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ORIGINAL PAPER

Second order duality for minmax fractional programming

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Abstract In the present paper, two types of second order dual models are formulated for a minmax fractional programming problem. The concept of η-bonvexity/ generalized η-bonvexity is adopted in order to discuss weak, strong and strict converse duality theorems.

Keywords Minmax programming · Fractional programming · Second order duality · η-bonvexity

1 Introduction

Optimization problems with minmax type functions arise in the design of electronic circuits, however, minmax fractional problems appear in the formulation of discrete and continuous rational approximation problems with respect to the Chebyshev norm [6], in continuous rational games [22], in multiobjective programming [23], in engineering design as well as in some portfolio selection problems discussed by Bajona-Xandri and Martinez-Legaz [5].

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In this paper, we consider the following minmax fractional programming problem:

$$\text{Minimize } \psi(x) = \sup_{y \in Y} \frac{f(x, y)}{h(x, y)} \tag{P}$$

$$\text{subject to } g(x) \leq 0, \quad x \in R^n,$$

where Y is a compact subset of R^l, $f(\cdot,\cdot) : R^n \times R^l \to R$, $h(\cdot,\cdot) : R^n \times R^l \to R$ are C^2 on $R^n \times R^l$ and $g(\cdot) : R^n \to R^m$ is C^2 on R^n. It is assumed that for each (x, y) in $R^n \times R^l$, $f(x, y) \geq 0$ and $h(x, y) > 0$.

For the case of convex differentiable minmax fractional programming, Yadav and Mukherjee [24] formulated two dual models for (P) and derived duality theorems. Chandra and Kumar [9] pointed out certain omissions and inconsistencies in the dual formulation of Yadav and Mukherjee [24]; they constructed two modified dual problems for fractional minmax programming problem and proved duality results. Liu and Wu [15,16], and Ahmad [1] obtained sufficient optimality conditions and duality theorems for (P) assuming the functions involved to be generalized convex. Recently, Yang and Hou [25] discussed optimality conditions and duality results for (P) involving generalized convexity assumptions.

Second order dual has computational advantages over the first order dual, as it provides tighter bounds for the value of the objective function, when approximations are used [10,18,20]. Mangasarian [18] first formulated the second order dual for a nonlinear programming problem by introducing an additional vector $p \in R^n$. Instead of imposing explicit condition on p, Mond [20] included p in a second order type convexity. Bector et al. [8] discussed second order duality results for minmax programming problems under generalized binvexity. Later on, Liu [14] extended these results involving second order generalized B-invexity assumptions.

In this paper, motivated by Chandra and Kumar [9], Bector et al. [8] and Liu [14], we formulate two second order duals for fractional minmax programming problem (P), and derive weak, strong and strict converse duality theorems under η-bonvexity/generalized η-bonvexity. This work generalizes the previous results appeared in [1,8,9,14–17,24,25].

2 Notations and preliminaries

Let $S = \{x \in R^n : g(x) \leq 0\}$ denote the set of all feasible solutions of (P). For each $(x, y) \in R^n \times R^l$, we define

$$J(x) = \{j \in M = \{1, 2, \ldots, m\} : g_j(x) = 0\},$$

$$Y(x) = \{y \in Y : f(x, y) + (x^T Bx)^{\frac{1}{2}} = \sup_{z \in Y} f(x, z) + (x^T Bx)^{\frac{1}{2}}\},$$

and

$$K(x) = \{(s, t, \tilde{y}) \in \mathbb{N} \times R^s_+ \times R^{ls} : 1 \le s \le n+1,\ t = (t_1, t_2, \ldots, t_s) \in R^s_+ \\ \text{with } \sum_{i=1}^{s} t_i = 1, \tilde{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_s) \text{ with } \bar{y}_i \in Y(x), i = 1, 2, \ldots, s\}.$$

Let $f : R^n \to R$ be a twice differentiable function.

