Also from (4.2), we have

$$-\sum_{j\in J_\beta}\mu_j g_j(z) + \frac{1}{2}p^{\mathrm{T}}\nabla^2\sum_{j\in J_\beta}\mu_j g_j(z)p \leqslant 0, \quad \beta = 1,2,\ldots,r. \tag{4.5}$$

The inequalities (4.4), (4.5), and the second order (F,α,ρ,d)-pseudoquasi-type-I assumption on $[\sum_{i=1}^{s} t_i f(\cdot,\bar{y}_i) + (\cdot)^{\mathrm{T}}Bu + \sum_{j\in J_0}\mu_j g_j(\cdot), \sum_{j\in J_\beta}\mu_j g_j(\cdot),\ \beta = 1,2,\ldots,r]$, at z imply

$$F\left(x,z;\alpha^1(x,z)\left\{\nabla\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + Bu + \nabla\sum_{j\in J_0}\mu_j g_j(z) + \nabla^2\sum_{i=1}^{s} t_i f(z,\bar{y}_i)p + \nabla^2\sum_{j\in J_0}\mu_j g_j(z)p\right\}\right) < -\rho_1^1 d^2(x,z),$$

$$F\left(x,z;\alpha^2(x,z)\left\{\nabla\sum_{j\in J_\beta}\mu_j g_j(z) + \nabla^2\sum_{j\in J_\beta}\mu_j g_j(z)p\right\}\right) \leqslant -\rho_\beta^2 d^2(x,z), \quad \beta = 1,2,\ldots,r.$$

As $\alpha^1(x,z) > 0$, $\alpha^2(x,z) > 0$ and F is sublinear, we get

$$F\left(x,z;\nabla\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + Bu + \nabla\sum_{j\in J_0}\mu_j g_j(z) + \nabla^2\sum_{i=1}^{s} t_i f(z,\bar{y}_i)p + \nabla^2\sum_{j\in J_0}\mu_j g_j(z)p\right) < -\frac{\rho_1^1}{\alpha^1(x,z)}d^2(x,z),$$

$$F\left(x,z;\nabla\sum_{j\in J_\beta}\mu_j g_j(z) + \nabla^2\sum_{j\in J_\beta}\mu_j g_j(z)p\right) \leqslant -\frac{\rho_\beta^2}{\alpha^2(x,z)}d^2(x,z), \quad \beta = 1,2,\ldots,r.$$

Now, by the sublinearity of F, we summarize to get

$$\begin{aligned} &F\left(x,z;\nabla\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + Bu + \nabla\sum_{j=1}^{m}\mu_j g_j(z) + \nabla^2\sum_{i=1}^{s} t_i f(z,\bar{y}_i)p + \nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p\right) \\ &\leqslant F\left(x,z;\nabla\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + Bu + \nabla\sum_{j\in J_0}\mu_j g_j(z) + \nabla^2\sum_{i=1}^{s} t_i f(z,\bar{y}_i)p + \nabla^2\sum_{j\in J_0}\mu_j g_j(z)p\right) \\ &\quad + \sum_{\beta=1}^{r} F\left(x,z;\nabla\sum_{j\in J_\beta}\mu_j g_j(z) + \nabla^2\sum_{j\in J_\beta}\mu_j g_j(z)p\right) \\ &< -\left(\frac{\rho_1^1}{\alpha^1(x,z)} + \frac{\sum_{\beta=1}^{r}\rho_\beta^2}{\alpha^2(x,z)}\right)d^2(x,z). \end{aligned}$$

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Since

$$\left(\frac{\rho_1^1}{\alpha^1(x,z)}+\frac{\sum_{\beta=1}^{r}\rho_\beta^2}{\alpha^2(x,z)}\right)\geqslant 0,$$

we have

$$F\left(x,z;\nabla\sum_{i=1}^{s}t_i f(z,\bar{y}_i)+Bu+\nabla\sum_{j=1}^{m}\mu_j g_j(z)+\nabla^2\sum_{i=1}^{s}t_i f(z,\bar{y}_i)p+\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p\right)<0,$$

which is a contradiction to (4.1), as $F(x,z;0)=0$. □

The proof of the following theorem is identical to that of Theorem 3.2 and hence, being omitted.

