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Second order symmetric duality in nondifferentiable multiobjective programming

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Abstract

A pair of Mond–Weir type second order symmetric nondifferentiable multiobjective programs is formulated. Weak, strong and converse duality theorems are established under η-pseudobonvexity assumptions. Special cases are discussed to show that this paper extends some work appeared in this area.

Keywords: Nondifferentiable programming; Multiobjective symmetric duality; η-Pseudobonvexity; Efficient solutions; Properly efficient solutions

1. Introduction

Symmetric duality in mathematical programming in which the dual of the dual is the primal problem was first introduced by Dorn [6]. Subsequently, Dantzig et al. [5], Mond [13] and Bazaraa and Goode [1] formulated a pair of symmetric dual programs and established duality under convexity–concavity

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assumptions. Later, Mond and Weir [16] presented a distinct pair of symmetric dual programs which allows the weakening of convexity–concavity conditions to pseudoconvexity–pseudoconcavity.

Weir and Mond [19] discussed symmetric duality in multiobjective programming by using the concept of efficiency. Chandra and Prasad [4] presented a pair of multiobjective programming problems by associating a vector valued infinite game to this pair. Gulati et al. [9] also established duality results for multiobjective symmetric dual problems without nonnegativity constraints.

Mangasarian [11] considered a nonlinear program and discussed second order duality under certain inequalities. Mond [14] assumed rather simple inequalities. Bector and Chandra [2] defined the functions satisfying the inequalities in [14] to be bonvex–boncave. To give examples of bonvex–boncave functions, Mond [14] has shown that a convex (concave) quadratic or linear function is bonvex (boncave). Mangasarian [11, p. 609] and Mond [14, p. 93] have also indicated possible computational advantages of the second order duals over the first order duals. Yang [20] also discussed second order Mangasarian type dual formulation under generalized representation conditions.

Gulati et al. [8] studied two distinct pairs of second order symmetric dual programs under η-bonvexity and η-pseudobonvexity. Recently, Hou and Yang [10] and Yang et al. [21] generalized the results in [8] to nondifferentiable programs on the lines of [15] involving second order F-convex and second order F-pseudoconvex functions.

In this paper, we formulate a new pair of second order symmetric nondifferentiable multiobjective dual programs of Mond–Weir type and prove duality theorems under η-pseudobonvexity assumptions. These results include, as special cases, recent duality results for multiobjective symmetric programs given by Suneja et al. [18] and for single objective symmetric programs studied by Mond and Schechter [15], Gulati et al. [8,9], Mishra [12], and Hou and Yang [10].

2. Preliminaries

Let F be a twice differentiable real valued function of x and y, where $x \in R^n$ and $y \in R^m$. Then $\nabla_x F$ and $\nabla_y F$ denote the gradient vectors with respect to x and y respectively. $\nabla_{xx} F$ and $\nabla_{yy} F$ are respectively, the $n \times n$ and $m \times m$ symmetric Hessian matrices. $(\nabla_{xx} Fr)_y$ denotes the matrix whose (i,j)th element is $\frac{\partial}{\partial y_i}(\nabla_{xx} Fr)_j$, where $r \in R^n$.

The following conventions for vectors in R^n will be used:

$$x \geqq u \iff x_i \geqq u_i, \quad i = 1, 2, \ldots, n;$$
$$x \geq u \iff x_i \geqq u_i, \quad i = 1, 2, \ldots, n, \text{ and } x \neq u;$$
$$x > u \iff x_i > u_i, \quad i = 1, 2, \ldots, n.$$

Definition 1. Let C be a compact convex set in R^n. The support function $s(x|C)$ of C is defined by

$$s(x|C) = \max\{x^t y : y \in C\}.$$

Definition 2. Let D be a nonempty convex set in R^n, and let $\psi : D \to R$ be convex. Then z is called a subgradient of ψ at $\bar{x} \in D$ if

$$\psi(x) \geqq \psi(\bar{x}) + z^t(x - \bar{x}) \quad \text{for all } x \in D.$$

A support function $s(x|C)$, being convex and everywhere finite, has a subdifferential, that is, there exists z such that $s(y|C) \geqq s(x|C) + z^t(y - x)$ for all $x \in C$. The set of all subdifferentials of $s(x|C)$ is given by

$$\partial s(x|C) = \{z \in C : z^t x = s(x|C)\}.$$

For a set S, the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) = \{y : y^t(z - x) \leqq 0 \quad \text{for all } z \in S\}.$$

When C is a compact convex set, then y is in $N_C(x)$ if and only if $s(y|C) = x^t y$, i.e., x is a subdifferential of s at y.