Definition 1 Function f is said to be η-bonvex at $\bar{x} \in R^n$, if there exists a certain mapping $\eta : R^n \times R^n \to R^n$ such that for all $x, p \in R^n$, we have

$$f(x) - f(\bar{x}) + \frac{1}{2} p^T \nabla^2 f(\bar{x}) p \ge \eta^T(x, \bar{x}) \left[\nabla f(\bar{x}) + \nabla^2 f(\bar{x}) p \right].$$

Definition 2 Function f is said to be (strictly) η-pseudobonvex at $\bar{x} \in R^n$, if there exists a certain mapping $\eta : R^n \times R^n \to R^n$ such that for all $x, p \in R^n$, we have

$$\eta^T(x, \bar{x}) \left[\nabla f(\bar{x}) + \nabla^2 f(\bar{x}) p \right] \ge 0 \Rightarrow f(x)(>) \ge f(\bar{x}) - \frac{1}{2} p^T \nabla^2 f(\bar{x}) p.$$

Definition 3 Function f is said to be (strictly) η-quasibonvex at $\bar{x} \in R^n$, if there exists a certain mapping $\eta : R^n \times R^n \to R^n$ such that for all $x, p \in R^n$, we have

$$f(x) \le f(\bar{x}) - \frac{1}{2} p^T \nabla^2 f(\bar{x}) p \Rightarrow \eta^T(x, \bar{x}) \left[\nabla f(\bar{x}) + \nabla^2 f(\bar{x}) p \right] (<) \le 0.$$

It has been revealed in [21] by means of an example that the above class of functions is an extension of bonvex functions [7].

The following theorem will be needed in the proofs of strong duality theorems:

Theorem 1 (*Necessary conditions*) [9] *Let* x^* *be a solution* (*local or global*) *of* (P) *and let* $\nabla g_j(x^*)$, $j \in J(x^*)$ *be linearly independent. Then there exist* $(s^*, t^*, \bar{y}^*) \in K(x^*)$, $\lambda^* \in R_+$, *and* $\mu^* \in R^m_+$ *such that*

$$\nabla \sum_{i=1}^{s^*} t_i^* (f(x^*, \bar{y}_i^*) - \lambda^* h(x^*, \bar{y}_i^*)) + \nabla \sum_{j=1}^{m} \mu_j^* g_j(x^*) = 0,$$

$$f(x^*, \bar{y}_i^*) - \lambda^* h(x^*, \bar{y}_i^*) = 0, \quad i = 1, 2, \ldots, s^*,$$

$$\sum_{j=1}^{m} \mu_j^* g_j(x^*) = 0,$$

$$t_i^* \ge 0, \quad \sum_{i=1}^{s^*} t_i^* = 1,\ \bar{y}_i^* \in Y(x^*), \quad i = 1, 2, \ldots, s^*.$$

3 First duality model

Making use of the optimality conditions of preceeding section, we formulate the following second order dual to (P) as follows:

$$\max_{(s,t,\bar{y})\in K(z)} \quad \sup_{(z,\mu,\lambda,p)\in H_1(s,t,\bar{y})} \lambda, \tag{MD}$$

where $H_1(s,t,\bar{y})$ denotes the set of all $(z,\mu,\lambda,p)\in R^n\times R^m_+\times R_+\times R^n$ satisfying

$$\nabla\sum_{i=1}^{s} t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)) + \nabla^2\sum_{i=1}^{s} t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i))p$$
$$+\nabla\sum_{j=1}^{m}\mu_j g_j(z) + \nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p = 0, \tag{3.1}$$

$$\sum_{i=1}^{s} t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)) - \frac{1}{2}p^T\nabla^2\sum_{i=1}^{s} t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i))p \geq 0, \tag{3.2}$$

$$\sum_{j=1}^{m}\mu_j g_j(z) - \frac{1}{2}p^T\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p \geq 0. \tag{3.3}$$

If, for a triplet $(s,t,\bar{y})\in K(z)$, the set $H_1(s,t,\bar{y})=\emptyset$, then we define the supremum over it to be $-\infty$.

Remark 3.1 If $p=0$, then (MD) becomes the dual considered in [1,15–17].

Theorem 2 (*Weak duality*) *Let x and $(z,\mu,\lambda,s,t,\bar{y},p)$ be feasible solutions of* (P) *and* (MD)*, respectively. Assume that*

(i) $\left[\sum_{i=1}^{s} t_i(f(\cdot,\bar{y}_i)-\lambda h(\cdot,\bar{y}_i))\right]$ *is η-pseudobonvex at z; and*

(ii) $\sum_{j=1}^{m}\mu_j g_j(\cdot)$ *is η-quasibonvex at z.*

Then

$$\sup_{y\in Y}\frac{f(x,y)}{h(x,y)} \geq \lambda.$$

Proof By the feasibility of x for (P), $\mu\geq 0$ and (3.3), we get

$$\sum_{j=1}^{m}\mu_j g_j(x) \leq 0 \leq \sum_{j=1}^{m}\mu_j g_j(z) - \frac{1}{2}p^T\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p.$$

