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SECOND ORDER DUALITY IN MULTIOBJECTIVE PROGRAMMING

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Abstract. A nonlinear multiobjective programming problem is considered. Weak, strong and strict converse duality theorems are established under generalized second order (F, α, ρ, d)-convexity for second order Mangasarian type and general Mond-Weir type vector duals.

1. Introduction

In recent years, there has been an increasing interest in generalizations of convexity in connection with sufficiency and duality in optimization problems. It has been found that only a few properties of convex functions are needed for establishing sufficiency and duality theorems. Using properties needed as definitions of new classes of functions, it is possible to generalize the notion of convexity and to extend the validity of theorems to larger classes of optimization problems. Consequently, several classes of generalized convexity have been introduced. More specifically, the concept of

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(F,ρ)-convexity was introduced by Preda [16], an extension of F-convexity [9] and ρ-convexity [17], and he used the concept to obtain some duality results for Wolfe vector dual, Mond-Weir vector dual and general Mond-Weir vector dual to multiobjective programming problem. Gulati and Islam [8] established sufficiency and duality results for multiobjective programming problems under generalized F-convexity. Later on, Aghezzaf and Hachimi [1] and Ahmad [2] generalized these results involving generalized (F,ρ)-convex functions. For a more comprehensive view of optimality conditions and duality results in multiobjective programming, we refer [6, 7, 18] and references cited therein.

Mangasarian [13] first formulated the second order dual for a nonlinear programming problem and established duality results under somewhat involved assumptions. Mond [14] reproved second order duality theorems under simpler assumptions than those previously used by Mangasarian [13], and showed that the second order dual has computational advantages over the first order dual. Zhang and Mond [19] extended the class of (F,ρ)-convex functions to second order (F,ρ)-convex functions and obtained duality results for Mangasarian type, Mond-Weir type and general Mond-Weir type multiobjective dual problems.

A newly introduced concept of generalized convexity, named as (F,α,ρ,d)-convexity can be viewed in [4, 10, 11], while (F,α,ρ,d)-pseudoconvexity and (F,α,ρ,d)-quasiconvexity can be found in [5]. Recently, Ahmad and Husain [3] introduced the class of generalized second order (F,α,ρ,d)-convex functions and discussed duality results for Mond-Weir type vector dual.

Consider the following nonlinear multiobjective programming problem:

(MP) Minimize $f(x) = [f_1(x), f_2(x), \ldots, f_k(x)]$
subject to $x \in S = \{x \in X \colon g(x) \leqq 0\}$,

where $f = (f_1, f_2, \ldots, f_k)\colon X \mapsto \mathbb{R}^k$, $g = (g_1, g_2, \ldots, g_m)\colon X \mapsto \mathbb{R}^m$ are assumed to be twice differentiable functions over X, an open subset of $\mathbb{R}^n$.

In this paper, we establish duality theorems under generalized second order (F,α,ρ,d)-convexity, for second order Mangasarian type and general Mond-Weir type duals associated with (MP). These results extend the results obtained by Mond and Zhang [15], Zhang and Mond [19] and Ahmad [2].

2. Notations and preliminaries

Throughout the paper, following convention for vectors $x, y \in \mathbb{R}^n$ will be followed: $x \geqq y$ if and only if $x_i \geqq y_i$, $i = 1, 2, \ldots, n$; $x \geq y$ if and only if $x \geqq y$ and $x \neq y$; $x > y$ if and only if $x_i > y_i$, $i = 1, 2, \ldots, n$.

Definition 2.1. A point $\bar{x} \in S$ is said to be an efficient solution of the vector minimum problem (MP), if there exists no other $x \in S$ such that

$$f(x) \leq f(\bar{x}).$$

In the sequel, we require the following definitions [3].

Definition 2.2. A functional $F\colon X \times X \times \mathbb{R}^n \mapsto \mathbb{R}$ is said to be sublinear in its third argument, if for all $x, \bar{x} \in X$

(i) $F(x, \bar{x}; a+b) \leqq F(x, \bar{x}; a) + F(x, \bar{x}; b)$, for all $a, b \in \mathbb{R}^n$,
(ii) $F(x, \bar{x}; \beta a) = \beta F(x, \bar{x}; a)$, for all $\beta \in \mathbb{R}$, $\beta \geqq 0$, and for all $a \in \mathbb{R}^n$.

Let F be sublinear and the scalar function $\phi\colon X \mapsto \mathbb{R}$ be twice differentiable at $\bar{x} \in X$ and $\rho \in \mathbb{R}$.

Definition 2.3. The function ϕ is said to be second order (F, α, ρ, d)-convex at $\bar{x}$ on X, if for all $x \in X$, there exist vector $p \in \mathbb{R}^n$, a real valued function $\alpha\colon X \times X \mapsto \mathbb{R}_+ \setminus \{0\}$, and a real valued function $d(\cdot,\cdot)\colon X \times X \mapsto \mathbb{R}$ such that

$$\phi(x) - \phi(\bar{x}) + \frac{1}{2} p^t \nabla^2 \phi(\bar{x}) p \geqq F\left(x, \bar{x}; \alpha(x, \bar{x}) \left\{\nabla \phi(\bar{x}) + \nabla^2 \phi(\bar{x}) p\right\}\right) + \rho d^2(x, \bar{x}).$$

If for all $x \in X$, $x \neq \bar{x}$, the above inequality holds as strict inequality, then ϕ is said to be strictly second order (F, α, ρ, d)-convex at $\bar{x}$ on X.