Consider the following multiobjective programming problem:

$$\begin{aligned} (\mathbf{P}) \quad & \text{Minimize} \quad f(x) = [f_1(x), f_2(x), \ldots, f_k(x)] \\ & \text{subject to} \quad x \in X = \{x \in R^n : g(x) \leqq 0\}, \end{aligned}$$

where $f: R^n \to R^k$ and $g: R^n \to R^m$.

Definition 3 [7]. A point $\bar{x} \in X$ is said to be an efficient solution of (P), if there exists no other $x \in X$ such that $f(x) \leqslant f(\bar{x})$.

A point $\bar{x}$ is said to be properly efficient solution of (P), if it is efficient and if there exists a scalar $M > 0$ such that, for each $i \in \{1, 2, \ldots, k\}$ and $x \in X$ satisfying $f_i(x) < f_i(\bar{x})$, we have

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leqq M,$$

for some j such that $f_j(x) > f_j(\bar{x})$.

Definition 4 [3]. A point $\bar{x} \in X$ is said to be a weak efficient solution of (P), if there exists no other $x \in X$ with $f(x) < f(\bar{x})$.

It readily follows that if $\bar{x} \in X$ is efficient, then it is also weak efficient.

Definition 5. A real twice differentiable function $F(x,y)$ defined on $X \times Y$, where X and Y are open sets in R^n and R^m respectively, is said to be η-pseudobonvex at $u \in X$ for fixed $v \in Y$, if there exists a function $\eta : X \times X \to R^n$ such that for $r \in R^n$, $x \in X$,

$$\eta^t(x,u)[\nabla_x F(u,v) + \nabla_{xx} F(u,v) r] \geqq 0 \quad \Rightarrow \quad F(x,v) \geqq F(u,v) - \tfrac{1}{2} r^t \nabla_{xx} F(u,v) r.$$

A real twice differentiable function $F(x,y) : X \times Y \to R$ is said to be η-pseudoboncave if $-F$ is η-pseudobonvex.

3. Mond–Weir type symmetric duality

We now state the following pair of second order nondifferentiable multiobjective symmetric programs and establish weak, strong and converse duality theorems.

Primal (**MP**):
Minimize $L(x,y,z,p) = [L_1(x,y,z_1,p_1), L_2(x,y,z_2,p_2), \ldots, L_k(x,y,z_k,p_k)]$
subject to

$$\sum_{i=1}^{k} \lambda_i \left[\nabla_y f_i(x,y) - z_i + \nabla_{yy} f_i(x,y) p_i\right] \leqq 0, \tag{1}$$

$$y^t \sum_{i=1}^{k} \lambda_i \left[\nabla_y f_i(x,y) - z_i + \nabla_{yy} f_i(x,y) p_i\right] \geqq 0, \tag{2}$$

$$\lambda > 0, \tag{3}$$

$$z_i \in D_i, \quad i = 1,2,\ldots,k. \tag{4}$$

Dual (**MD**):
Maximize $H(u,v,w,r) = [H_1(u,v,w_1,r_1), H_2(u,v,w_2,r_2), \ldots, H_k(u,v,w_k,r_k)]$
subject to

$$\sum_{i=1}^{k} \lambda_i [\nabla_x f_i(u,v) + w_i + \nabla_{xx} f_i(u,v) r_i] \geqq 0, \tag{5}$$

$$u^t \sum_{i=1}^{k} \lambda_i [\nabla_x f_i(u,v) + w_i + \nabla_{xx} f_i(u,v) r_i] \leqq 0, \tag{6}$$

$$\lambda > 0, \tag{7}$$

$$w_i \in C_i, \quad i = 1,2,\ldots,k, \tag{8}$$

where

$$L_i(x,y,z_i,p_i) = f_i(x,y) + s(x|C_i) - y^t z_i - \tfrac{1}{2} p_i^t \nabla_{yy} f_i(x,y) p_i,$$

$$H_i(u, v, w_i, r_i) = f_i(u, v) - s(v|D_i) + u^t w_i - \tfrac{1}{2} r_i^t \nabla_{xx} f_i(u, v) r_i,$$

