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Symmetric duality for multiobjective fractional variational problems with generalized invexity

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Abstract

The concept of symmetric duality for multiobjective fractional problems has been extended to the class of multiobjective variational problems. Weak, strong and converse duality theorems are proved under generalized invexity assumptions. A close relationship between these problems and multiobjective fractional symmetric dual problems is also presented.

Keywords: Multiobjective programming; Symmetric duality; Fractional variational programming; Pseudoinvexity; Properly efficient solutions

1. Introduction

In mathematical programming, a pair of primal and dual problems is called symmetric if the dual of the dual is the primal problem; that is, if the dual problem is expressed in the form of the primal problem, then its dual is the pri-

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mal problem. However, the majority of dual formulations in nonlinear programming do not possess this property. The first symmetric dual formulation for quadratic programming was proposed by Dorn [8]. Subsequently, Dantzig et al. [7] discussed symmetric dual programs and established symmetric duality under convexity–concavity assumption.

Mond and Hanson [15] first formulated a pair of symmetric dual variational problems by providing continuous analogue of the symmetric dual pair of Dantzig et al. [7] and proved usual duality theorems under convexity–concavity assumption. Later on, Smart and Mond [19] studied symmetric duality with invexity for variational problems, by omitting the non-negativity constraints those in [15]. They also reduced these problems to the static symmetric dual programs by relaxing the time dependency. In [10], Gulati et al. extended the results of Mond and Hanson [15] to a class of minimax mixed integer programming problems by constraining some of the primal and dual variables to belong to some arbitrary sets of integers.

Since the duality results for convex programming do not apply to fractional programs in general [16], duality concepts for such non-convex programs were defined seperately; e.g., [16,17]. Most duals in fractional programming are not symmetric [16–18]. Chandra et al. [3] first introduced a symmetric dual in nonlinear fractional programming. Further Gulati et al. [11] generalized these results to static and continuous nonlinear fractional programming. For the multiobjective case of static nonlinear fractional program, symmetric duality was introduced by Weir [20], and weak and strong duality theorems were derived under convexity assumptions. Recently, Yang et al. [21] extended the results in [20] to non-differentiable case involving support functions.

Kim and Lee [13] presented a pair of multiobjective symmetric dual variational problems and discussed duality results assuming invexity. In [12], Gulati et al. gave a different pair of multiobjective symmetric dual variational programs in which duality results are obtained under pseudoconvexity–pseudoconcavity assumption. Chen [5] derived symmetric duality results for multiobjective variational mixed integer programming problems. Recently, Chen [4] and Kim et al. [14] discussed duality results for multiobjective symmetric fractional variational problems involving invex functions.

The purpose of this paper is to introduce a continuous analogue of the (static) symmetric multiobjective fractional programs introduced by Weir [20] without non-negativity constraints for a class of multiobjective variational programming problems, and to establish duality results involving pseudoinvex functions.

2. Notations and preliminaries

Let $I = [a,b]$ be a real interval, and $f^i(t,x(t),\dot{x}(t),y(t),\dot{y}(t))$ and $g^i(t,x(t),\dot{x}(t),y(t),\dot{y}(t))$, where x: $I \to R^n$ and y: $I \to R^m$, with derivatives $\dot{x}$ and $\dot{y}$, are twice continuously differentiable functions, for $i \in \{1, 2, \ldots, k\}$. Superscripts denote vector components; subscripts denote partial derivatives. f^i_x, $f^i_{\dot{x}}$, f^i_y and $f^i_{\dot{y}}$ denote gradient vectors of the scalar function $f^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ with respect to x, $\dot{x}$, y and $\dot{y}$, i.e.,

$$f^i_x = \left(\frac{\partial f^i}{\partial x^1}, \ldots, \frac{\partial f^i}{\partial x^n}\right)^{\mathrm{T}}, \quad f^i_{\dot{x}} = \left(\frac{\partial f^i}{\partial \dot{x}^1}, \ldots, \frac{\partial f^i}{\partial \dot{x}^n}\right)^{\mathrm{T}},$$

$$f^i_y = \left(\frac{\partial f^i}{\partial y^1}, \ldots, \frac{\partial f^i}{\partial y^m}\right)^{\mathrm{T}}, \quad f^i_{\dot{y}} = \left(\frac{\partial f^i}{\partial \dot{y}^1}, \ldots, \frac{\partial f^i}{\partial \dot{y}^m}\right)^{\mathrm{T}}$$

for $i \in \{1, 2, \ldots, k\}$. Similarly g^i_x, $g^i_{\dot{x}}$, g^i_y and $g^i_{\dot{y}}$ denote the gradient vectors of $g^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ with respect to x, $\dot{x}$, y and $\dot{y}$.

The following observations are used for proving the strong duality theorem (Theorem 2):

$$\frac{\mathrm{d}}{\mathrm{d}t} f^i_{\dot{y}} = f^i_{\dot{y}t} + f^i_{\dot{y}y}\dot{y} + f^i_{\dot{y}\dot{y}}\ddot{y} + f^i_{\dot{y}x}\dot{x} + f^i_{\dot{y}\dot{x}}\ddot{x}.$$

Consequently,

$$\frac{\partial}{\partial y}\left(\frac{\mathrm{d}}{\mathrm{d}t} f^i_{\dot{y}}\right) = \frac{\mathrm{d}}{\mathrm{d}t} f^i_{\dot{y}y}, \frac{\partial}{\partial \dot{y}}\left(\frac{\mathrm{d}}{\mathrm{d}t} f^i_{\dot{y}}\right) = \frac{\mathrm{d}}{\mathrm{d}t} f^i_{\dot{y}\dot{y}} + f^i_{\dot{y}y},$$

$$\frac{\partial}{\partial \ddot{y}}\left(\frac{\mathrm{d}}{\mathrm{d}t} f^i_{\dot{y}}\right) = f^i_{\dot{y}\dot{y}}, \frac{\partial}{\partial x}\left(\frac{\mathrm{d}}{\mathrm{d}t} f^i_{\dot{y}}\right) = \frac{\mathrm{d}}{\mathrm{d}t} f^i_{\dot{y}x},$$

$$\frac{\partial}{\partial \dot{x}}\left(\frac{\mathrm{d}}{\mathrm{d}t} f^i_{\dot{y}}\right) = \frac{\mathrm{d}}{\mathrm{d}t} f^i_{\dot{y}\dot{x}} + f^i_{\dot{y}x}, \quad \frac{\partial}{\partial \ddot{x}}\left(\frac{\mathrm{d}}{\mathrm{d}t} f^i_{\dot{y}}\right) = f^i_{\dot{y}\dot{x}}, \qquad \text{for } i \in \{1, 2, \ldots, k\}.$$

Similarly, $\frac{\mathrm{d}}{\mathrm{d}t} g^i_{\dot{y}}$ can be defined.