Definition 2.4. The function ϕ is said to be second order (F, α, ρ, d)-pseudoconvex at $\bar{x}$ on X, if for all $x \in X$, there exist vector $p \in \mathbb{R}^n$, a real valued function $\alpha\colon X \times X \mapsto \mathbb{R}_+ \setminus \{0\}$, and a real valued function $d(\cdot,\cdot)\colon X \times X \mapsto \mathbb{R}$ such that

$$\phi(x) < \phi(\bar{x}) - \frac{1}{2} p^t \nabla^2 \phi(\bar{x}) p \Rightarrow F\left(x, \bar{x}; \alpha(x, \bar{x}) \left\{\nabla \phi(\bar{x}) + \nabla^2 \phi(\bar{x}) p\right\}\right) < -\rho d^2(x, \bar{x}).$$

Definition 2.5. The function ϕ is said to be strictly second order (F, α, ρ, d)-pseudoconvex at $\bar{x}$ on X, if for all $x \in X$, $x \neq \bar{x}$, there exist vector $p \in \mathbb{R}^n$, a real valued function $\alpha\colon X \times X \mapsto \mathbb{R}_+ \setminus \{0\}$, and a real valued function $d(\cdot,\cdot)\colon X \times X \mapsto \mathbb{R}$ such that

$$F\left(x, \bar{x}; \alpha(x, \bar{x}) \left\{\nabla \phi(\bar{x}) + \nabla^2 \phi(\bar{x}) p\right\}\right) \geqq -\rho d^2(x, \bar{x}) \Rightarrow \phi(x) > \phi(\bar{x}) - \frac{1}{2} p^t \nabla^2 \phi(\bar{x}) p,$$

or equivalently

$$\phi(x) \leqq \phi(\bar{x}) - \frac{1}{2}p^t\nabla^2\phi(\bar{x})p \Rightarrow F\left(x, \bar{x}; \alpha(x, \bar{x})\left\{\nabla\phi(\bar{x}) + \nabla^2\phi(\bar{x})p\right\}\right) < -\rho d^2(x, \bar{x}).$$

Definition 2.6. The function ϕ is said to be second order (F, α, ρ, d)-quasiconvex at $\bar{x}$ on X, if for all $x \in X$, there exist vector $p \in \mathbb{R}^n$, a real valued function $\alpha\colon X \times X \mapsto \mathbb{R}_+ \setminus \{0\}$, and a real valued function $d(\cdot, \cdot)\colon X \times X \mapsto \mathbb{R}$ such that

$$\phi(x) \leqq \phi(\bar{x}) - \frac{1}{2}p^t\nabla^2\phi(\bar{x})p \Rightarrow F\left(x, \bar{x}; \alpha(x, \bar{x})\left\{\nabla\phi(\bar{x}) + \nabla^2\phi(\bar{x})p\right\}\right) \leqq -\rho d^2(x, \bar{x}),$$

or equivalently

$$F\left(x, \bar{x}; \alpha(x, \bar{x})\left\{\nabla\phi(\bar{x}) + \nabla^2\phi(\bar{x})p\right\}\right) > -\rho d^2(x, \bar{x}) \Rightarrow \phi(x) > \phi(\bar{x}) - \frac{1}{2}p^t\nabla^2\phi(\bar{x})p.$$

A k-dimensional vector function $\psi = (\psi_1, \psi_2, \dots, \psi_k)$ is said to be second order (F, α, ρ, d)-convex, if each ψ_i, $i = 1, 2, \dots, k$, is second order (F, α, ρ_i, d)-convex for the same sublinear functional F. Other definitions follow similarly.

Remark 2.1. Let $\alpha(x, \bar{x}) = 1$. Then second order (F, α, ρ, d)-convexity becomes the second order (F, ρ)-convexity introduced by Zhang and Mond [19]. In addition, if we set second order term equal to zero i.e., $p = 0$, it reduces to (F, ρ)-convexity in [2, 16].

In [12], Maeda derived the following necessary conditions for a feasible solution x^* to be an efficient solution of (MP) under generalized Guignard constraint qualification (GGCQ). We need these conditions in the proof of strong duality theorems.

Theorem 2.1 (Kuhn-Tucker Type Necessary Conditions)**.** *Assume that x^* is an efficient solution for (MP) at which the generalized Guignard constraint qualification (GGCQ) is satisfied. Then there exist $\lambda^* \in \mathbb{R}^k$ and $y^* \in \mathbb{R}^m$, such that*

$$\begin{aligned}
&\lambda^{*t}\nabla f(x^*) + y^{*t}\nabla g(x^*) = 0,\\
&y^{*t}g(x^*) = 0,\\
&y^* \geqq 0,\\
&\lambda^* > 0,\ \lambda^{*t}e = 1.
\end{aligned}$$

3. Mangasarian type second order duality

In this section, we consider the following Mangasarian type second order dual associated with multiobjective problem (MP) and establish weak, strong and strict converse duality theorems under generalized second order (F, α, ρ, d)-convexity:

(WD) Maximize
$$\left(f_1(u) + y^t g(u) - \frac{1}{2}p^t \nabla^2 \left[f_1(u) + y^t g(u)\right] p, \ldots,\right.$$
$$\left. f_k(u) + y^t g(u) - \frac{1}{2}p^t \nabla^2 \left[f_k(u) + y^t g(u)\right] p\right)$$

subject to

$$\nabla \lambda^t f(u) + \nabla^2 \lambda^t f(u)p + \nabla y^t g(u) + \nabla^2 y^t g(u)p = 0, \tag{3.1}$$
$$y \geqq 0, \tag{3.2}$$
$$\lambda > 0, \tag{3.3}$$
$$\lambda^t e = 1, \tag{3.4}$$

where $e = (1, 1, \ldots, 1) \in \mathbb{R}^k$, λ is a k-dimensional vector, and is an m-dimensional vector.

Theorem 3.1 (Weak Duality)**.** *Suppose that for all feasible x in (MP) and all feasible (u, y, λ, p) in (WD)*

(i) *f_i, $i = 1, 2, \ldots, k$, is second order (F, α, ρ_i, d)-convex at u, and g_j, $j = 1, 2, \ldots, m$, is second order (F, α, σ_j, d)-convex at u, and*
$$\frac{1}{\alpha(x,u)}\left(\sum_{i=1}^{k} \lambda_i \rho_i + \sum_{j=1}^{m} \sigma_j y_j\right) \geqq 0;$$
or

(ii) *$\lambda^t f + y^t g$ is second order (F, α, ρ, d)-pseudoconvex at u, and*
$$\frac{\rho}{\alpha(x,u)} \geqq 0.$$