$\lambda_i \in R$, $p_i \in R^m$, $r_i \in R^n$, $i = 1, 2, \ldots, k$, and f_i, $i = 1, 2, \ldots, k$ are thrice differentiable functions from $R^n \times R^m$ to R, C_i and D_i, $i = 1, 2, \ldots, k$ are compact convex sets in R^n and R^m. Also we take $p = (p_1, p_2, \ldots, p_k)$, $r = (r_1, r_2, \ldots, r_k)$, $w = (w_1, w_2, \ldots, w_k)$ and $z = (z_1, z_2, \ldots, z_k)$.

Theorem 1 (Weak duality). *Let* (x, y, λ, z, p) *be feasible for* (MP) *and* (u, v, λ, w, r) *be feasible for* (MD). *Let*

(i) $\sum_{i=1}^{k} \lambda_i [f_i(\cdot, v) + (\cdot)^t w_i]$ *be* η_1*-pseudobonvex at* u,
(ii) $\sum_{i=1}^{k} \lambda_i [f_i(x, \cdot) - (\cdot)^t z_i]$ *be* η_2*-pseudoboncave at* y,
(iii) $\eta_1(x, u) + u \geqq 0$, *and*
(iv) $\eta_2(v, y) + y \geqq 0$.

Then

$$L(x, y, z, p) \nleq H(u, v, w, r).$$

Proof. From (5) and hypothesis (iii), we have

$$\eta_1^t(x, u) \sum_{i=1}^{k} \lambda_i [\nabla_x f_i(u, v) + w_i + \nabla_{xx} f_i(u, v) r_i]$$
$$\geqq -u^t \sum_{i=1}^{k} \lambda_i [\nabla_x f_i(u, v) + w_i + \nabla_{xx} f_i(u, v) r_i] \geqq 0 \quad \text{(by (6))}.$$

Therefore, hypothesis (i) implies

$$\sum_{i=1}^{k} \lambda_i [f_i(x, v) + x^t w_i] \geqq \sum_{i=1}^{k} \lambda_i \left[f_i(u, v) + u^t w_i - \frac{1}{2} r_i^t \nabla_{xx} f_i(u, v) r_i \right]. \tag{9}$$

From (1) and hypothesis (iv), it follows that

$$\eta_2^t(v, y) \sum_{i=1}^{k} \lambda_i \left[\nabla_y f_i(x, y) - z_i + \nabla_{yy} f_i(x, y) p_i \right]$$
$$\leqq -y^t \sum_{i=1}^{k} \lambda_i \left[\nabla_y f_i(x, y) - z_i + \nabla_{yy} f_i(x, y) p_i \right] \leqq 0 \quad \text{(using (2))},$$

which, in view of hypothesis (ii) gives

$$\sum_{i=1}^{k} \lambda_i [f(x, v) - v^t z_i] \leqq \sum_{i=1}^{k} \lambda_i \left[f_i(x, y) - y^t z_i - \frac{1}{2} p_i^t \nabla_{yy} f_i(x, y) p_i \right]. \tag{10}$$

The relations (9) and (10) yield

$$\sum_{i=1}^{k}\lambda_i\left[f_i(x,y)+x^tw_i-y^tz_i-\frac{1}{2}p_i^t\nabla_{yy}f_i(x,y)p_i\right]$$
$$\geqq\sum_{i=1}^{k}\lambda_i\left[f_i(u,v)+u^tw_i-v^tz_i-\frac{1}{2}r_i^t\nabla_{xx}f(u,v)r_i\right].$$

Finally, using $x^tw_i\leqq s(x|C_i)$, $w_i\in C_i$, $i=1,2,\ldots,k$ and $v^tz_i\leqq s(v|D_i)$, $z_i\in D_i$, $i=1,2,\ldots,k$, we obtain

$$\sum_{i=1}^{k}\lambda_i\left[f_i(x,y)+s(x|C_i)-y^tz_i-\frac{1}{2}p_i^t\nabla_{yy}f_i(x,y)p_i\right]$$
$$\geqq\sum_{i=1}^{k}\lambda_i\left[f_i(u,v)-s(v|D_i)+u^tw_i-\frac{1}{2}r_i^t\nabla_{xx}f(u,v)r_i\right].$$