Let $C(I, R^n)$ denote the space of piecewise smooth functions x with norm $\|x\| = \|x\|_\infty + \|\mathrm{D}x\|_\infty$, where the differentiation operator D is given by

$$u = \mathrm{D}x \Longleftrightarrow x(t) = \alpha + \int_0^t u(s)\mathrm{d}s,$$

where α is a given boundary value. Therefore $\frac{\mathrm{d}}{\mathrm{d}t} \equiv \mathrm{D}$ except at discontinuities.

Consider the following multiobjective variational problem studied by Bector and Husain [1]:

$$(\mathbf{P}) \quad \text{Minimize} \quad \left(\int_a^b \phi^1(t, x(t), \dot{x}(t))\mathrm{d}t, \ldots, \int_a^b \phi^k(t, x(t), \dot{x}(t))\mathrm{d}t\right)$$
$$\text{subject to} \quad x(a) = \alpha, \quad x(b) = \beta,$$
$$h(t, x(t), \dot{x}(t)) \leqq 0, \quad t \in I.$$

Let X denote the set of all feasible solutions of (P), i.e.,

$$X = \{x \in C(I, R^n) | x(a) = \alpha, x(b) = \beta, h(t, x(t), \dot{x}(t)) \leqq 0, \quad t \in I\}.$$

Definition 1 (*Geoffrion* [9]). A point $x^0 \in X$ is said to be an efficient solution of (P), if for all $x \in X$,

$$\int_a^b \phi^i(t, x^0(t), \dot{x}^0(t))\,\mathrm{d}t \geqq \int_a^b \phi^i(t, x(t), \dot{x}(t))\,\mathrm{d}t, \quad \text{for all} \quad i \in \{1, 2, \ldots, k\}$$

$$\Rightarrow \int_a^b \phi^i(t, x^0(t), \dot{x}^0(t))\,\mathrm{d}t = \int_a^b \phi^i(t, x(t), \dot{x}(t))\,\mathrm{d}t,$$

$$\text{for all} \quad i \in \{1, 2, \ldots, k\}.$$

The efficient point x^0 is said to be a properly efficient solution of (P), if there exists a scalar $M > 0$ such that, for all $i \in \{1, 2, \ldots, k\}$,

$$\int_a^b \phi^i(t, x^0(t), \dot{x}^0(t))\,\mathrm{d}t - \int_a^b \phi^i(t, x(t), \dot{x}(t))\,\mathrm{d}t \leqq M\left(\int_a^b \phi^j(t, x(t), \dot{x}(t))\,\mathrm{d}t - \int_a^b \phi^j(t, x^0(t), \dot{x}^0(t))\,\mathrm{d}t\right)$$

for some j, such that

$$\int_a^b \phi^j(t, x(t), \dot{x}(t))\,\mathrm{d}t > \int_a^b \phi^j(t, x^0(t), \dot{x}^0(t))\,\mathrm{d}t,$$

whenever $x \in X$, and

$$\int_a^b \phi^i(t, x(t), \dot{x}(t))\,\mathrm{d}t < \int_a^b \phi^i(t, x^0(t), \dot{x}^0(t))\,\mathrm{d}t.$$

An efficient solution that is not properly efficient is said to be improperly efficient. Thus for x^0 to be improperly efficient means that to every sufficiently large $M > 0$, there is an $x \in X$ and an $i \in \{1,2,\ldots,k\}$ such that

$$\int_a^b \phi^i(t, x(t), \dot{x}(t))\,\mathrm{d}t < \int_a^b \phi^i(t, x^0(t), \dot{x}^0(t))\,\mathrm{d}t$$

and

$$\left(\int_a^b \phi^i(t,x^0(t),\dot{x}^0(t))\,\mathrm{d}t - \int_a^b \phi^i(t,x(t),\dot{x}(t))\,\mathrm{d}t\right)$$
$$> M\left(\int_a^b \phi^j(t,x(t),\dot{x}(t))\,\mathrm{d}t - \int_a^b \phi^j(t,x^0(t),\dot{x}^0(t))\,\mathrm{d}t\right)$$

for all $j \in \{1,2,\ldots,k\}$ satisfying

$$\int_a^b \phi^j(t,x(t),\dot{x}(t))\,\mathrm{d}t > \int_a^b \phi^j(t,x^0(t),\dot{x}^0(t))\,\mathrm{d}t.$$

Definition 2 (*Borwein* [2]). A point $x^0 \in X$ is said to be a weak minimum for (P) if there exists no other $x \in X$ for which

$$\int_a^b \phi^i(t,x^0(t),\dot{x}^0(t))\,\mathrm{d}t > \int_a^b \phi^i(t,x(t),\dot{x}(t))\,\mathrm{d}t \quad \text{for all} \quad i \in \{1,2,\ldots,k\}.$$

From this, it follows that if $x^0 \in X$ is efficient for (P), then it is also a weak minimum for (P).