Then, the following cannot hold

$$f_i(x) \leqq f_i(u) + y^t g(u) - \frac{1}{2}p^t \nabla^2 \left[f_i(u) + y^t g(u)\right] p, \text{ for all } i \in K, \tag{3.5}$$

and

$$f_j(x) < f_j(u) + y^t g(u) - \frac{1}{2}p^t \nabla^2 \left[f_j(u) + y^t g(u)\right] p, \text{ for some } j \in K. \tag{3.6}$$

Proof. Let x be feasible for (MP) and (u, y, λ, p) feasible for (WD). Suppose contrary to the result that (3.5) and (3.6) hold. By $y \geqq 0$ and $g(x) \leqq 0$, we have

$$f_i(x) + y^t g(x) \leqq f_i(u) + y^t g(u) - \frac{1}{2} p^t \nabla^2 \left[f_i(u) + y^t g(u) \right] p, \quad \text{for all } i \in K, \tag{3.7}$$

and

$$f_j(x) + y^t g(x) < f_j(u) + y^t g(u) - \frac{1}{2} p^t \nabla^2 \left[f_j(u) + y^t g(u) \right] p, \quad \text{for some } j \in K. \tag{3.8}$$

(i) In view of the hypothesis $\lambda > 0$ and $\lambda^t e = 1$, we get

$$\lambda^t f(x) + y^t g(x) < \lambda^t f(u) + y^t g(u) - \frac{1}{2} p^t \nabla^2 \left[\lambda^t f(u) + y^t g(u) \right] p. \tag{3.9}$$

The second order (F, α, ρ_i, d)-convexity of f_i, $i = 1, 2, \ldots, k$, and the second order (F, α, σ_j, d)-convexity of g_j, $j = 1, 2, \ldots, m$, at u imply

$$\begin{aligned} & f_i(x) - f_i(u) + \frac{1}{2} p^t \nabla^2 f_i(u) p \\ & \geqq F\left(x, u; \alpha(x, u) \left\{ \nabla f_i(u) + \nabla^2 f_i(u) p \right\}\right) + \rho_i d^2(x, u), \end{aligned}$$

$i = 1, 2, \ldots, k$, and

$$\begin{aligned} & g_j(x) - g_j(u) + \frac{1}{2} p^t \nabla^2 g_j(u) p \\ & \geqq F\left(x, u; \alpha(x, u) \left\{ \nabla g_j(u) + \nabla^2 g_j(u) p \right\}\right) + \sigma_j d^2(x, u), \end{aligned}$$

$j = 1, 2, \ldots, m$.

On multiplying the first inequality by $\lambda_i > 0$ and second by $y_j \geqq 0$, and then summing up to get

$$\begin{aligned} & \lambda^t f(x) + y^t g(x) - \lambda^t f(u) - y^t g(u) + \frac{1}{2} p^t \nabla^2 \left[\lambda^t f(u) + y^t g(u) \right] p \\ & \geqq F\left(x, u; \alpha(x, u) \left\{ \nabla \lambda^t f(u) + \nabla^2 \lambda^t f(u) p \right\}\right) \\ & + F\left(x, u; \alpha(x, u) \left\{ \nabla y^t g(u) + \nabla^2 y^t g(u) p \right\}\right) \\ & + \left(\sum_{i=1}^{k} \lambda_i \rho_i + \sum_{j=1}^{m} \sigma_j y_j \right) d^2(x, u), \end{aligned}$$

which in view of (3.9) and the sublinearity of F with $\alpha(x, u) > 0$ gives

$$\begin{aligned} & F\left(x, u; \nabla \lambda^t f(u) + \nabla^2 \lambda^t f(u) p + \nabla y^t g(u) + \nabla^2 y^t g(u) p\right) \\ & < -\frac{1}{\alpha(x, u)} \left(\sum_{i=1}^{k} \lambda_i \rho_i + \sum_{j=1}^{m} \sigma_j y_j \right) d^2(x, u). \end{aligned}$$

Since

$$\frac{1}{\alpha(x,u)}\left(\sum_{i=1}^{k}\lambda_i\rho_i+\sum_{j=1}^{m}\sigma_j y_j\right)\geqq 0,$$

the above inequality implies

$$F\left(x,u;\nabla\lambda^t f(u)+\nabla^2\lambda^t f(u)p+\nabla y^t g(u)+\nabla^2 y^t g(u)p\right)<0,$$

a contradiction to (3.1), since $F\,(x,u;0)=0$.

(ii) The second order (F,α,ρ,d)-pseudoconvexity of $\lambda^t f+y^t g$ at u along with (3.9) yields

$$F\left(x,u;\alpha(x,u)\left\{\nabla\lambda^t f(u)+\nabla^2\lambda^t f(u)p+\nabla y^t g(u)+\nabla^2 y^t g(u)p\right\}\right)$$
$$<-\rho d^2(x,u),$$

which together with the sublinearity of F and $\alpha(x,u)>0$ gives

$$F\left(x,u;\nabla\lambda^t f(u)+\nabla^2\lambda^t f(u)p+\nabla y^t g(u)+\nabla^2 y^t g(u)p\right)$$
$$<-\frac{\rho}{\alpha(x,u)}d^2(x,u).$$

Since

$$\frac{\rho}{\alpha(x,u)}\geqq 0,$$

then we have

$$F\left(x,u;\nabla\lambda^t f(u)+\nabla^2\lambda^t f(u)p+\nabla y^t g(u)+\nabla^2 y^t g(u)p\right)<0,$$

which again contradicts (3.1), since $F\,(x,u;0)=0$. □

Theorem 3.2 (Strong Duality)**.** *Let $\bar{x}$ be an efficient solution of (MP) at which the generalized Guignard constraint qualification $(GGCQ)$ is satisfied. Then there exist $\bar{y}\in\mathbb{R}^m$ and $\bar{\lambda}\in\mathbb{R}^k$, such that $(\bar{x},\bar{y},\bar{\lambda},\bar{p}=0)$ is feasible for (WD) and the corresponding objective values of (MP) and (WD) are equal.*

If, in addition, the assumptions of weak duality (Theorem 3.1) hold for all feasible solutions of (MP) and (WD), then $(\bar{x},\bar{y},\bar{\lambda},\bar{p}=0)$ is an efficient solution of (WD).