Hence

$$L(x,y,z,p)\nleq H(u,v,w,r).\qquad\square$$

Theorem 2 (Strong duality). *Let f be thrice differentiable on $R^n\times R^m$. Let $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ be a weak efficient solution for* (MP); *fix $\lambda=\bar{\lambda}$ in* (MD) *and suppose that*

(A1) *$\nabla yy\, f_i$ is nonsingular for all $i=1,2,\ldots,k$,*
(A2) *the matrix $\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{yy}f_i\bar{p}_i)_y$ is positive or negative definite, and*
(A3) *the set $\left(\nabla_y f_1-\bar{z}_1+\nabla_{yy}f_1\bar{p}_1,\nabla_y f_2-\bar{z}_2+\nabla_{yy}f_2\bar{p}_2,\ldots,\nabla_y f_k-\bar{z}_k+\nabla_{yy}f_k\bar{p}_k\right)$ is linearly independent,*

then there exist $\bar{w}_i\in R^n$, $i=1,2,\ldots,k$ such that $\bar{p}=0$, $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{r}=0)$ is feasible for (MD) *and*

$$L(\bar{x},\bar{y},\bar{z},\bar{p})=H(\bar{x},\bar{y},\bar{w},\bar{r}).$$

Also, if the hypotheses of Theorem 1 *are satisfied for all feasible solutions of* (MP) *and* (MD), *then $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{r})$ is a properly efficient solution for* (MD).

Proof. Since $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ is a weak efficient solution of (MP), by the Fritz–John conditions [17], there exist $\alpha\in R^k$, $\beta\in R^m$, $v\in R$, $\delta\in R^k$ and $\bar{w}_i\in R^n$, $i=1,2,\ldots,k$ such that

$$\sum_{i=1}^{k}\alpha_i\left[\nabla_x f_i+\bar{w}_i-\frac{1}{2}(\nabla_{yy}f_i\bar{p}_i)_x\bar{p}_i\right]$$
$$+\sum_{i=1}^{k}\bar{\lambda}_i\left[\nabla_{yx}f_i+\left(\nabla_{yy}f_i\bar{p}_i\right)_x\right](\beta-v\bar{y})=0,\qquad(11)$$

$$\bar{w}_i \in C_i, \quad \bar{x}^t\bar{w}_i = s(\bar{x}|C_i), \qquad i = 1,2,\ldots,k, \tag{12}$$

$$\sum_{i=1}^{k}(\alpha_i - v\bar{\lambda}_i)[\nabla_y f_i - \bar{z}_i] + \sum_{i=1}^{k}\bar{\lambda}_i[\nabla_{yy}f_i](\beta - v\bar{y} - v\bar{p}_i) + \sum_{i=1}^{k}(\nabla_{yy}f_i\bar{p}_i)_y\left[(\beta - v\bar{y})\bar{\lambda}_i - \frac{1}{2}\alpha_i\bar{p}_i\right] = 0, \tag{13}$$

$$(\beta - v\bar{y})^t\left[\nabla_y f_i - \bar{z}_i + \nabla_{yy}f_i\bar{p}_i\right] - \delta_i = 0, \quad i = 1,2,\ldots,k, \tag{14}$$

$$[(\beta - v\bar{y})\bar{\lambda}_i - \alpha_i\bar{p}_i]^t\nabla_{yy}f_i, \quad i = 1,2,\ldots,k, \tag{15}$$

$$\alpha_i\bar{y} + \bar{\lambda}_i(\beta - v\bar{y}) \in N_{D_i}(\bar{z}_i), \quad i = 1,2,\ldots,k, \tag{16}$$

$$\beta^t\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i - \bar{z}_i + \nabla_{yy}f_i\bar{p}_i) = 0, \tag{17}$$

$$v\bar{y}^t\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i - \bar{z}_i + \nabla_{yy}f_i\bar{p}_i) = 0, \tag{18}$$

$$\delta^t\bar{\lambda} = 0, \tag{19}$$

$$(\alpha,\beta,v,\delta) \geqq 0, \tag{20}$$

$$(\alpha,\beta,v,\delta) \neq 0. \tag{21}$$

In view of $\bar{\lambda} > 0$ and $\delta \geqq 0$, it readily follows from (19) that $\delta = 0$. Therefore from (14), we have

$$(\beta - v\bar{y})^t\left[\nabla_y f_i - \bar{z}_i + \nabla_{yy}f_i\bar{p}_i\right] = 0, \quad i = 1,2,\ldots,k. \tag{22}$$