Definition 3. The functional $\int_a^b f(t,x(t),\dot{x}(t),y(t),\dot{y}(t))\mathrm{d}t$ is said to be pseudoinvex in x and $\dot{x}$ if there exists a vector function $\eta \in R^n$ with $\eta = 0$ at t if $x(t) = u(t)$ such that for each y and $\dot{y}$

$$\int_a^b [\eta^{\mathrm{T}} f_x(t,u,\dot{u},y,\dot{y}) + \frac{\mathrm{d}\eta^{\mathrm{T}}}{\mathrm{d}t} f_{\dot{x}}(t,u,\dot{u},y,\dot{y})]\,\mathrm{d}t \geqq 0$$
$$\Rightarrow \int_a^b f(t,x,\dot{x},y,\dot{y})\,\mathrm{d}t \geqq \int_a^b f(t,u,\dot{u},y,\dot{y})\,\mathrm{d}t,$$

for all x,u: $I \to R^n$.

Similarly, the functional $-\int_a^b f(t,x(t),\dot{x}(t),y(t),\dot{y}(t))\mathrm{d}t$ is said to be pseudoinvex in y and $\dot{y}$ if there exists a vector function $\xi \in R^m$ with $\xi = 0$ at t if $y(t) = v(t)$ such that for each x and $\dot{x}$

$$\int_a^b [\xi^{\mathrm{T}} f_y(t,x,\dot{x},y,\dot{y}) + \frac{\mathrm{d}\xi^{\mathrm{T}}}{\mathrm{d}t} f_{\dot{y}}(t,x,\dot{x},y,\dot{y})]\,\mathrm{d}t \leqq 0$$
$$\Rightarrow \int_a^b f(t,x,\dot{x},v,\dot{v})\,\mathrm{d}t \leqq \int_a^b f(t,x,\dot{x},y,\dot{y})\,\mathrm{d}t$$

for all y,v: $I \to R^m$.

In the sequel, we will write $\eta(t,x,u)$ for $\eta(t,x,\dot{x},u,\dot{u})$ and $\xi(t,v,y)$ for $\xi(t,v,\dot{v},y,\dot{y})$.

3. Symmetric duality

Now we introduce the continuous analogue of the static symmetric multiobjective fractional dual programs [20]:

Primal (FP)

$$\text{Minimize}\quad \left[\frac{\int_a^b f^1(t,x,\dot{x},y,\dot{y})\mathrm{d}t}{\int_a^b g^1(t,x,\dot{x},y,\dot{y})\mathrm{d}t},\ldots,\frac{\int_a^b f^k(t,x,\dot{x},y,\dot{y})\mathrm{d}t}{\int_a^b g^k(t,x,\dot{x},y,\dot{y})\mathrm{d}t}\right]$$

$$\begin{aligned}
\text{subject to}\quad & x(a)=0=x(b),\quad y(a)=0=y(b),\\
& \dot{x}(a)=0=\dot{x}(b),\quad \dot{y}(a)=0=\dot{y}(b),\\
& \sum_{i=1}^{k}\lambda^i\left\{G^i(x,y)\left(f_y^i-\mathrm{D}f_{\dot{y}}^i\right)-F^i(x,y)\left(g_y^i-\mathrm{D}g_{\dot{y}}^i\right)\right\}\leqq 0,\quad t\in I,\\
& y(t)^{\mathrm{T}}\sum_{i=1}^{k}\lambda^i\left\{G^i(x,y)\left(f_y^i-\mathrm{D}f_{\dot{y}}^i\right)-F^i(x,y)\left(g_y^i-\mathrm{D}g_{\dot{y}}^i\right)\right\}\geqq 0,\\
& \quad t\in I,\quad \lambda>0.
\end{aligned}$$

Dual (FD)

$$\text{Maximize}\quad \left[\frac{\int_a^b f^1(t,u,\dot{u},v,\dot{v})\mathrm{d}t}{\int_a^b g^1(t,u,\dot{u},v,\dot{v})\mathrm{d}t},\ldots,\frac{\int_a^b f^k(t,u,\dot{u},v,\dot{v})\mathrm{d}t}{\int_a^b g^k(t,u,\dot{u},v,\dot{v})\mathrm{d}t}\right]$$

$$\begin{aligned}
\text{subject to}\quad & u(a)=0=u(b),\quad v(a)=0=v(b),\\
& \dot{u}(a)=0=\dot{u}(b),\quad \dot{v}(a)=0=\dot{v}(b),\\
& \sum_{i=1}^{k}\lambda^i\left\{G^i(u,v)\left(f_x^i-\mathrm{D}f_{\dot{x}}^i\right)-F^i(u,v)\left(g_x^i-\mathrm{D}g_{\dot{x}}^i\right)\right\}\geqq 0,\quad t\in I,\\
& u(t)^{\mathrm{T}}\sum_{i=1}^{k}\lambda^i\left\{G^i(u,v)\left(f_x^i-\mathrm{D}f_{\dot{x}}^i\right)-F^i(u,v)\left(g_x^i-\mathrm{D}g_{\dot{x}}^i\right)\right\}\leqq 0,\\
& \quad t\in I,\quad \lambda>0,
\end{aligned}$$

where, for $i=1,2,\ldots,k$, f^i: $I\times R^n\times R^n\times R^m\times R^m\to R_+$, and g^i: $I\times R^n\times R^n\times R^m\times R^m\to R_+\backslash\{0\}$ are continuously differentiable functions and

$$F^i(x,y)=\int_a^b f^i(t,x,\dot{x},y,\dot{y})\,\mathrm{d}t,\quad G^i(x,y)=\int_a^b g^i(t,x,\dot{x},y,\dot{y})\,\mathrm{d}t.$$

It may be noted that unlike Kim et al. [14], we do not include the constraint $\lambda^{\mathrm{T}}e=1$ in the problems (FP) and (FD). This constraint does not play any role to prove the duality theorems. Its presence in the problems creates difficulties to establish strong and converse duality theorems, as the strong and converse duality theorems proved here could not be obtained with its presence in the problems (see Remark 2 after the strong duality theorem).

Remark 1. If $G^i(x,y) = 1$, $i = 1,2,\ldots,k$, then the fractional dual programs (FP) and (FD) become symmetric variational dual programs of Gulati et al. [12] wherein primal and dual problems include both the non-negative constraints $x(t) \geqq 0$ and $v(t) \geqq 0$. Moreover, if $k = 1$, the problems (FP) and (FD) reduce to the single objective symmetric dual fractional variational problems considered by Gulati et al. [11].