Proof. Since $\bar{x}$ is an efficient solution of (MP) at which the generalized Guignard constraint qualification (GGCQ) is satisfied, then by Theorem 2.1, there exist $\bar{y}\in\mathbb{R}^m$ and $\bar{\lambda}\in\mathbb{R}^k$, such that

$$\bar{\lambda}^t\nabla f(\bar{x})+\bar{y}^t\nabla g(\bar{x})=0,$$
$$\bar{y}^t g(\bar{x})=0,$$
$$\bar{y}\geqq 0,$$

$$\bar{\lambda} > 0,\ \bar{\lambda}^t e = 1.$$

Therefore, $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0)$ is feasible for (WD) and the corresponding objective values of (MP) and (WD) are equal. The efficiency of this feasible solution for (WD) thus follows from weak duality (Theorem 3.1). □

Theorem 3.3 (Strict Converse Duality)**.** *Let $\bar{x}$ and $(\bar{u}, \bar{y}, \bar{\lambda}, \bar{p})$ be the efficient solutions of (MP) and (WD) respectively, such that*

$$\bar{\lambda}^t f(\bar{x}) = \bar{\lambda}^t f(\bar{u}) + \bar{y}^t g(\bar{u}) - \frac{1}{2}\bar{p}^t \nabla^2 \left[\bar{\lambda}^t f(\bar{u}) + \bar{y}^t g(\bar{u})\right] \bar{p}. \tag{3.10}$$

Suppose that f_i, $i = 1, 2, \ldots, k$, is strictly second order (F, α, ρ_i, d)-convex at $\bar{u}$, and g_j, $j = 1, 2, \ldots, m$, is second order (F, α, σ_j, d)-convex at $\bar{u}$, and

$$\frac{1}{\alpha(\bar{x}, \bar{u})} \left(\sum_{i=1}^{k} \bar{\lambda}_i \rho_i + \sum_{j=1}^{m} \sigma_j \bar{y}_j \right) \geqq 0.$$

Then $\bar{x} = \bar{u}$; that is, $\bar{u}$ is an efficient solution of (MP).

Proof. We assume that $\bar{x} \neq \bar{u}$ and exhibit a contradiction. Since f_i, $i = 1, 2, \ldots, k$, is strictly second order (F, α, ρ_i, d)-convex at $\bar{u}$, and g_j, $j = 1, 2, \ldots, m$, is second order (F, α, σ_j, d)-convex at $\bar{u}$, we have

$$\begin{aligned} &f_i(\bar{x}) - f_i(\bar{u}) + \frac{1}{2}\bar{p}^t \nabla^2 f_i(\bar{u})\bar{p} \\ &> F\left(\bar{x}, \bar{u}; \alpha(\bar{x}, \bar{u}) \left\{\nabla f_i(\bar{u}) + \nabla^2 f_i(\bar{u})\bar{p}\right\}\right) + \rho_i d^2(\bar{x}, \bar{u}), \end{aligned}$$

$i = 1, 2, \ldots, k$, and

$$\begin{aligned} &g_j(\bar{x}) - g_j(\bar{u}) + \frac{1}{2}\bar{p}^t \nabla^2 g_j(\bar{u})\bar{p} \\ &\geqq F\left(\bar{x}, \bar{u}; \alpha(\bar{x}, \bar{u}) \left\{\nabla g_j(\bar{u}) + \nabla^2 g_j(\bar{u})\bar{p}\right\}\right) + \sigma_j d^2(\bar{x}, \bar{u}), \end{aligned}$$

$j = 1, 2, \ldots, m$.

On multiplying the first inequality by $\bar{\lambda}_i > 0$ and second by $\bar{y}_j \geqq 0$ and then summing up to get

$$\begin{aligned} &\bar{\lambda}^t f(\bar{x}) + \bar{y}^t g(\bar{x}) - \bar{\lambda}^t f(\bar{u}) - \bar{y}^t g(\bar{u}) + \frac{1}{2}\bar{p}^t \nabla^2 \left[\bar{\lambda}^t f(\bar{u}) + \bar{y}^t g(\bar{u})\right] \bar{p} \\ &> F\left(\bar{x}, \bar{u}; \alpha(\bar{x}, \bar{u}) \left\{\nabla \bar{\lambda}^t f(\bar{u}) + \nabla^2 \bar{\lambda}^t f(\bar{u})\bar{p}\right\}\right) \\ &+ F\left(\bar{x}, \bar{u}; \alpha(\bar{x}, \bar{u}) \left\{\nabla \bar{y}^t g(\bar{u}) + \nabla^2 \bar{y}^t g(\bar{u})\bar{p}\right\}\right) \\ &+ \left(\sum_{i=1}^{k} \bar{\lambda}_i \rho_i + \sum_{j=1}^{m} \sigma_j \bar{y}_j \right) d^2(\bar{x}, \bar{u}), \end{aligned}$$

which in view of (3.10) and the feasibility of $\bar{x}$ for (MP) implies

$$F\left(\bar{x},\bar{u};\alpha(\bar{x},\bar{u})\left\{\nabla\bar{\lambda}^t f(\bar{u})+\nabla^2\bar{\lambda}^t f(\bar{u})\bar{p}\right\}\right)$$
$$+F\left(\bar{x},\bar{u};\alpha(\bar{x},\bar{u})\left\{\nabla\bar{y}^t g(\bar{u})+\nabla^2\bar{y}^t g(\bar{u})\bar{p}\right\}\right)$$
$$<-\left(\sum_{i=1}^{k}\bar{\lambda}_i\rho_i+\sum_{j=1}^{m}\sigma_j\bar{y}_j\right)d^2(\bar{x},\bar{u}).$$