Since $\nabla_{yy}f_i$ is nonsingular for $i = 1,2,\ldots,k$, (15) yields

$$(\beta - v\bar{y})\bar{\lambda}_i = \alpha_i\bar{p}_i, \quad i = 1,2,\ldots,k. \tag{23}$$

Now from (13),

$$\sum_{i=1}^{k}(\alpha_i - v\bar{\lambda}_i)(\nabla_y f_i - \bar{z}_i + \nabla_{yy}f_i\bar{p}_i) + \frac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{yy}f_i\bar{p}_i)_y(\beta - v\bar{y}) = 0. \tag{24}$$

On multiplying (24) by $(\beta - v\bar{y})^t$ from the left and using (22), we get

$$(\beta - v\bar{y})^t\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{yy}f_i\bar{p}_i)_y(\beta - v\bar{y}) = 0,$$

which by the hypothesis (A2) implies

$$\beta = v\bar{y}. \tag{25}$$

Therefore, from (24)

$$\sum_{i=1}^{k}(\alpha_i - v\bar{\lambda}_i)(\nabla_y f_i - \bar{z}_i + \nabla_{yy} f_i \bar{p}_i) = 0,$$

which by the hypothesis (A3) yields

$$\alpha_i = v\bar{\lambda}_i, \quad i = 1, 2, \ldots, k. \tag{26}$$

Suppose $v = 0$. Then from (25) and (26), we get $\beta = 0$ and $\alpha = 0$ respectively. Thus $(\alpha, \beta, v, \delta) = 0$, a contradiction to (21). Hence

$$v > 0. \tag{27}$$

Since $\bar{\lambda}_i > 0$, $i = 1, 2, \ldots, k$, from (26) and (27), we get

$$\alpha_i > 0, \quad i = 1, 2, \ldots, k.$$

Using (25) in (23), we have

$$\alpha_i \bar{p}_i = 0, \quad i = 1, 2, \ldots, k,$$

and hence

$$\bar{p}_i = 0, \quad i = 1, 2, \ldots, k. \tag{28}$$

Now using relations (25) and (28) in (11), it follows that

$$\sum_{i=1}^{k} \alpha_i [\nabla_x f_i + \bar{w}_i] = 0,$$

which by (26) gives

$$\sum_{i=1}^{k} \bar{\lambda}_i [\nabla_x f_i + \bar{w}_i] = 0,$$

and hence, we also have

$$\bar{x}^t \sum_{i=1}^{k} \bar{\lambda}_i [\nabla_x f_i + \bar{w}_i] = 0.$$

Therefore $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{r} = 0)$ is feasible for (MD).

From (16) and (25), we obtain

$$\bar{y}^t \bar{z}_i = s(\bar{y}|D_i), \quad i = 1, 2, \ldots, k. \tag{29}$$

By (12) and (29), we get

$$\begin{aligned} &f_i(\bar{x}, \bar{y}) + s(\bar{x}|C_i) - \bar{y}^t \bar{z}_i - \tfrac{1}{2}\bar{p}_i^t \nabla_{yy} f_i(\bar{x}, \bar{y})\bar{p}_i \\ &\quad = f_i(\bar{x}, \bar{y}) - s(\bar{y}|D_i) + \bar{x}^t \bar{w}_i - \tfrac{1}{2}\bar{r}_i^t \nabla_{xx} f_i(\bar{x}, \bar{y})\bar{r}_i, \quad i = 1, 2, \ldots, k. \end{aligned}$$

Thus $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{r} = 0)$ is a feasible solution of the dual problem (MD) and

$$L(\bar{x}, \bar{y}, \bar{z}, \bar{p}) = H(\bar{x}, \bar{y}, \bar{w}, \bar{r}). \tag{30}$$

