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On multiobjective second order symmetric duality with cone constraints

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ABSTRACT

A pair of Wolfe type multiobjective second order symmetric dual programs with cone constraints is formulated and usual duality results are established under second order invexity assumptions. These results are then used to investigate symmetric duality for minimax version of multiobjective second order symmetric dual programs wherein some of the primal and dual variables are constrained to belong to some arbitrary sets, i.e., the sets of integers. This paper points out certain omissions and inconsistencies in the earlier work of Mishra [S.K. Mishra, Multiobjective second order symmetric duality with cone constraints, European Journal of Operational Research 126 (2000) 675–682] and Mishra and Wang [S.K. Mishra, S.Y. Wang, Second order symmetric duality for nonlinear multiobjective mixed integer programming, European Journal of Operational Research 161 (2005) 673–682].

1. Introduction

Unlike linear programming, the majority of dual formulations in nonlinear programming do not possess the symmetry property. The first symmetric dual formulation for quadratic programming was proposed by Dorn [6]. Dantzig et al. [4] formulated a pair of symmetric dual programs and discussed duality results involving convex/concave functions. The same results were generalized by Bazaraa and Goode [2] to arbitrary cones. Nanda and Das [17] attempted to construct a pair of symmetric dual programming problems for arbitrary cones and derived symmetric duality under pseudoinvexity assumptions. Chandra and Kumar [3] pointed out that the dual construction of Nanda and Das [17] is not correct and certain assumptions made by them are highly restricted. They also emphasized that for studying symmetric duality under pseudoinvexity, the dual formulation has to be in the spirit of Mond and Weir [16], and not on the lines of Dantzig et al. [4].

In [5], Devi formulated a pair of symmetric dual nonlinear programming problems over arbitrary cones and established usual duality results under η-pseudobonvexity (second order pseudoinvexity) assumptions. Recently, Gulati et al. [9] traced out various mistakes in the definitions, models and duality results of Devi [5]. Mishra [14] formulated a pair of multiobjective second order symmetric dual nonlinear programs over arbitrary cones and established weak, strong, converse and self duality theorems under second order (strict) pseudoinvexity. This work was in turn an attempt to extend the second order symmetric duality results of Devi [5] to multiobjective case.

This paper consists five sections. In Section 1, we collect some references related to the present study. In Section 2, notations and definitions are given and some logical shortcomings in the definitions and dual formulation of Mishra [14] are pointed out. In Section 3, we present a pair of Wolfe type multiobjective second order symmetric dual programs over arbitrary cones and establish weak, strong and converse duality theorems under second order invexity assumptions. It is also shown that the conditions imposed on the functions $\eta_1(x,u)$ and $\eta_2(v,y)$ in proving symmetric duality theorems by Mishra [14] are incorrect. In Section 4, we formulate minimax mixed integer edition of the symmetric dual pair presented in Section 3, and discuss symmetric duality by using separability of the kernel function $K(x,y)$. Some mistakes in the work of Mishra and Wang [15] are also emphasized. Section 5 is the concluding one.

2. Notations and preliminaries

The following conventions for vectors in R^n will be followed throughout this paper: $x \geqq y \Longleftrightarrow x_i \geqq y_i, i = 1,2,\ldots,n;\ x \geqslant y \Longleftrightarrow x \geqq y$, and $x \neq y;\ x > y \Longleftrightarrow x_i > y_i,\ i = 1,2,\ldots,n.\ x \ngeqslant y$ is the negation of $x \geqslant y$.

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Consider the following multiobjective programming problem:

$$(\mathbf{P})\quad \text{Minimize}\ \ f(x)$$
$$\text{subject to}\ g(x) \leqq 0,$$

where $f : R^n \to R^k$ and $g : R^n \to R^m$.

Definition 1. A feasible point $\bar{x}$ of (P) is said to be a weakly efficient solution of (P), if there exists no other feasible x such that

$$f(x) < f(\bar{x}).$$

Definition 2. A set C of R^n is called a cone, if for each $x \in C$ and $\lambda \in R,\ \lambda \geqq 0$, we have $\lambda x \in C$. Moreover, if C is convex, then it is convex cone.

Definition 3. $C^* = \{z \in R^n | z^T x \leqq 0,\ \text{for all}\ x \in C\}$ is called the polar of the cone C.

Let $C_1 \subset R^n,\ C_2 \subset R^m$ be closed convex cones with nonempty interiors having polars C_1^* and C_2^*, respectively. Let $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ be open and $S_1 \times S_2 \subseteq R^n \times R^m$. Let $C_1 \times C_2 \subseteq S_1 \times S_2$and let $K(x,y) : S_1 \times S_2 \to R$ be twice differentiable.