In order to simplify notation, we rewrite the primal and dual problems as follows:

Primal (FP′)

$$\begin{aligned}
\text{Minimize}\quad & q=(q^1,q^2,\ldots,q^k)\\
\text{subject to}\quad & x(a)=0=x(b),\quad y(a)=0=y(b), && (1)\\
& \dot{x}(a)=0=\dot{x}(b),\quad \dot{y}(a)=0=\dot{y}(b), && (2)\\
& \int_a^b f^i(t,x,\dot{x},y,\dot{y})\mathrm{d}t-q^i\int_a^b g^i(t,x,\dot{x},y,\dot{y})\mathrm{d}t=0,\\
& \quad i=1,2,\ldots,k, && (3)\\
& \sum_{i=1}^k \lambda^i\left\{\left(f_y^i-\mathrm{D}f_{\dot{y}}^i\right)-q^i\left(g_y^i-\mathrm{D}g_{\dot{y}}^i\right)\right\}\leqq 0,\quad t\in I, && (4)\\
& y(t)^{\mathrm{T}}\sum_{i=1}^k \lambda^i\left\{\left(f_y^i-\mathrm{D}f_{\dot{y}}^i\right)-q^i\left(g_y^i-\mathrm{D}g_{\dot{y}}^i\right)\right\}\geqq 0,\quad t\in I, && (5)\\
& \lambda>0. && (6)
\end{aligned}$$

Dual (FD′)

$$\begin{aligned}
\text{Maximize}\quad & p=(p^1,p^2,\ldots,p^k)\\
\text{subject to}\quad & u(a)=0=u(b),\quad v(a)=0=v(b), && (7)\\
& \dot{u}(a)=0=\dot{u}(b),\quad \dot{v}(a)=0=\dot{v}(b), && (8)\\
& \int_a^b f^i(t,u,\dot{u},v,\dot{v})\,\mathrm{d}t-p^i\int_a^b g^i(t,u,\dot{u},v,\dot{v})\,\mathrm{d}t=0,\\
& \quad i=1,2,\ldots,k, && (9)\\
& \sum_{i=1}^k \lambda^i\left\{\left(f_x^i-\mathrm{D}f_{\dot{x}}^i\right)-p^i\left(g_x^i-\mathrm{D}g_{\dot{x}}^i\right)\right\}\geqq 0,\quad t\in I, && (10)\\
& u(t)^{\mathrm{T}}\sum_{i=1}^k \lambda^i\left\{\left(f_x^i-\mathrm{D}f_{\dot{x}}^i\right)-p^i\left(g_x^i-\mathrm{D}g_{\dot{x}}^i\right)\right\}\leqq 0,\quad t\in I, && (11)\\
& \lambda>0. && (12)
\end{aligned}$$

We shall use H and Z for the sets of feasible solutions for the primal and dual multiobjective fractional variational problems, respectively.

The following weak and strong duality theorems are discussed in terms of (FP') and (FD'), but apply equally to (FP) and (FD).

Theorem 1 (Weak duality). *Let* $(x(t), y(t), \lambda, q)$ *be feasible for* (FP') *and* $(u(t), v(t), \lambda, p)$ *for* (FD'). *Assume that*

(i) $\sum_{i=1}^{k}\lambda^i \int_a^b \{f^i(t,\ldots,y,\dot{y}) - p^i g^i(t,\ldots,y,\dot{y})\}\mathrm{d}t$ *is pseudoinvex in* x *and* $\dot{x}$ *with* $\eta(t,x,u) + u(t) \geqq 0,\ t \in I$, *and*
(ii) $-\sum_{i=1}^{k}\lambda^i \int_a^b \{f^i(t,x,\dot{x},\ldots,) - q^i g^i(t,x,\dot{x},\ldots,\}\mathrm{d}t$ *is pseudoinvex in* y *and* $\dot{y}$ *with* $\xi(t,v,y) + y(t) \geqq 0,\ t \in I$.

Then

$$q \nleq p.$$

Proof. The relation (10) together with $\eta(t,x,u) + u(t) \geqq 0,\ t \in I$ implies

$$[\eta(t,x,u) + u(t)]^{\mathrm{T}}\left[\sum_{i=1}^{k}\lambda^i\{(f_x^i - \mathrm{D}f_{\dot{x}}^i) - p^i(g_x^i - \mathrm{D}g_{\dot{x}}^i)\}\right] \geqq 0,\ \ t \in I$$

or,

$$(\eta(t,x,u))^{\mathrm{T}}\left[\sum_{i=1}^{k}\lambda^i\{(f_x^i - \mathrm{D}f_{\dot{x}}^i) - p^i(g_x^i - \mathrm{D}g_{\dot{x}}^i)\}\right] \geqq 0,\ \ t \in I \quad \text{(using (11))},$$

which implies

$$\begin{aligned} 0 &\leqq \int_a^b \eta(t,x,u)^{\mathrm{T}}\left[\sum_{i=1}^{k}\lambda^i\{(f_x^i - p^i g_x^i) - D(f_{\dot{x}}^i - p^i g_{\dot{x}}^i)\}\right]\mathrm{d}t \\ &= \int_a^b \left[\eta(t,x,u)^{\mathrm{T}}\sum_{i=1}^{k}\lambda^i(f_x^i - p^i g_x^i) + \frac{\mathrm{d}\eta(t,x,u)^{\mathrm{T}}}{\mathrm{d}t}\sum_{i=1}^{k}\lambda^i(f_{\dot{x}}^i - p^i g_{\dot{x}}^i)\right]\mathrm{d}t \\ &\quad - \eta(t,x,u)^{\mathrm{T}}\sum_{i=1}^{k}\lambda^i(f_{\dot{x}}^i - p^i g_{\dot{x}}^i)\Bigg|_{t=a}^{b}, \quad \text{(integrating the second term by parts).} \end{aligned}$$