The above inequality together with hypothesis (ii) implies

$$\eta^T(x,z)\left[\nabla\sum_{j=1}^{m}\mu_j g_j(z) + \nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p\right] \leq 0. \tag{3.4}$$

From (3.1) and (3.4), we have

$$\eta^T(x,z)\left[\nabla\sum_{i=1}^{s}t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i))+\nabla^2\sum_{i=1}^{s}t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i))p\right]\geq 0,$$

which by the virtue of hypothesis (i) yields

$$\begin{aligned}&\sum_{i=1}^{s}t_i(f(x,\bar{y}_i)-\lambda h(x,\bar{y}_i))\\&\quad\geq\sum_{i=1}^{s}t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i))-\frac{1}{2}p^T\nabla^2\sum_{i=1}^{s}t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i))p\\&\quad\geq 0\quad\text{(by (3.2))}.\end{aligned}$$

Therefore, there exists a certain $i_\circ$ such that

$$f(x,\bar{y}_{i_\circ})-\lambda h(x,\bar{y}_{i_\circ})\geq 0.$$

Hence

$$\sup_{y\in Y}\frac{f(x,y)}{h(x,y)}\geq\frac{f(x,\bar{y}_{i_\circ})}{h(x,\bar{y}_{i_\circ})}\geq\lambda.$$

□

Theorem 3 (*Strong duality*) *Assume that x^* is an optimal solution of* (P) *and $\nabla g_j(x^*)$, $j\in J(x^*)$, are linearly independent. Then there exist $(s^*,t^*,\bar{y}^*)\in K(x^*)$ and $(x^*,\mu^*,\lambda^*,p^*=0)\in H_1(s^*,t^*,\bar{y}^*)$ such that $(x^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*=0)$ is a feasible solution of* (MD) *and the two objectives have the same values. If, in addition, the assumptions of weak duality* (*Theorem* 2) *hold for all feasible solutions $(z,\mu,\lambda,s,t,\bar{y},p)$ of* (MD)*, then $(x^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*=0)$ is an optimal solution of* (MD).

Proof Since x^* is an optimal solution of (P) and $\nabla g_j(x^*)$, $j\in J(x^*)$ are linearly independent, then by Theorem 1, there exist $(s^*,t^*,\bar{y}^*)\in K(x^*)$ and $(x^*,\mu^*,\lambda^*,p^*=0)\in H_1(s^*,t^*,\bar{y}^*)$ such that $(x^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*=0)$ is a feasible solution of (MD) and the two objectives have same values. Optimality of $(x^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*=0)$ for (MD), thus follows from weak duality (Theorem 2). □

Theorem 4 (*Strict converse duality*) *Let x^* and $(z^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*)$ be optimal solutions of* (P) *and* (MD)*, respectively. Suppose that*

(i) $\nabla g_j(x^*)$, $j\in J(x^*)$, *are linearly independent,*

(ii) $\left[\sum_{i=1}^{s^*}t_i^*(f(\cdot,\bar{y}_i^*)-\lambda^*h(\cdot,\bar{y}_i^*))\right]$ *is strictly η-pseudobonvex at z^*; and*

(iii) $\sum_{j=1}^{m}\mu_j^*g_j(\cdot)$ *is η-quasibonvex at z^*.*

Then $z^*=x^*$.

Proof Suppose to the contrary that $z^* \neq x^*$, and we will derive a contradiction. Since x^* and $(z^*, \mu^*, \lambda^*, s^*, t^*, \bar{y}^*, p^*)$ are optimal solutions of (P) and (MD), respectively, and $\nabla g_j(x^*),\ j \in J(x^*)$ are linearly independent, therefore, by Theorem 3, we have

$$\sup_{y^* \in Y} \frac{f(x^*, y^*)}{h(x^*, y^*)} = \lambda^*. \tag{3.5}$$

The feasibility of x^* for (P), $\mu^* \geq 0$ and (3.3) imply

$$\sum_{j=1}^{m} \mu_j^* g_j(x^*) \leq 0 \leq \sum_{j=1}^{m} \mu_j^* g_j(z^*) - \frac{1}{2} p^{*T} \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j(z^*) p^*,$$

which along with hypothesis (iii) gives

$$\eta^T(x^*, z^*) \left[\nabla \sum_{j=1}^{m} \mu_j^* g_j(z^*) + \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j(z^*) p^* \right] \leq 0. \tag{3.6}$$