Definition 4 [9]. $K(x,y)$ is said to be second order invex (η_1-bonvex) in the first variable at $u \in S_1$, if there exists a vector function $\eta_1 : S_1 \times S_1 \to R^n$ such that for $v \in S_2,\ q \in R^n$ and $x \in S_1$

$$K(x,v) - K(u,v) \geqq \eta_1^T(x,u)[\nabla_x K(u,v) + \nabla_{xx} K(u,v)q] - \frac{1}{2} q^T \nabla_{xx} K(u,v) q,$$

and $K(x,y)$ is said to be second order invex (η_2-bonvex) in the second variable at $v \in S_2$, if there exists a vector function $\eta_2 : S_2 \times S_2 \to R^m$ such that for $u \in S_1,\ p \in R^m$ and $y \in S_2$

$$K(u,y) - K(u,v) \geqq \eta_2^T(y,v)[\nabla_y K(u,v) + \nabla_{yy} K(u,v)p] - \frac{1}{2} p^T \nabla_{yy} K(u,v) p.$$

Remark 1. It may be noted that like Devi [5], the functions η_1 and η_2 involved in the definitions of second order invexity/pseudoinvexity given by Mishra [14], are incorrectly taken from $C_1 \times C_1 \to C_1$ and $C_2 \times C_2 \to C_2$ as can be seen in [9].

Mishra [14] considered the following pair of multiobjective second order symmetric dual problems:

$$\begin{aligned}
(\mathbf{P})\quad & \text{Minimize} && K(x,y) - [y^T \nabla_y (\lambda^T K)(x,y)]e - \tfrac{1}{2} p_0^T \nabla_{yy} (\lambda^T K)(x,y) p_0 \\
& \text{subject to} && (x,y) \in C_1 \times C_2, \\
& && \nabla_y (\lambda^T K)(x,y), \quad \nabla_{yy} (\lambda^T K)(x,y) p_0 \in C_2^*, \\
& && \lambda \geqslant 0, \quad \lambda^T e = 1. \\
(\mathbf{D})\quad & \text{Maximize} && K(u,v) - [u^T \nabla_x (\lambda^T K)(u,v)]e - \tfrac{1}{2} p_1^T \nabla_{xx} (\lambda^T K)(u,v) p_1 \\
& \text{subject to} && (u,v) \in C_1 \times C_2, \\
& && -\nabla_x (\lambda^T K)(u,v), \quad -\nabla_{xx} (\lambda^T K)(u,v) p_1 \in C_1^*, \\
& && \lambda \geqslant 0, \quad \lambda^T e = 1,
\end{aligned}$$

where K is a p–vector, and $e = (1,1,\ldots,1) \in R^p$.

Remark 2. Following points can be observed about the above problems:

(i) In view of the fact of Chandra and Kumar [3], the above pair does no longer reduce to the symmetric dual pair of Mond and Weir [16]. However, the symmetric duality was discussed under second order pseudoinvexity assumptions.

(ii) In each of the objective functions, the terms with unequal dimensions have been added, which seems to be absurd and leads to an incorrect formulation of the problems.

(iii) It may be remarked that in (P), two constraints

$$\nabla_y (\lambda^T K)(x,y) \in C_2^* \quad \text{and} \quad \nabla_{yy} (\lambda^T K)(x,y) p_0 \in C_2^*$$

have been taken instead of one constraint

$$\nabla_y (\lambda^T K)(x,y) + \nabla_{yy} (\lambda^T K)(x,y) p_0 \in C_2^*.$$

However, in the proof of strong duality theorem, these constraints have been inaccurately taken together in the form of $g(z)$, i.e.,

$$g(z) = \nabla_y (\lambda^T K)(x,y) + \nabla_{yy} (\lambda^T K)(x,y) p_0.$$

In case, if the constraints are taken as in (P) and (D) along with two multipliers, the strong duality theorem cannot be proved on the lines of Devi [5] (see Remark 3.1 in [9]).

As the overall paper is based on the definitions and problems formulation, therefore symmetric duality results in [14] seem to be incorrect.