Now we show proper efficiency of $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{r})$ for (MD) by exhibiting a contradiction. If $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{r})$ is not efficient for (MD) then there exists a feasible solution $(\bar{u},\bar{v},\bar{\lambda},\bar{w},\bar{r})$ for (MD) such that

$$H(\bar{x},\bar{y},\bar{w},\bar{r}) \leqslant H(\bar{u},\bar{v},\bar{w},\bar{r}).$$

In view of (30), it follows that

$$L(\bar{x},\bar{y},\bar{z},\bar{p}) \leqslant H(\bar{u},\bar{v},\bar{w},\bar{r}),$$

which contradicts Theorem 1. If $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{r})$ is not properly efficient for (MD), then for some feasible $(\bar{u},\bar{v},\bar{\lambda},\bar{w},\bar{r})$ of (MD) and for some i,

$$\begin{aligned} &f_i(\bar{u},\bar{v}) - s(\bar{v}|D_i) + \bar{u}^t\bar{w}_i - \frac{1}{2}\bar{r}_i^t\nabla_{xx}f_i(\bar{u},\bar{v})\bar{r}_i \\ &\quad > f_i(\bar{x},\bar{y}) - s(\bar{y}|D_i) + \bar{x}^t\bar{w}_i - \frac{1}{2}\bar{r}_i^t\nabla_{xx}f_i(\bar{x},\bar{y})\bar{r}_i \end{aligned}$$

and

$$\begin{aligned} &\left[f_i(\bar{u},\bar{v}) - s(\bar{v}|D_i) + \bar{u}^t\bar{w}_i - \frac{1}{2}\bar{r}_i^t\nabla_{xx}f_i(\bar{u},\bar{v})\bar{r}_i\right] \\ &\quad - \left[f_i(\bar{x},\bar{y}) - s(\bar{y}|D_i) + \bar{x}^t\bar{w}_i - \frac{1}{2}\bar{r}_i^t\nabla_{xx}f_i(\bar{x},\bar{y})\bar{r}_i\right] \\ &\quad > M\left[\left\{f_j(\bar{x},\bar{y}) - s(\bar{y}|D_j) + \bar{x}^t\bar{w}_j - \frac{1}{2}\bar{r}_j^t\nabla_{xx}f_j(\bar{x},\bar{y})\bar{r}_j\right\}\right. \\ &\qquad \left. - \left\{f_j(\bar{u},\bar{v}) - s(\bar{v}|D_j) + \bar{u}^t\bar{w}_j - \frac{1}{2}\bar{r}_j^t\nabla_{xx}f_j(\bar{u},\bar{v})\bar{r}_j\right\}\right] \end{aligned}$$

for any $M>0$, and all j satisfying

$$\begin{aligned} &f_j(\bar{x},\bar{y}) - s(\bar{y}|D_j) + \bar{x}^t\bar{w}_j - \frac{1}{2}\bar{r}_j^t\nabla_{xx}f_j(\bar{x},\bar{y})\bar{r}_j \\ &\quad > f_j(\bar{u},\bar{v}) - s(\bar{v}|D_j) + \bar{u}^t\bar{w}_j - \frac{1}{2}\bar{r}_j\nabla_{xx}f_j(\bar{u},\bar{v})\bar{r}_j, \end{aligned}$$

$\bar{x}^t\bar{w}_i = s(\bar{x}|C_i)$ and $\bar{y}^t\bar{z}_i = s(\bar{y}|D_i)$, $i=1,2,\ldots,k$.

This means that

$$\begin{aligned} &\left[f_i(\bar{u},\bar{v}) - s(\bar{v}|D_i) + \bar{u}^t\bar{w}_i - \frac{1}{2}\bar{r}_i^t\nabla_{xx}f_i(\bar{u},\bar{v})\bar{r}_i\right] \\ &\quad - \left[f_i(\bar{x},\bar{y}) + s(\bar{x}|C_i) - \bar{y}^t\bar{z}_i - \frac{1}{2}\bar{r}_i^t\nabla_{xx}f_i(\bar{x},\bar{y})\bar{r}_i\right] \end{aligned}$$

can be made arbitrarily large. Thus for any $\bar{\lambda} > 0$,

$$\begin{aligned} &\sum_{i=1}^k \bar{\lambda}_i\left[f_i(\bar{u},\bar{v}) - s(\bar{v}|D_i) + \bar{u}^t\bar{w}_i - \frac{1}{2}\bar{r}_i^t\nabla_{xx}f_i(\bar{u},\bar{v})\bar{r}_i\right] \\ &\quad > \sum_{i=1}^k \bar{\lambda}_i\left[f_i(\bar{x},\bar{y}) + s(\bar{x}|C_i) - \bar{y}^t\bar{z}_i - \frac{1}{2}\bar{r}_i^t\nabla_{xx}f_i(\bar{x},\bar{y})\bar{r}_i\right]. \end{aligned}$$

This again contradicts Theorem 1. $\square$

A converse duality theorem may be merely stated as its proof would run analogously to that of Theorem 2.