Since F is sublinear and $\alpha(\bar{x},\bar{u})>0$, then

$$F\left(\bar{x},\bar{u};\nabla\bar{\lambda}^t f(\bar{u})+\nabla^2\bar{\lambda}^t f(\bar{u})\bar{p}+\nabla\bar{y}^t g(\bar{u})+\nabla^2\bar{y}^t g(\bar{u})\bar{p}\right)$$
$$<-\frac{1}{\alpha(\bar{x},\bar{u})}\left(\sum_{i=1}^{k}\bar{\lambda}_i\rho_i+\sum_{j=1}^{m}\sigma_j\bar{y}_j\right)d^2(\bar{x},\bar{u}),$$

which in view of

$$\frac{1}{\alpha(\bar{x},\bar{u})}\left(\sum_{i=1}^{k}\bar{\lambda}_i\rho_i+\sum_{j=1}^{m}\sigma_j\bar{y}_j\right)\geqq 0$$

yields

$$F\left(\bar{x},\bar{u};\nabla\bar{\lambda}^t f(\bar{u})+\nabla^2\bar{\lambda}^t f(\bar{u})\bar{p}+\nabla\bar{y}^t g(\bar{u})+\nabla^2\bar{y}^t g(\bar{u})\bar{p}\right)<0,$$

a contradiction to (3.1), since $F(\bar{x},\bar{u};0)=0$. Hence, $\bar{x}=\bar{u}$. □

4. General Mond-Weir type second order duality

In this section, we consider the following general Mond-Weir type second order dual associated with multiobjective problem (MP):

(GMD) Maximize
$$\left(f_1(u)+\sum_{i\in I_0}y_i g_i(u)-\frac{1}{2}p^t\nabla^2\left[f_1(u)+\sum_{i\in I_0}y_i g_i(u)\right]p,\right.$$
$$\left.\ldots,f_k(u)+\sum_{i\in I_0}y_i g_i(u)-\frac{1}{2}p^t\nabla^2\left[f_k(u)+\sum_{i\in I_0}y_i g_i(u)\right]p\right)$$

subject to

$$\nabla\lambda^t f(u)+\nabla^2\lambda^t f(u)p+\nabla y^t g(u)+\nabla^2 y^t g(u)p=0, \tag{4.1}$$
$$\sum_{i\in I_\beta}y_i g_i(u)-\frac{1}{2}p^t\nabla^2\sum_{i\in I_\beta}y_i g_i(u)p\geqq 0,\ \beta=1,2,\ldots,r, \tag{4.2}$$
$$y\geqq 0, \tag{4.3}$$
$$\lambda>0, \tag{4.4}$$
$$\lambda^t e=1, \tag{4.5}$$

where $I_\beta \subseteq M = \{1, 2, \ldots, m\}$, $\beta = 0, 1, 2, \ldots, r$, with $I_\beta \cap I_\gamma = \emptyset$ if $\beta \neq \gamma$ and $\bigcup_{\beta=0}^{r} I_\beta = M$.

Theorem 4.1 (Weak Duality)**.** *Suppose that for all feasible x in (MP) and all feasible (u, y, λ, p) in (GMD)*

(i) $\sum_{i \in I_\beta} y_i g_i$, $\beta = 1, 2, \ldots, r$, *is second order $(F, \alpha, \sigma_\beta, d)$-quasiconvex at u, and assume that any one of the following conditions holds:*

(ii) $I_0 \neq M$, *for all $i \in K$, $f_i + \sum_{i \in I_0} y_i g_i$ is second order (F, α_1, ρ_i, d)-quasiconvex and for some $j \in K$, $f_j + \sum_{i \in I_0} y_i g_i$ is second order (F, α_1, ρ_j, d)-pseudoconvex at u, and*

$$\left[\frac{1}{\alpha(x,u)} \sum_{\beta=1}^{r} \sigma_\beta + \frac{1}{\alpha_1(x,u)} \sum_{i=1}^{k} \lambda_i \rho_i\right] \geqq 0;$$

(iii) $I_0 \neq M$, *$\lambda^t f + \sum_{i \in I_0} y_i g_i$ is second order (F, α_2, ρ, d)-pseudoconvex at u, and*

$$\left[\frac{1}{\alpha(x,u)} \sum_{\beta=1}^{r} \sigma_\beta + \frac{\rho}{\alpha_2(x,u)}\right] \geqq 0.$$

Then, the following cannot hold

$$f_i(x) \leqq f_i(u) + \sum_{i \in I_0} y_i g_i(u) - \frac{1}{2} p^t \nabla^2 \left[f_i(u) + \sum_{i \in I_0} y_i g_i(u)\right] p, \quad \text{for all } i \in K, \tag{4.6}$$

and

$$f_j(x) < f_j(u) + \sum_{i \in I_0} y_i g_i(u) - \frac{1}{2} p^t \nabla^2 \left[f_j(u) + \sum_{i \in I_0} y_i g_i(u)\right] p, \quad \text{for some } j \in K. \tag{4.7}$$

Proof. (i) Let x be any feasible solution in (MP) and (u, y, λ, p) be any feasible solution in (GMD). Then $y \geqq 0, g(x) \leqq 0$ and (4.2) yields

$$\sum_{i \in I_\beta} y_i g_i(x) \leqq 0 \leqq \sum_{i \in I_\beta} y_i g_i(u) - \frac{1}{2} p^t \nabla^2 \sum_{i \in I_\beta} y_i g_i(u) p, \quad \beta = 1, 2, \ldots, r. \tag{4.8}$$

Since $\sum_{i\in I_\beta} y_i g_i$, $\beta = 1, 2, \dots, r$, is second order $(F, \alpha, \sigma_\beta, d)$-quasiconvex at u, then (4.8) gives

$$F\left(x, u; \alpha(x,u)\left\{\nabla \sum_{i\in I_\beta} y_i g_i(u) + \nabla^2 \sum_{i\in I_\beta} y_i g_i(u) p\right\}\right) \leqq -\sigma_\beta d^2(x,u), \quad \beta = 1, 2, \dots, r.$$

The sublinearity of F with $\alpha(x,u) > 0$ implies

$$\begin{aligned} &F\left(x, u; \nabla \sum_{i\in M\setminus I_0} y_i g_i(u) + \nabla^2 \sum_{i\in M\setminus I_0} y_i g_i(u) p\right) \\ &\leqq \sum_{\beta=1}^{r} F\left(x, u; \nabla \sum_{i\in I_\beta} y_i g_i(u) + \nabla^2 \sum_{i\in I_\beta} y_i g_i(u) p\right) \\ &\leqq -\frac{1}{\alpha(x,u)}(\sum_{\beta=1}^{r} \sigma_\beta) d^2(x,u). \end{aligned} \tag{4.9}$$