3. Wolfe type second order symmetric duality

We formulate the following pair of multiobjective second order symmetric dual programs over arbitrary cones:

$$(\mathbf{WP})\quad \text{Minimize} \quad K(x,y) - [y^T\nabla_y(\lambda^TK)(x,y)]e - [y^T\nabla_{yy}(\lambda^TK)(x,y)p]e - \frac{1}{2}[p^T\nabla_{yy}(\lambda^TK)(x,y)p]e$$

$$\text{subject to} \quad \nabla_y(\lambda^TK)(x,y) + \nabla_{yy}(\lambda^TK)(x,y)p \in C_2^*, \tag{1}$$

$$x \in C_1, \tag{2}$$

$$\lambda > 0, \quad \lambda^Te = 1. \tag{3}$$

$$(\mathbf{WD})\quad \text{Maximize} \quad K(u,v) - [u^T\nabla_x(\lambda^TK)(u,v)]e - [u^T\nabla_{xx}(\lambda^TK)(u,v)q]e - \frac{1}{2}[q^T\nabla_{xx}(\lambda^TK)(u,v)q]e$$

$$\text{subject to} \quad -\nabla_x(\lambda^TK)(u,v) - \nabla_{xx}(\lambda^TK)(u,v)q \in C_1^*, \tag{4}$$

$$v \in C_2, \tag{5}$$

$$\lambda > 0, \quad \lambda^Te = 1, \tag{6}$$

where $K(x,y) : S_1 \times S_2 \to R^k,\ p \in R^m,\ q \in R^n,\ \lambda \in R^k$, and $e = (1,1,\ldots,1)^T \in R^k$.

Remark 3

(i) For $C_1 = R^n_+$ and $C_2 = R^m_+$, the above pair becomes the symmetric dual pair [11,19] with the omission of $x \in C_1$ and $v \in C_2$ from (WP) and (WD).
(ii) For $k = 1$, the above pair reduces to Wolfe type second order symmetric dual pair of Gulati et al. [9]. If, in addition, $C_1 = R^n_+$ and $C_2 = R^m_+$, we get the dual pair of Gulati et al. [8].

We shall discuss the symmetric duality results for the problems (WP) and (WD) under the following assumptions similar to those in [3,10]:

$$\left.\begin{aligned} &\eta_1(x,u) + u \in C_1, \quad \text{for all } (x,u) \in C_1 \times C_1, \\ &\eta_2(v,y) + y \in C_2, \quad \text{for all } (v,y) \in C_2 \times C_2. \end{aligned}\right\} \tag{7}$$

Remark 4. It may be noted that these assumptions have not been taken by Mishra [14] in any of the symmetric duality theorems. However, these were erroneously used as $x, u \in C_1$ and $\eta_1(x,u) \in C_1$, and also $v, y \in C_2$ and $\eta_2(v,y) \in C_2$. For, in particular, if $\eta_1(x,u) = x - u$ and $C_1 = R^n_+$, then it amounts to assuming that $x \geqq 0,\ u \geqq 0$ implies $x - u \geqq 0$, which is not always true.

Theorem 1 (Weak duality). *Let (x,y,λ,p) and (u,v,λ,q) be feasible solutions of (WP) and (WD), respectively. Let*

(i) $\lambda^TK(\cdot,v)$ be second order invex in x with respect to η_1; and
(ii) $-\lambda^TK(x,\cdot)$ be second order invex in y with respect to η_2.

Then

$$\left[K(x,y) - [y^T\nabla_y(\lambda^TK)(x,y)]e - [y^T\nabla_{yy}(\lambda^TK)(x,y)p]e - \frac{1}{2}[p^T\nabla_{yy}(\lambda^TK)(x,y)p]e\right]$$
$$\not\leq \left[K(u,v) - [u^T\nabla_x(\lambda^TK)(u,v)]e - [u^T\nabla_{xx}(\lambda^TK)(u,v)q]e - \frac{1}{2}[q^T\nabla_{xx}(\lambda^TK)(u,v)q]e\right].$$

Proof. By second order invexity of $\lambda^TK(\cdot,v)$ and $-\lambda^TK(x,\cdot)$ with respect to η_1 and η_2, respectively, we have

$$\lambda^TK(x,v) - \lambda^TK(u,v) \geqq \eta_1^T(x,u)[\nabla_x(\lambda^TK)(u,v) + \nabla_{xx}(\lambda^TK)(u,v)q] - \frac{1}{2}q^T\nabla_{xx}(\lambda^TK)(u,v)q,$$

and

$$-\lambda^TK(x,v) + \lambda^TK(x,y) \geqq -\eta_2^T(v,y)[\nabla_y(\lambda^TK)(x,y) + \nabla_{yy}(\lambda^TK)(x,y)p] + \frac{1}{2}p^T\nabla_{yy}(\lambda^TK)(x,y)p.$$