Theorem 3 (Converse duality). *Let f be thrice differentiable on* $R^n \times R^m$. *Let* $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{w}, \bar{r})$ *be a weak efficient solution for* (MD); *fix* $\lambda = \bar{\lambda}$ *in* (MP) *and suppose that*

(B1) $\nabla_{xx} f_i$ *is nonsingular for all* $i = 1, 2, \ldots, k$,
(B2) *the matrix* $\sum_{i=1}^{k} \bar{\lambda}_i (\nabla_{xx} f_i \bar{r}_i)_x$ *is positive or negative definite, and*
(B3) *the set* $(\nabla_x f_1 - \bar{w}_1 + \nabla_{xx} f_1 \bar{r}_1, \nabla_x f_2 - \bar{w}_2 + \nabla_{xx} f_2 \bar{r}_2, \ldots, \nabla_x f_k - \bar{w}_k + \nabla_{xx} f_k \bar{r}_k)$ *is linearly independent*,

then there exist $\bar{z}_i \in R^m$, $i = 1, 2, \ldots, k$ *such that* $\bar{r} = 0$, $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}, \bar{p} = 0)$ *is feasible for* (MP) *and*

$$L(\bar{u}, \bar{v}, \bar{z}, \bar{p}) = H(\bar{u}, \bar{v}, \bar{w}, \bar{r}).$$

Also, if the hypotheses of Theorem 1 *are satisfied for all feasible solutions of* (MP) *and* (MD), *then* $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}, \bar{p})$ *is a properly efficient solution for* (MP).

4. Special cases

(i) Let $C_i = \{0\}$ and $D_i = \{0\}$, $i = 1, 2, \ldots, k$. Then (MP) and (MD) are reduced to the second order multiobjective symmetric dual programs of Suneja et al. [18]. If in addition $p = 0$, $r = 0$. Then we get the multiobjective symmetric dual pair of Gulati et al. [9].
(ii) If $k = 1$ in (MP) and (MD), then we obtain nondifferentiable symmetric dual programs studied by Hou and Yang [10].
(iii) If in (MP) and (MD), $k = 1$, $C_i = \{0\}$ and $D_i = \{0\}$, $i = 1, 2, \ldots, k$, then we get the symmetric dual programs of Gulati et al. [8] and Mishra [12] with the addition of nonnegativity constraints $x \geqq 0$ and $y \geqq 0$ in (MP) and (MD) respectively.
(iv) If $k = 1$, $p = 0$ and $r = 0$, then we obtain symmetric dual multiobjective programming problems studied by Mond and Schechter [15].
(v) From the symmetry of primal and dual problems (MP) and (MD), we can construct other new symmetric dual pairs. For example, if we take $C_i = \{A_i y : y^t A_i y \leqq 1$ and $D_i = \{B_i x : x^t B_i x \leqq 1$, $i = 1, 2, \ldots, k$, where A_i and B_i, $i = 1, 2, \ldots, k$ are positive semidefinite matrices, then it can be easily verified that $(x^t A_i x)^{\frac{1}{2}} = s(x|C_i)$ and $(y^t B_i y)^{\frac{1}{2}} = s(y|D_i)$, $i = 1, 2, \ldots, k$. Thus, a number of new symmetric dual pairs and duality results can be obtained.

5. Conclusion

In this article, a new pair of Mond–Weir type nondifferentiable multiobjective second order symmetric dual programs is presented and duality relations between primal and dual problems are established. The nondifferentiability terms in the form of support functions have been included in the objective functions of each problem. The results developed in this paper improve and generalize a number of existing results in the literature. These results can be further generalized for minimax mixed integer programs, wherein some of the primal and dual variables are constrained to belong to some arbitrary sets e.g., the sets of integers.

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