Since $\eta(t,x,u) = 0$, at $t = a$ and $t = b$, the above inequality gives

$$\int_a^b \left[\eta(t,x,u)^{\mathrm{T}}\sum_{i=1}^{k}\lambda^i(f_x^i - p^i g_x^i) + \frac{\mathrm{d}\eta(t,x,u)^{\mathrm{T}}}{\mathrm{d}t}\sum_{i=1}^{k}\lambda^i(f_{\dot{x}}^i - p^i g_{\dot{x}}^i)\right]\mathrm{d}t \geqq 0.$$

This, because of pseudoinvexity condition (i), yields

$$\begin{aligned} &\sum_{i=1}^{k}\lambda^i\int_a^b\{f^i(t,x,\dot{x},v,\dot{v}) - p^i g^i(t,x,\dot{x},v,\dot{v})\}\mathrm{d}t \\ &\quad \geqq \sum_{i=1}^{k}\lambda^i\int_a^b\{f^i(t,u,\dot{u},v,\dot{v}) - p^i g^i(t,u,\dot{u},v,\dot{v})\}\mathrm{d}t. \end{aligned}$$

In view of (9), the above inequality gives

$$\sum_{i=1}^{k}\lambda^i\int_a^b\{f^i(t,x,\dot{x},v,\dot{v})-p^ig^i(t,x,\dot{x},v,\dot{v})\}\mathrm{d}t\geqq 0. \tag{13}$$

Now inequalities (4), (5) and $\xi(t,v,y)+y(t)\geqq 0$, $t\in I$ give

$$0\geqq\xi(t,v,y)^{\mathrm{T}}\left[\sum_{i=1}^{k}\lambda^i\{(f_y^i-\mathrm{D}f_{\dot{y}}^i)-q^i(g_{\dot{y}}^i-\mathrm{D}g_{\dot{y}}^i)\}\right],$$

which implies

$$\begin{aligned}0&\geqq\int_a^b\xi(t,v,y)^{\mathrm{T}}\left[\sum_{i=1}^{k}\lambda^i\{(f_y^i-q^ig_{\dot{y}}^i)-\mathrm{D}(f_{\dot{y}}^i-q^ig_{\dot{y}}^i)\}\right]\mathrm{d}t\\&=\int_a^b\left[\xi(t,v,y)^{\mathrm{T}}\sum_{i=1}^{k}\lambda^i(f_y^i-q^ig_y^i)+\frac{\mathrm{d}\xi(t,v,y)^{\mathrm{T}}}{\mathrm{d}t}\sum_{i=1}^{k}\lambda^i(f_{\dot{y}}^i-q^ig_{\dot{y}}^i)\right]\mathrm{d}t\\&\quad-\xi(t,v,y)^{\mathrm{T}}\sum_{i=1}^{k}\lambda^i(f_{\dot{y}}^i-q^ig_{\dot{y}}^i)\Big|_{t=a}^{b},\quad\text{(integrating by parts)}.\end{aligned}$$

Because $\xi(t,v,y)=0$, at $t=a$ and $t=b$, therefore we have

$$\int_a^b\left[\xi(t,v,y)^{\mathrm{T}}\sum_{i=1}^{k}\lambda^i(f_y^i-q^ig_y^i)+\frac{\mathrm{d}\xi(t,v,y)^{\mathrm{T}}}{\mathrm{d}t}\sum_{i=1}^{k}\lambda^i(f_{\dot{y}}^i-q^ig_{\dot{y}}^i)\right]\mathrm{d}t\leqq 0.$$

This, in view of pseudoinvexity condition (ii), implies

$$\begin{aligned}&\sum_{i=1}^{k}\lambda^i\int_a^b\{f^i(t,x,\dot{x},v,\dot{v})-q^ig^i(t,x,\dot{x},v,\dot{v})\}\,\mathrm{d}t\\&\quad\leqq\sum_{i=1}^{k}\lambda^i\int_a^b\{f^i(t,x,\dot{x},y,\dot{y})-q^ig^i(t,x,\dot{x},y,\dot{y})\}\,\mathrm{d}t,\end{aligned}$$

or

$$\sum_{i=1}^{k}\lambda^i\int_a^b f^i(t,x,\dot{x},v,\dot{v})-q^ig^i(t,x,\dot{x},v,\dot{v})\}\,\mathrm{d}t\leqq 0,\quad\text{(using (3))}.$$

The above inequality alongwith (13) yields

$$\sum_{i=1}^{k}\lambda^i(q^i-p^i)\int_a^b g^i(t,x,\dot{x},v,\dot{v})\,\mathrm{d}t\geqq 0. \tag{14}$$

If, for some $i\in\{1,2,\ldots,k\}$, $q^i<p^i$ and for all $j\in\{1,2,\ldots,k\}$ with $j\neq i$, $q^j\leqq p^j$, then since $\int_a^b g^i(t,x,\dot{x},v,\dot{v})\mathrm{d}t>0$ and $\lambda>0$, one would obtain a contradiction to (14); hence $q\nleq p$. □

Theorem 2 (Strong duality). *Let the hypotheses of Theorem* 1 *be satisfied. Assume that*

($A1$) $(\bar{x},\bar{y},\bar{\lambda},\bar{q})$ *is a properly efficient solution of* (FP').

($A2$) *the system* $[\phi(t)^{\mathrm{T}}\{\sum_{i=1}^{k}\bar{\lambda}^i(f^i_{yy}-\bar{q}^ig^i_{yy})-\mathrm{D}\sum_{i=1}^{k}\bar{\lambda}^i(f^i_{\dot{y}y}-\bar{q}^ig^i_{\dot{y}y})\}+\mathrm{D}\{\phi(t)^{\mathrm{T}}\mathrm{D}\sum_{i=1}^{k}\bar{\lambda}^i(f^i_{\dot{y}\dot{y}}-\bar{q}^ig^i_{\dot{y}\dot{y}})\}+\mathrm{D}^2\{-\phi(t)^{\mathrm{T}}\sum_{i=1}^{k}\bar{\lambda}^i(f^i_{\ddot{y}\dot{y}}-\bar{q}^ig^i_{\ddot{y}\dot{y}})\}]\phi(t)=0$ *implies* $\phi(t)=0$, $t\in I$, *and*

($A3$) *the set* $\{((f^1_y-\bar{q}^1g^1_y)-\mathrm{D}(f^1_{\dot{y}}-\bar{q}^1g^1_{\dot{y}})),\ldots,((f^k_y-\bar{q}^kg^k_y)-\mathrm{D}(f^k_{\dot{y}}-\bar{q}^kg^k_{\dot{y}}))\}$ *is linearly independent.*

Then $(\bar{x},\bar{y},\bar{q})$ *is a properly efficient solution of* (FD') *with* $\lambda=\bar{\lambda}$.