Now suppose contrary to the result that (4.6) and (4.7) hold. By $y \geqq 0$ and $g(x) \leqq 0$, it follows that

$$f_i(x) + \sum_{i\in I_0} y_i g_i(x) \leqq f_i(u) + \sum_{i\in I_0} y_i g_i(u) - \frac{1}{2} p^t \nabla^2 \left[f_i(u) + \sum_{i\in I_0} y_i g_i(u)\right] p, \quad \text{for all } i \in K, \tag{4.10}$$

and

$$f_j(x) + \sum_{i\in I_0} y_i g_i(x) < f_j(u) + \sum_{i\in I_0} y_i g_i(u) - \frac{1}{2} p^t \nabla^2 \left[f_j(u) + \sum_{i\in I_0} y_i g_i(u)\right] p, \quad \text{for some } j \in K. \tag{4.11}$$

(ii) Using the second order (F, α_1, ρ_i, d)-quasiconvexity of $f_i + \sum_{i\in I_0} y_i g_i$, for all $i \in K$, and the second order (F, α_1, ρ_j, d)-pseudoconvexity of $f_j + \sum_{i\in I_0} y_i g_i$, for some $j \in K$, we have from (4.10) and (4.11)

$$F\left(x, u; \alpha_1(x,u)\left\{\nabla f_i(u) + \nabla^2 f_i(u) p + \nabla \sum_{i\in I_0} y_i g_i(u) + \nabla^2 \sum_{i\in I_0} y_i g_i(u) p\right\}\right)$$
$$\leqq -\rho_i d^2(x,u),$$

for all $i \in K$, and

$$F\left(x, u; \alpha_1(x,u)\left\{\nabla f_j(u) + \nabla^2 f_j(u)p + \nabla \sum_{i\in I_0} y_i g_i(u) + \nabla^2 \sum_{i\in I_0} y_i g_i(u)p\right\}\right)$$
$$< -\rho_j d^2(x,u),$$

for some $j \in K$.

The sublinearity of F, $\alpha_1(x,u) > 0$, $\lambda > 0$ and $\lambda^t e = 1$ imply

$$F\left(x, u; \nabla\lambda^t f(u) + \nabla^2\lambda^t f(u)p + \nabla \sum_{i\in I_0} y_i g_i(u) + \nabla^2 \sum_{i\in I_0} y_i g_i(u)p\right)$$
$$< -\frac{1}{\alpha_1(x,u)}\left(\sum_{i=1}^{k} \lambda_i\rho_i\right) d^2(x,u). \tag{4.12}$$

Using (4.9), (4.12) and the sublinearity of F, we get

$$\begin{aligned}
&F\left(x, u; \nabla\lambda^t f(u) + \nabla^2\lambda^t f(u)p + \nabla y^t g(u) + \nabla^2 y^t g(u)p\right)\\
&\leqq F\left(x, u; \nabla\lambda^t f(u) + \nabla^2\lambda^t f(u)p + \nabla \sum_{i\in I_0} y_i g_i(u) + \nabla^2 \sum_{i\in I_0} y_i g_i(u)p\right)\\
&+ F\left(x, u; \nabla \sum_{i\in M\setminus I_0} y_i g_i(u) + \nabla^2 \sum_{i\in M\setminus I_0} y_i g_i(u)p\right)\\
&< -\left[\frac{1}{\alpha(x,u)}\sum_{\beta=1}^{r}\sigma_\beta + \frac{1}{\alpha_1(x,u)}\sum_{i=1}^{k}\lambda_i\rho_i\right] d^2(x,u)\\
&\leqq 0 \qquad \left(\text{since } \left[\frac{1}{\alpha(x,u)}\sum_{\beta=1}^{r}\sigma_\beta + \frac{1}{\alpha_1(x,u)}\sum_{i=1}^{k}\lambda_i\rho_i\right] \geqq 0\right),
\end{aligned}$$

which is a contradiction to (4.1), since $F(x, u; 0) = 0$.

(iii) By $\lambda > 0$ and $\lambda^t e = 1$, (4.10) and (4.11) imply

$$\lambda^t f(x) + \sum_{i\in I_0} y_i g_i(x) < \lambda^t f(u) + \sum_{i\in I_0} y_i g_i(u) - \frac{1}{2}p^t\nabla^2\left[\lambda^t f(u) + \sum_{i\in I_0} y_i g_i(u)\right]p,$$

which by the second order (F, α_2, ρ, d)-pseudoconvexity of $\lambda^t f + \sum_{i\in I_0} y_i g_i$ at u gives

$$F\left(x,u;\alpha_2(x,u)\left\{\nabla\lambda^t f(u)+\nabla^2\lambda^t f(u)p+\nabla\sum_{i\in I_0} y_i g_i(u)+\nabla^2\sum_{i\in I_0} y_i g_i(u)p\right\}\right)$$
$$< -\rho d^2(x,u). \tag{4.13}$$

Using (4.9), (4.13) and the sublinearity of F with $\alpha_2(x,u) > 0$, we get

$$F\left(x,u;\nabla\lambda^t f(u)+\nabla^2\lambda^t f(u)p+\nabla y^t g(u)+\nabla^2 y^t g(u)p\right)$$
$$\leqq F\left(x,u;\nabla\lambda^t f(u)+\nabla^2\lambda^t f(u)p+\nabla\sum_{i\in I_0} y_i g_i(u)+\nabla^2\sum_{i\in I_0} y_i g_i(u)p\right)$$
$$+F\left(x,u;\nabla\sum_{i\in M\setminus I_0} y_i g_i(u)+\nabla^2\sum_{i\in M\setminus I_0} y_i g_i(u)p\right)$$
$$< -\left[\frac{1}{\alpha(x,u)}\sum_{\beta=1}^{r}\sigma_\beta+\frac{\rho}{\alpha_2(x,u)}\right]d^2(x,u).$$

Since

$$\left[\frac{1}{\alpha(x,u)}\sum_{\beta=1}^{r}\sigma_\beta+\frac{\rho}{\alpha_2(x,u)}\right]\geqq 0,$$

we have

$$F\left(x,u;\nabla\lambda^t f(u)+\nabla^2\lambda^t f(u)p+\nabla y^t g(u)+\nabla^2 y^t g(u)p\right)<0,$$

again a contradiction to (4.1), since $F(x,u;0)=0$. □

We now merely state the following strong duality theorem as its proof would run analogously to that of Theorem 3.2.