Theorem 4.2 (*Strong duality*). *Assume that* x^* *is an optimal solution of* (P) *and* $\nabla g_j(x^*)$, $j\in J(x^*)$ *are linearly independent. Then there exist* $(s^*,t^*,\bar{y}^*)\in K(x^*)$ *and* $(x^*,u^*,\mu^*,p^*{=}0)\in H_2(s^*,t^*,\bar{y}^*)$ *such that* $(x^*,u^*,\mu^*,s^*,t^*,\bar{y}^*,p^*=0)$ *is a feasible solution of* (MD) *and the two objectives have the same values. Further, if the weak duality* (Theorem 4.1) *holds for all feasible solutions* $(z,u,\mu,s,t,\bar{y},p)$ *of* (MD), *then* $(x^*,u^*,\mu^*,s^*,t^*,\bar{y}^*,p^*=0)$ *is an optimal solution of* (MD).

Theorem 4.3 (*Strict converse duality*). *Let* x^* *and* $(z^*,u^*,\mu^*,s^*,t^*,\bar{y}^*,p^*)$ *be the optimal solutions of* (P) *and* (MD), *respectively. Suppose that* $[\sum_{i=1}^{s^*}t_i^* f(\cdot,\bar{y}_i^*)+(\cdot)^{\mathrm{T}}Bu^*+\sum_{j\in J_0}\mu_j^* g_j(\cdot),\ \sum_{j\in J_\beta}\mu_j^* g_j(\cdot),\ \beta=1,2,\ldots,r]$ *is second-order* (F,α,ρ,d)*-strictly pseudoquasi-type*-I *at* z^* *with*

$$\left(\frac{\rho_1^1}{\alpha^1(x^*,z^*)}+\frac{\sum_{\beta=1}^{r}\rho_\beta^2}{\alpha^2(x^*,z^*)}\right)\geqslant 0,$$

and $\nabla g_j(x^*)$, $j\in J(x^*)$ *are linearly independent. Then* $z^*=x^*$, *that is,* z^* *is an optimal solution of* (P).

Proof. It can be proved by a contradiction. □

Remark 4.1. If we take $\alpha^1(x,z)=\alpha^2(x,z)=1$; $\rho_1^1=0$, $\rho_\beta^2=0$, $\beta=1,2,\ldots,r$, and $F(x,z;a)=\eta^{\mathrm{T}}(x,z)a$, for a certain mapping $\eta\colon S\times X\to R^n$, in Theorems 4.1–4.3, we get Theorems 3.1–3.3 in [17].

5. Special cases

(i) Let $B=0$. Then (P) and (WD) reduce to one of the pairs discussed in [6].
(ii) If $B=0$ and $p=0$, then (P) and (WD) become the problems proposed by Tanimoto [22].
(iii) Let $B=0$. Then (P) and (MD) reduce to the primal and dual problems of Liu [14].
(iv) If we set $B=0$ and $J_0=\emptyset$ in (MD), then we get another dual obtained in [6].

6. Concluding remarks

In this paper, we discussed second-order duality results for two types of dual models of a nondifferentiable minmax programming problem involving generalized (F,α,ρ,d)-type-I functions. The present work can be further extended to a class of nondifferentiable minmax fractional programming problems [3,10].

The question arises as to whether the second-order duality results developed in this paper hold for the following complex minmax programming problem:

$$\text{(CP)}\quad \text{Minimize}\quad f(\xi)=\sup_{v\in W}\operatorname{Re}[\phi(\xi,v)+(z^{\mathrm{H}}Bz)^{1/2}]$$
$$\text{subject to}\quad \xi\in S^0=\{\xi\in \boldsymbol{C}^{2n}\colon -g(\xi)\in S\},$$

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where $\xi = (z, \bar{z})$, $v = (\omega, \bar{\omega})$ for $z \in \boldsymbol{C}^n$, $\omega \in \boldsymbol{C}^l$. $\phi(\cdot, \cdot)\colon \boldsymbol{C}^{2n} \times \boldsymbol{C}^{2l} \to \boldsymbol{C}$ is analytic with respect to ξ, W is a specified compact subset in $\boldsymbol{C}^{2l}$, S is a polyhedral cone in $\boldsymbol{C}^m$ and $g\colon \boldsymbol{C}^{2n} \to \boldsymbol{C}^m$ is analytic. Also $B \in \boldsymbol{C}^{n\times n}$ is a positive semidefinite Hermitian matrix.

It may be noted that for $B = 0$, (CP) is a complex minmax programming problem considered in [18].

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