Proof. Since $(\bar{x},\bar{y},\bar{\lambda},\bar{q})$ is a properly efficient solution of (FP'), it is also weak efficient. Hence, applying the Fritz John optimality conditions [6], there exist $\alpha\in R^k$, $\beta\in R^k$, piecewise smooth γ: $I\to R^m$, ξ: $I\to R$, and $\delta\in R^k$ such that

$$L=\sum_{i=1}^{k}\alpha^i\bar{q}^i+\sum_{i=1}^{k}\beta^i(f^i-\bar{q}^ig^i)+[\gamma(t)-\xi(t)\bar{y}(t)]^{\mathrm{T}}\left[\sum_{i=1}^{k}\bar{\lambda}^i\{(f^i_y-\bar{q}^ig^i_y)-\mathrm{D}(f^i_{\dot{y}}-\bar{q}^ig^i_{\dot{y}})\}\right]-\delta^{\mathrm{T}}\bar{\lambda}$$

satisfies the following conditions at $(\bar{x},\bar{y},\bar{\lambda},\bar{q})$:

$$L_x-\mathrm{D}L_{\dot{x}}+\mathrm{D}^2L_{\ddot{x}}=0,\quad t\in I,\tag{15}$$

$$L_y-\mathrm{D}L_{\dot{y}}+\mathrm{D}^2L_{\ddot{y}}=0,\quad t\in I,\tag{16}$$

$$L_\lambda=0,\quad t\in I,\tag{17}$$

$$L_q=0,\quad t\in I,\tag{18}$$

$$\int_a^b\beta^i(f^i-\bar{q}^ig^i)\mathrm{d}t=0,\quad i\in\{1,2,\ldots,k\},\quad t\in I,\tag{19}$$

$$\gamma(t)^{\mathrm{T}}\sum_{i=1}^{k}\bar{\lambda}^i\{(f^i_y-\bar{q}^ig^i_y)-\mathrm{D}(f^i_{\dot{y}}-\bar{q}^ig^i_{\dot{y}})\}=0,\quad t\in I,\tag{20}$$

$$\xi(t)\bar{y}(t)^{\mathrm{T}}\sum_{i=1}^{k}\bar{\lambda}^i\{(f^i_y-\bar{q}^ig^i_y)-\mathrm{D}(f^i_{\dot{y}}-\bar{q}^ig^i_{\dot{y}})\}=0,\quad t\in I,\tag{21}$$

$$\delta^{\mathrm{T}}\bar{\lambda}=0,\tag{22}$$

$$(\alpha, \gamma(t), \xi(t), \delta) \geqq 0, \quad t \in I, \tag{23}$$

$$(\alpha, \beta, \gamma(t), \xi(t), \delta) \neq 0, \quad t \in I. \tag{24}$$

The above relations hold throughout the interval I, except at the corners of $(\bar{x}(t), \bar{y}(t), \bar{\lambda}, \bar{q})$, where (15) and (16) hold for unique right and left hand limits. The piecewise smooth functions γ and ξ are continuously differentiable except possibly at corners of $(\bar{x}(t), \bar{y}(t), \bar{\lambda}, \bar{q})$. □

Using the observations on $Df^i_{\dot{y}}$ and $Dg^i_{\dot{y}}$, $i \in \{1,2,\ldots,k\}$, from Section 2, Eqs. (15)–(18) become

$$\begin{aligned}
&\sum_{i=1}^{k} \beta^i \{(f^i_x - \bar{q}^i g^i_x) - D(f^i_{\dot{x}} - \bar{q}^i g^i_{\dot{x}})\} + (\gamma(t) - \xi(t)\bar{y}(t))^{\mathrm{T}} \\
&\quad \times \sum_{i=1}^{k} \bar{\lambda}^i \{(f^i_{yx} - \bar{q}^i g^i_{yx}) - D(f^i_{\dot{y}x} - \bar{q}^i g^i_{\dot{y}x})\} - D(\gamma(t) - \xi(t)\bar{y}(t))^{\mathrm{T}} \\
&\quad \times \sum_{i=1}^{k} \bar{\lambda}^i \{(f^i_{y\dot{x}} - \bar{q}^i g^i_{y\dot{x}}) - D(f^i_{\dot{y}\dot{x}} - \bar{q}^i g^i_{\dot{y}\dot{x}}) - (f^i_{\dot{y}x} - \bar{q}^i g^i_{\dot{y}x})\} \\
&\quad + D^2 \left\{ -(\gamma(t) - \xi(t)\bar{y}(t))^{\mathrm{T}} \sum_{i=1}^{k} \bar{\lambda}^i (f^i_{\dot{y}\dot{x}} - \bar{q}^i g^i_{\dot{y}\dot{x}}) \right\} = 0, \quad t \in I,
\end{aligned} \tag{25}$$

$$\begin{aligned}
&\sum_{i=1}^{k} (\beta^i - \xi(t)\bar{\lambda}^i) \{(f^i_y - \bar{q}^i g^i_y) - D(f^i_{\dot{y}} - \bar{q}^i g^i_{\dot{y}})\} + (\gamma(t) \\
&\quad - \xi(t)\bar{y}(t))^{\mathrm{T}} \left[\sum_{i=1}^{k} \bar{\lambda}^i \{(f^i_{yy} - \bar{q}^i g^i_{yy}) - D(f^i_{\dot{y}y} - \bar{q}^i g^i_{\dot{y}y})\} \right] \\
&\quad + D \left[(\gamma(t) - \xi(t)\bar{y}(t))^{\mathrm{T}} \sum_{i=1}^{k} \bar{\lambda}^i \{D(f^i_{\dot{y}\dot{y}} - \bar{q}^i g^i_{\dot{y}\dot{y}})\} \right] \\
&\quad + D^2 \left\{ -(\gamma(t) - \xi(t)\bar{y}(t))^{\mathrm{T}} \sum_{i=1}^{k} \bar{\lambda}^i (f^i_{\dot{y}\dot{y}} - \bar{q}^i g^i_{\dot{y}\dot{y}}) \right\} = 0, \quad t \in I,
\end{aligned} \tag{26}$$