Theorem 4.2 (Strong Duality)**.** *Let $\bar{x}$ be an efficient solution of (MP) at which the generalized Guignard constraint qualification $(GGCQ)$ is satisfied. Then there exist $\bar{y}\in\mathbb{R}^m$ and $\bar{\lambda}\in\mathbb{R}^k$, such that $(\bar{x},\bar{y},\bar{\lambda},\bar{p}=0)$ is feasible for (GMD) and the corresponding objective values of (MP) and (GMD) are equal.*

If, in addition, the assumptions of weak duality (Theorem 4.1) hold for all feasible solutions of (MP) and (GMD), then $(\bar{x},\bar{y},\bar{\lambda},\bar{p}=0)$ is an efficient solution of (GMD).

Theorem 4.3 (Strict Converse Duality)**.** *Let $\bar{x}$ and $(\bar{u},\bar{y},\bar{\lambda},\bar{p})$ be the efficient solutions of (MP) and (GMD) respectively, such that*

$$\bar{\lambda}^t f(\bar{x}) = \bar{\lambda}^t f(\bar{u}) + \sum_{i \in I_0} \bar{y}_i g_i(\bar{u}) - \frac{1}{2}\bar{p}^t \nabla^2 \left[\bar{\lambda}^t f(\bar{u}) + \sum_{i \in I_0} \bar{y}_i g_i(\bar{u})\right] \bar{p}. \tag{4.14}$$

Suppose that any one of the following conditions is satisfied:

(i) $I_0 \neq M$, $\sum_{i \in I_\beta} \bar{y}_i g_i$, $\beta = 1, 2, \ldots, r$ *is second order* $(F, \alpha, \sigma_\beta, d)$-*quasiconvex at* $\bar{u}$ *and* $\bar{\lambda}^t f + \sum_{i \in I_0} \bar{y}_i g_i$ *is strictly second order* (F, α_1, ρ, d)-*pseudoconvex at* $\bar{u}$, *and*

$$\left[\frac{1}{\alpha(\bar{x}, \bar{u})} \sum_{\beta=1}^{r} \sigma_\beta + \frac{\rho}{\alpha_1(\bar{x}, \bar{u})}\right] \geqq 0;$$

(ii) $I_0 \neq M$, $\sum\limits_{i \in I_\beta} \bar{y}_i g_i$, $\beta = 1, 2, \ldots, r$, *is strictly second order* $(F, \alpha, \sigma_\beta, d)$-*pseudoconvex at* $\bar{u}$ *and* $\bar{\lambda}^t f + \sum_{i \in I_0} \bar{y}_i g_i$ *is second order* (F, α_1, ρ, d)-*quasiconvex at* $\bar{u}$, *and*

$$\left[\frac{1}{\alpha(\bar{x}, \bar{u})} \sum_{\beta=1}^{r} \sigma_\beta + \frac{\rho}{\alpha_1(\bar{x}, \bar{u})}\right] \geqq 0.$$

Then $\bar{x} = \bar{u}$; *that is,* $\bar{u}$ *is an efficient solution of* (MP).

Proof. We assume that $\bar{x} \neq \bar{u}$ and exhibit a contradiction. Let $\bar{x}$ be feasible for (MP) and $(\bar{u}, \bar{y}, \bar{\lambda}, \bar{p})$ be feasible for (GMD). Then $\bar{y} \geqq 0, g(\bar{x}) \leqq 0$ and (4.2) yields

$$\sum_{i \in I_\beta} \bar{y}_i g_i(\bar{x}) \leqq 0 \leqq \sum_{i \in I_\beta} \bar{y}_i g_i(\bar{u}) - \frac{1}{2}\bar{p}^t \nabla^2 \sum_{i \in I_\beta} \bar{y}_i g_i(\bar{u}) \bar{p}, \quad \beta = 1, 2, \ldots, r, \tag{4.15}$$

which by the second order $(F, \alpha, \sigma_\beta, d)$-quasiconvexity of $\sum_{i \in I_\beta} \bar{y}_i g_i$ at $\bar{u}$ gives

$$F\left(\bar{x}, \bar{u}; \alpha(\bar{x}, \bar{u}) \left\{\nabla \sum_{i \in I_\beta} \bar{y}_i g_i(\bar{u}) + \nabla^2 \sum_{i \in I_\beta} \bar{y}_i g_i(\bar{u}) \bar{p}\right\}\right) \leqq -\sigma_\beta d^2(\bar{x}, \bar{u}), \quad \beta = 1, 2, \ldots, r. \tag{4.16}$$

The sublinearity of F and (4.16) with $\alpha(\bar{x}, \bar{u}) > 0$ imply

$$F(\bar{x}, \bar{u}; \nabla \sum_{i \in M \setminus I_0} \bar{y}_i g_i(\bar{u}) + \nabla^2 \sum_{i \in M \setminus I_0} \bar{y}_i g_i(\bar{u}) \bar{p})$$
$$\leqq \sum_{\beta=1}^{r} F(\bar{x}, \bar{u}; \nabla \sum_{i \in I_\beta} \bar{y}_i g_i(\bar{u}) + \nabla^2 \sum_{i \in I_\beta} \bar{y}_i g_i(\bar{u}) \bar{p})$$
$$\leqq -\frac{1}{\alpha(\bar{x}, \bar{u})} (\sum_{\beta=1}^{r} \sigma_\beta) d^2(\bar{x}, \bar{u}).$$