Therefore, inequality (3.1) along with (3.6) yields

$$\eta^T(x^*, z^*) \left[\nabla \sum_{i=1}^{s^*} t_i^* (f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)) + \nabla^2 \sum_{i=1}^{s^*} t_i^* (f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)) p^* \right] \geq 0,$$

which by hypothesis (ii) and inequality (3.2) gives

$$\sum_{i=1}^{s^*} t_i^* (f(x^*, \bar{y}_i^*) - \lambda^* h(x^*, \bar{y}_i^*)) > 0.$$

For a certain $i_\circ$, this implies

$$\sup_{y^* \in Y} \frac{f(x^*, y^*)}{h(x^*, y^*)} \geq \frac{f(x^*, \bar{y}_{i_\circ}^*)}{h(x^*, \bar{y}_{i_\circ}^*)} > \lambda^*,$$

which is a contradiction to (3.5). Hence $z^* = x^*$. □

4 Second duality model

In this section, we formulate the following second order dual to (P) and discuss duality results.

$$\max_{(s,t,\bar{y})\in K(z)} \quad \sup_{(z,\mu,\lambda,p)\in H_2(s,t,\bar{y})} \lambda, \tag{GMD}$$

where $H_2(s,t,\bar{y})$ denotes the set of all $(z,\mu,\lambda,p)\in R^n\times R_+^m\times R_+\times R^n$ satisfying

$$\nabla\sum_{i=1}^{s} t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i))+\nabla^2\sum_{i=1}^{s} t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i))p$$
$$+\nabla\sum_{j=1}^{m}\mu_j g_j(z)+\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p=0, \tag{4.1}$$

$$\sum_{i=1}^{s} t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i))+\sum_{j\in J_\circ}\mu_j g_j(z)$$
$$-\frac{1}{2}p^T\nabla^2\left[\sum_{i=1}^{s} t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i))+\sum_{j\in J_\circ}\mu_j g_j(z)\right]p\geq 0, \tag{4.2}$$

$$\sum_{j\in J_\alpha}\mu_j g_j(z)-\frac{1}{2}p^T\nabla^2\sum_{j\in J_\alpha}\mu_j g_j(z)p\geq 0,\quad \alpha=1,2,\ldots,r, \tag{4.3}$$

where $J_\alpha\subseteq M,\ \alpha=0,1,2,\ldots,r$, with $\bigcup_{\alpha=0}^{r} J_\alpha=M$ and $J_\alpha\cap J_\beta=\emptyset$, if $\alpha\neq\beta$. If, for a triplet $(s,t,\bar{y})\in K(z)$, the set $H_2(s,t,\bar{y})=\emptyset$, then we define the supremum over it to be $-\infty$.

Theorem 5 (*Weak duality*) *Let x and $(z,\mu,\lambda,s,t,\bar{y},p)$ be feasible solutions of* (P) *and* (GMD)*, respectively. Assume that*

(i) $\left[\sum_{i=1}^{s} t_i(f(\cdot,\bar{y}_i)-\lambda h(\cdot,\bar{y}_i))+\sum_{j\in J_\circ}\mu_j g_j(\cdot)\right]$ *is η-pseudobonvex at z; and*

(ii) $\sum_{j\in J_\alpha}\mu_j g_j(\cdot),\ \alpha=1,2,\ldots,r,$ *is η-quasibonvex at z.*

Then

$$\sup_{y\in Y}\frac{f(x,y)}{h(x,y)}\geq\lambda.$$

Proof By the feasibility of x for (P), $\mu\geq 0$ and (4.3), we have

$$\sum_{j\in J_\alpha}\mu_j g_j(x)\leq 0\leq\sum_{j\in J_\alpha}\mu_j g_j(z)-\frac{1}{2}p^T\nabla^2\sum_{j\in J_\alpha}\mu_j g_j(z)p,\quad \alpha=1,2,\ldots,r. \tag{4.4}$$

The inequality (4.4) and hypothesis (ii) give

$$\eta^T(x,z)\left[\nabla\sum_{j\in J_\alpha}\mu_j g_j(z)+\nabla^2\sum_{j\in J_\alpha}\mu_j g_j(z)p\right]\leq 0,\quad \alpha=1,2,\ldots,r,$$

which together with (4.1) yields

$$\eta^T(x,z)\left[\nabla\sum_{i=1}^{s}t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i))+\nabla^2\sum_{i=1}^{s}t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i))p\right.$$
$$\left.+\nabla\sum_{j\in J_\circ}\mu_j g_j(z)+\nabla^2\sum_{j\in J_\circ}\mu_j g_j(z)p\right]\geq 0.$$