$$\begin{aligned}
&\{(f^i_y - \bar{q}^i g^i_y) - D(f^i_{\dot{y}} - \bar{q}^i g^i_{\dot{y}})\}(\gamma(t) - \xi(t)\bar{y}(t)) - \delta^i = 0, \\
&\quad i \in \{1, 2, \ldots, k\}, \quad t \in I
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
&\alpha^i - \beta^i g^i - (\gamma(t) - \xi(t)\bar{y}(t))^{\mathrm{T}} \bar{\lambda}^i (g^i_y - Dg^i_{\dot{y}}) = 0, \\
&\quad i \in \{1, 2, \ldots, k\}, \quad t \in I.
\end{aligned} \tag{28}$$

Since $\delta \geqq 0$ and $\bar{\lambda} > 0$, (22) yields $\delta = 0$. Consequently Eq. (27) becomes

$$\begin{aligned}
&\{(f^i_y - \bar{q}^i g^i_y) - D(f^i_{\dot{y}} - \bar{q}^i g^i_{\dot{y}})\}(\gamma(t) - \xi(t)\bar{y}(t)) = 0, \\
&\quad i \in \{1, 2, \ldots, k\}, \quad t \in I.
\end{aligned} \tag{29}$$

Multiplying (26) by $(\gamma(t)-\xi(t)\bar{y}(t))$ and using (29), we have

$$(\gamma(t)-\xi(t)\bar{y}(t))^{\mathrm{T}}\sum_{i=1}^{k}\bar{\lambda}^{i}\{(f_{yy}^{i}-\bar{q}^{i}g_{yy}^{i})-\mathrm{D}(f_{\dot{y}y}^{i}-\bar{q}^{i}g_{\dot{y}y}^{i})\}\cdot(\gamma(t) \\ -\xi(t)\bar{y}(t))+\mathrm{D}\left[(\gamma(t)-\xi(t)\bar{y}(t))^{\mathrm{T}}\sum_{i=1}^{k}\bar{\lambda}^{i}\{\mathrm{D}(f_{\dot{y}\dot{y}}^{i}-\bar{q}^{i}g_{\dot{y}\dot{y}}^{i})\}(\gamma(t)-\xi(t)\bar{y}(t))\right] \\ +\mathrm{D}^{2}\left[-(\gamma(t)-\xi(t)\bar{y}(t))^{\mathrm{T}}\sum_{i=1}^{k}\bar{\lambda}^{i}(f_{\dot{y}\dot{y}}^{i}-\bar{q}^{i}g_{\dot{y}\dot{y}}^{i})(\gamma(t)-\xi(t)\bar{y}(t))\right]=0. \tag{30}$$

This, in view of Hypothesis (A2), yields

$$\gamma(t)-\xi(t)\bar{y}(t)=0,\quad t\in I. \tag{31}$$

From (26), we have

$$\sum_{i=1}^{k}(\beta^{i}-\xi(t)\bar{\lambda}^{i})\{(f_{y}^{i}-\bar{q}^{i}g_{y}^{i})-D(f_{\dot{y}}^{i}-\bar{q}^{i}g_{\dot{y}}^{i})\}=0,$$

which, because of the hypothesis (A3), implies

$$\beta^{i}-\xi(t)\bar{\lambda}^{i}=0,\quad i\in\{1,2,\ldots,k\},$$

or

$$\beta=\xi(t)\bar{\lambda}. \tag{32}$$

If, for $t\in I$, $\xi(t)=0$, then from Eqs. (31) and (32) we get $\gamma(t)=0$ and $\beta=0$, respectively. Also, Eq. (28) gives $\alpha=0$. Hence $(\alpha,\beta,\gamma(t),\xi(t),\delta)=0$, contradicting the Fritz John condition (24). Thus $\xi(t)>0$, $t\in I$.

Now, Eqs. (25), (31) and (32) yield

$$\sum_{i=1}^{k}\bar{\lambda}^{i}\{(f_{x}^{i}-\bar{q}^{i}g_{x}^{i})-\mathrm{D}(f_{\dot{x}}^{i}-\bar{q}^{i}g_{\dot{x}}^{i})\}=0,\quad t\in I. \tag{33}$$

From (31), we get

$$\bar{y}(t)=\frac{\gamma(t)}{\xi(t)}\geqq 0,\quad t\in I. \tag{34}$$

Thus from (33) and (34), it follows that $(\bar{x},\bar{y},\bar{\lambda},\bar{q})\in Z$ i.e., $(\bar{x},\bar{y},\bar{q})$ is feasible for (FD') with $\lambda=\bar{\lambda}$, and the two objective functionals are obviously equal.

If $(\bar{x},\bar{y},\bar{q})$ is not efficient for (FD') with $\lambda=\bar{\lambda}$, then there exists a point (u^{0},v^{0},p^{0}) feasible for (FD') such that

$$p^{0}\geqslant\bar{q},$$

which contradicts the conclusion of Theorem 1. Now it remains to show that $(\bar{x},\bar{y},\bar{\lambda},\bar{q})$ is properly efficient. If it is not so, then for some $(u^0,v^0,\bar{\lambda},p^0)\in Z$ and some i, $p^{0i}-\bar{q}^i>M$ for any $M>0$.

Since $\int_a^b g^i(t,\bar{x},\dot{\bar{x}},v^0,\dot{v}^0)>0$ and $\bar{\lambda}^i>0,\ i\in\{1,2,\ldots,k\}$, it follows

$$\sum_{i=1}^{k}\bar{\lambda}^i(\bar{q}^i-p^{0i})\int_a^b g^i(t,\bar{x},\dot{\bar{x}},v^0,\dot{v}^0)\mathrm{d}t<0,$$

which contradicts the weak duality relation (14). Thus $(\bar{x},\bar{y},\bar{q})$ is properly efficient for (FD') with $\lambda=\bar{\lambda}$.