The first dual constraint and the above inequality along with the sublinearity of F imply

$$F\left(\bar{x}, \bar{u}; \nabla \bar{\lambda}^t f(\bar{u}) + \nabla^2 \bar{\lambda}^t f(\bar{u}) \bar{p} + \nabla \sum_{i \in I_0} \bar{y}_i g_i(\bar{u}) + \nabla^2 \sum_{i \in I_0} \bar{y}_i g_i(\bar{u}) \bar{p}\right)$$
$$\geqq \frac{1}{\alpha(\bar{x}, \bar{u})} \left(\sum_{\beta=1}^{r} \sigma_\beta) d^2(\bar{x}, \bar{u}\right).$$

Since

$$\left[\frac{1}{\alpha(\bar{x}, \bar{u})} \sum_{\beta=1}^{r} \sigma_\beta + \frac{\rho}{\alpha_1(\bar{x}, \bar{u})}\right] \geqq 0,$$

we have

$$F\left(\bar{x}, \bar{u}; \nabla \bar{\lambda}^t f(\bar{u}) + \nabla^2 \bar{\lambda}^t f(\bar{u}) \bar{p} + \nabla \sum_{i \in I_0} \bar{y}_i g_i(\bar{u}) + \nabla^2 \sum_{i \in I_0} \bar{y}_i g_i(\bar{u}) \bar{p}\right)$$
$$\geqq -\frac{\rho}{\alpha_1(\bar{x}, \bar{u})} d^2(\bar{x}, \bar{u}).$$

Since $\alpha_1(\bar{x}, \bar{u}) > 0$, we obtain

$$F\left(\bar{x}, \bar{u}; \alpha_1(\bar{x}, \bar{u}) \left\{\nabla \bar{\lambda}^t f(\bar{u}) + \nabla^2 \bar{\lambda}^t f(\bar{u}) \bar{p} + \nabla \sum_{i \in I_0} \bar{y}_i g_i(\bar{u}) +! + \nabla^2 \sum_{i \in I_0} \bar{y}_i g_i(\bar{u}) \bar{p}\right\}\right)$$
$$\geqq -\rho d^2(\bar{x}, \bar{u}),$$

which by the strict second order (F, α_1, ρ, d)-pseudoconvexity of $\bar{\lambda}^t f + \sum_{i \in I_0} \bar{y}_i g_i$ at $\bar{u}$ yields

$$\bar{\lambda}^t f(\bar{x}) + \sum_{i \in I_0} \bar{y}_i g_i(\bar{x}) > \bar{\lambda}^t f(\bar{u}) + \sum_{i \in I_0} \bar{y}_i g_i(\bar{u}) - \frac{1}{2} \bar{p}^t \nabla^2 \left[\bar{\lambda}^t f(\bar{u}) + \sum_{i \in I_0} \bar{y}_i g_i(\bar{u})\right] \bar{p}.$$

The above inequality in view of $\sum_{i \in I_0} \bar{y}_i g_i(\bar{x}) \leqq 0$ contradicts (4.14).

On the other hand, when hypothesis (ii) holds, the strict second order $(F, \alpha, \sigma_\beta, d)$-pseudoconvexity of $\sum_{i \in I_\beta} \bar{y}_i g_i$ and (4.15) yield

$$F\left(\bar{x}, \bar{u}; \alpha(\bar{x}, \bar{u}) \left\{\nabla \sum_{i \in I_\beta} \bar{y}_i g_i(\bar{u}) + \nabla^2 \sum_{i \in I_\beta} \bar{y}_i g_i(\bar{u}) \bar{p}\right\}\right) < -\sigma_\beta d^2(\bar{x}, \bar{u}), \quad \beta = 1, 2, \dots, r.$$

Using the sublinearity of F and $\alpha(\bar{x}, \bar{u}) > 0$, it follows from the above inequality that

$$F\left(\bar{x}, \bar{u}; \nabla \sum_{i \in M \setminus I_0} \bar{y}_i g_i(\bar{u}) + \nabla^2 \sum_{i \in M \setminus I_0} \bar{y}_i g_i(\bar{u}) \bar{p}\right) < -\frac{1}{\alpha(\bar{x}, \bar{u})} (\sum_{\beta=1}^{r} \sigma_\beta) d^2(\bar{x}, \bar{u})$$
$$\leqq \frac{\rho}{\alpha_1(\bar{x}, \bar{u})}) d^2(\bar{x}, \bar{u}).$$

Therefore, from the first dual constraint and the sublinearity of F with $\alpha_1(\bar{x}, \bar{u}) > 0$, we have

$$F\left(\bar{x}, \bar{u}; \alpha_1(\bar{x}, \bar{u}) \left\{\nabla \bar{\lambda}^t f(\bar{u}) + \nabla^2 \bar{\lambda}^t f(\bar{u}) + \nabla \sum_{i \in I_0} \bar{y}_i g_i(\bar{u}) + \nabla^2 \sum_{i \in I_0} \bar{y}_i g_i(\bar{u}) \bar{p}\right\}\right)$$
$$> -\rho d^2(\bar{x}, \bar{u}),$$

which by the virtue of second order (F, α_1, ρ, d)-quasiconvexity of $\bar{\lambda}^t f + \sum_{i \in I_0} \bar{y}_i g_i$ at $\bar{u}$ becomes

$$\bar{\lambda}^t f(\bar{x}) + \sum_{i \in I_0} \bar{y}_i g_i(\bar{x}) > \bar{\lambda}^t f(\bar{u}) + \sum_{i \in I_0} \bar{y}_i g_i(\bar{u}) - \frac{1}{2} \bar{p}^t \nabla^2 \left[\bar{\lambda}^t f(\bar{u}) + \sum_{i \in I_0} \bar{y}_i g_i(\bar{u})\right] \bar{p}.$$

Since $\bar{x}$ is feasible for (MP) and $\bar{y} \geqq 0$, we have

$$\bar{\lambda}^t f(\bar{x}) > \bar{\lambda}^t f(\bar{u}) + \sum_{i \in I_0} \bar{y}_i g_i(\bar{u}) - \frac{1}{2} \bar{p}^t \nabla^2 \left[\bar{\lambda}^t f(\bar{u}) + \sum_{i \in I_0} \bar{y}_i g_i(\bar{u})\right] \bar{p},$$

again contradicting (4.14). Hence, in both cases $\bar{x} = \bar{u}$. □

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