By using hypothesis (i), the above inequality implies

$$\sum_{i=1}^{s}t_i(f(x,\bar{y}_i)-\lambda h(x,\bar{y}_i))+\sum_{j\in J_\circ}\mu_j g_j(x)$$
$$\geq\sum_{i=1}^{s}t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i))+\sum_{j\in J_\circ}\mu_j g_j(z)$$
$$-\frac{1}{2}p^T\nabla^2\left[\sum_{i=1}^{s}t_i(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i))+\sum_{j\in J_\circ}\mu_j g_j(z)\right]p.$$

By $\mu\geq 0,\ g(x)\leq 0$ and (4.2), it follows that

$$\sum_{i=1}^{s}t_i(f(x,\bar{y}_i)-\lambda h(x,\bar{y}_i))\geq 0.$$

Therefore, there exists a certain $i_\circ$ such that

$$f(x,\bar{y}_{i_\circ})-\lambda h(x,\bar{y}_{i_\circ})\geq 0.$$

Hence

$$\sup_{y\in Y}\frac{f(x,y)}{h(x,y)}\geq\frac{f(x,\bar{y}_{i_\circ})}{h(x,\bar{y}_{i_\circ})}\geq\lambda.$$

□

The proof of the following theorem is similar to that of Theorem 3, and hence, is omitted.

Theorem 6 (*Strong duality*) *Assume that* x^* *is an optimal solution of* (P) *and* $\nabla g_j(x^*)$, $j \in J(x^*)$, *are linearly independent. Then there exist* $(s^*, t^*, \bar{y}^*) \in K(x^*)$ *and* $(x^*, \mu^*, \lambda^*, p^* = 0) \in H_2(s^*, t^*, \bar{y}^*)$ *such that* $(x^*, \mu^*, \lambda^*, s^*, t^*, \bar{y}^*, p^* = 0)$ *is a feasible solution of* (GMD) *and the two objectives have the same values. If, in addition, the assumptions of weak duality* (*Theorem* 5) *hold for all feasible solutions* $(z, \mu, \lambda, s, t, \bar{y}, p)$ *of* (GMD), *then* $(x^*, \mu^*, \lambda^*, s^*, t^*, \bar{y}^*, p^* = 0)$ *is an optimal solution of* (GMD).

Theorem 7 (*Strict converse duality*). *Let* x^* *and* $(z^*, \mu^*, \lambda^*, s^*, t^*, \bar{y}^*, p^*)$ *be optimal solutions of* (P) *and* (GMD), *respectively. Suppose that*

(i) $\nabla g_j(x^*)$, $j \in J(x^*)$, *are linearly independent,*
(ii) $\left[\sum_{i=1}^{s^*} t_i^*(f(\cdot, \bar{y}_i^*) - \lambda^* h(\cdot, \bar{y}_i^*)) + \sum_{j\in J_\circ} \mu_j^* g_j(\cdot)\right]$ *is strictly* η*-pseudobonvex at* z^*; *and*
(iii) $\sum_{j\in J_\alpha} \mu_j^* g_j(\cdot)$, $\alpha = 1, 2, \ldots, r$, *is* η*-quasibonvex at* z^*.

Then $z^* = x^*$.

Proof It can be proved similarly to Theorem 4. □

Remark 4.1 If, we take $J_\circ = \emptyset$, then Theorems 5–7 reduce to Theorems 2–4 of Sect. 3.

5 Further developments

The question arises as to whether the second order duality results developed in this paper hold for the following nondifferentiable minmax fractional programming problem [2,11–13,19]:

$$\text{Minimize } \sup_{y\in Y} \frac{f(x,y) + (x^T Bx)^{\frac{1}{2}}}{h(x,y) - (x^T Dx)^{\frac{1}{2}}} \tag{NP}$$
$$\text{subject to } g(x) \leq 0, \ x \in R^n,$$

where Y is a compact subset of R^l, B and D are $n \times n$ positive semidefinite symmetric matices, $f(\cdot,\cdot) : R^n \times R^l \to R$, $h(\cdot,\cdot) : R^n \times R^l \to R$ are C^2 on $R^n \times R^l$ and $g(\cdot) : R^n \to R^m$ is C^2 on R^n. If $h(x, y) = 1$ and $D = 0$, then (NP) reduces to the problem considered in [4].

Moreover, the present work can easily be extended to a unified class of functions, i.e., second order (F, α, ρ, d)-convex/second order generalized (F, α, ρ, d)-convex functions [3].

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