Remark 2. As stated in Section 3, unlike Kim et al. [14], we do not include the constraint $\lambda^{\mathrm{T}}e=1$ in the primal problem. If it is added in primal problem, then we need one more Lagrangian multiplier $\tau\in R$ corresponding to this constraint and therefore Eq. (27) is obtained as

$$\{(f_y^i-\bar{q}^ig_y^i)-\mathrm{D}(f_{\dot{y}}^i-\bar{q}^ig_{\dot{y}}^i)\}(\gamma(t)-\xi(t)\bar{y}(t))-\delta^i+\tau=0,$$
$$i\in\{1,2,\ldots,k\},\ t\in I,$$

As shown above, we get $\delta=0$ but we are unable to show $\tau=0$ and therefore we do not obtain Eq. (29), which plays a key role to derive the strong duality theorem.

A converse duality theorem may be stated as its proof would be analogous to Theorem 2.

Theorem 3 (Converse duality). *Let the hypotheses of Theorem* 1 *be satisfied. Assume that*

($A1$) $(\bar{u},\bar{v},\bar{\lambda},\bar{p})$ *is a properly efficient solution of* (FD').
($A2$) *the system* $[\phi(t)^{\mathrm{T}}\{\sum_{i=1}^k\bar{\lambda}^i\{(f_{xx}^i-\bar{p}^ig_{xx}^i)\}-\mathrm{D}\{\sum_{i=1}^k\bar{\lambda}^i(f_{\dot{x}x}^i-\bar{p}^ig_{\dot{x}x}^i)\}\}+\mathrm{D}\{\phi(t)^{\mathrm{T}}\mathrm{D}\{\sum_{i=1}^k\bar{\lambda}^i(f_{\dot{x}\dot{x}}^i-\bar{p}^ig_{\dot{x}\dot{x}}^i)\}+\mathrm{D}^2\{-\phi(t)^{\mathrm{T}}\sum_{i=1}^k\bar{\lambda}^i(f_{\dot{x}\dot{x}}^i-\bar{p}^ig_{\dot{x}\dot{x}}^i)\}]\phi(t)=0$ *implies* $\phi(t)=0$, $t\in I$, *and*
($A3$) *the set* $\{((f_x^1-\bar{p}^1g_x^1)-\mathrm{D}(f_{\dot{x}}^1-\bar{p}^1g_{\dot{x}}^1)),\ldots,((f_x^k-\bar{p}^kg_x^k)-\mathrm{D}(f_{\dot{x}}^k-\bar{p}^kg_{\dot{x}}^k))\}$ *is linearly independent.*

Then $(\bar{u},\bar{v},\bar{p})$ *is a properly efficient solution of* (FP') *with* $\lambda=\bar{\lambda}$.

4. Static multiobjective symmetric dual programs

If the time dependency of problems (FP) and (FD) is removed, then we obtain the following multiobjective fractional symmetric dual pair:

Primal (SFP)

$$\text{Minimize} \quad \left[\frac{f^1(x,y)}{g^1(x,y)}, \frac{f^2(x,y)}{g^2(x,y)}, \ldots, \frac{f^k(x,y)}{g^k(x,y)}\right]$$

$$\text{subject to} \quad \sum_{i=1}^{k} \lambda^i \left(g^i f_y^i - f^i g_y^i\right) \leqq 0,$$

$$y^{\mathrm{T}} \sum_{i=1}^{k} \lambda^i \left(g^i f_y^i - f^i g_y^i\right) \geqq 0, \quad \lambda > 0.$$

Dual (SFD)

$$\text{Maximize} \quad \left[\frac{f^1(u,v)}{g^1(u,v)}, \frac{f^2(u,v)}{g^2(u,v)}, \ldots, \frac{f^k(u,v)}{g^k(u,v)}\right]$$

$$\text{subject to} \quad \sum_{i=1}^{k} \lambda^i \left(g^i f_x^i - f^i g_x^i\right) \geqq 0,$$

$$u^{\mathrm{T}} \sum_{i=1}^{k} \lambda^i \left(g^i f_x^i - f^i g_x^i\right) \leqq 0, \quad \lambda > 0.$$

These are the programs considered in Weir [20] with the removal of $x \geqq 0$, $v \geqq 0$ from (SFP) and (SFD) respectively. The problems (SFP) and (SFD) do not include the constraint $\lambda^{\mathrm{T}} e = 1$, where $e = (1, 1, \ldots, 1) \in R^k$ as taken in [20], as it is not required in the validation of any of the theorems cited therein. Furthermore, the condition (A2) in Theorem 2 reduces in static case, to the condition that

$$\phi^{\mathrm{T}} \left[\sum_{i=1}^{k} \bar{\lambda}^i \{f_{yy}^i - \bar{q}^i g_{yy}^i\}\right] \phi = 0 \Rightarrow \phi = 0.$$

This is, of course, equivalent to the requirement that $\sum_{i=1}^{k} \bar{\lambda}^i \{f_{yy}^i - \bar{q}^i g_{yy}^i\}$ be positive or negative definite assumed in [20].

5. Conclusion

In this paper, we have presented symmetric duality results under generalized invexity assumptions for multiobjective fractional variational problems. Since we have used the weaker invexity assumptions to prove the duality relations, our results are significantly different and stronger than those which are appeared in [4,14]. In fact, in our paper, strong and converse duality theorems are established with hypotheses (A2) and (A3). The assumption (A2) is similar to assumption (I) in [4], while assumption (A3) is quite different from (II) in [4]. Also, in the present article (A3) is a linear independence assumption, and such a condition seems natural in the light of earlier studies in the static and dynamic case. Thus the results developed here improve and generalize the results of Chen [4] and Kim et al. [14]. It appears that these results can be further extended to other classes of multiobjective fractional variational programming problems including problems with non-smooth functions and minimax mixed integer programming problems etc.

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