Adding these inequalities, we get

$$\left[\lambda^TK(x,y) - \frac{1}{2}p^T\nabla_{yy}(\lambda^TK)(x,y)p\right] - \left[\lambda^TK(u,v) - \frac{1}{2}q^T\nabla_{xx}(\lambda^TK)(u,v)q\right] \geqq \eta_1^T(x,u)[\nabla_x(\lambda^TK)(u,v) + \nabla_{xx}(\lambda^TK)(u,v)q]$$
$$- \eta_2^T(v,y)[\nabla_y(\lambda^TK)(x,y) + \nabla_{yy}(\lambda^TK)(x,y)p]. \tag{8}$$

Since $\eta_1(x,u) + u \in C_1$ and $-[\nabla_x(\lambda^TK)(u,v) + \nabla_{xx}(\lambda^TK)(u,v)q] \in C_1^*$, then

$$[\eta_1(x,u) + u]^T[\nabla_x(\lambda^TK)(u,v) + \nabla_{xx}(\lambda^TK)(u,v)q] \leqq 0,$$

implying thereby

$$\eta_1^T(x,u)[\nabla_x(\lambda^TK)(u,v) + \nabla_{xx}(\lambda^TK)(u,v)q] \leqq -u^T[\nabla_x(\lambda^TK)(u,v) + \nabla_{xx}(\lambda^TK)(u,v)q]. \tag{9}$$

Similarly, $\eta_2^T(v,y) + y \in C_2$ and $[\nabla_y(\lambda^TK)(x,y) + \nabla_{yy}(\lambda^TK)(x,y)p] \in C_2^*$ yield

$$-\eta_2^T(v,y)[\nabla_y(\lambda^TK)(x,y) + \nabla_{yy}(\lambda^TK)(x,y)p] \leqq y^T[\nabla_y(\lambda^TK)(x,y) + \nabla_{yy}(\lambda^TK)(x,y)p]. \tag{10}$$

Now, using Eqs. (9) and (10) in (8), we obtain

$$\lambda^T K(x,y) - y^T\nabla_y(\lambda^T K)(x,y) - y^T\nabla_{yy}(\lambda^T K)(x,y)p - \frac{1}{2}p^T\nabla_{yy}(\lambda^T K)(x,y)p$$
$$\geqq \lambda^T K(u,v) - u^T\nabla_x(\lambda^T K)(u,v) - u^T\nabla_{xx}(\lambda^T K)(u,v)q - \frac{1}{2}q^T\nabla_{xx}(\lambda^T K)(u,v)q.$$

Since $\lambda^T e = 1$, the above inequality can be written as

$$\lambda^T K(x,y) - (\lambda^T e)y^T\nabla_y(\lambda^T K)(x,y) - (\lambda^T e)y^T\nabla_{yy}(\lambda^T K)(x,y)p - \frac{1}{2}(\lambda^T e)p^T\nabla_{yy}(\lambda^T K)(x,y)p$$
$$\geqq \lambda^T K(u,v) - (\lambda^T e)u^T\nabla_x(\lambda^T K)(u,v) - (\lambda^T e)u^T\nabla_{xx}(\lambda^T K)(u,v)q - \frac{1}{2}(\lambda^T e)q^T\nabla_{xx}(\lambda^T K)(u,v)q.$$

That is

$$\lambda^T\Big[K(x,y) - \big[y^T\nabla_y(\lambda^T K)(x,y)\big]e - \big[y^T\nabla_{yy}(\lambda^T K)(x,y)p\big]e - \frac{1}{2}\big[p^T\nabla_{yy}(\lambda^T K)(x,y)p\big]e\Big]$$
$$\geqq \lambda^T\Big[K(u,v) - \big[u^T\nabla_x(\lambda^T K)(u,v)\big]e - \big[u^T\nabla_{xx}(\lambda^T K)(u,v)q\big]e - \frac{1}{2}\big[q^T\nabla_{xx}(\lambda^T K)(u,v)q\big]e\Big],$$

or

$$\Big[K(x,y) - \big[y^T\nabla_y(\lambda^T K)(x,y)\big]e - \big[y^T\nabla_{yy}(\lambda^T K)(x,y)p\big]e - \frac{1}{2}\big[p^T\nabla_{yy}(\lambda^T K)(x,y)p\big]e\Big]$$
$$\nleq \Big[K(u,v) - \big[u^T\nabla_x(\lambda^T K)(u,v)\big]e - \big[u^T\nabla_{xx}(\lambda^T K)(u,v)q\big]e - \frac{1}{2}\big[q^T\nabla_{xx}(\lambda^T K)(u,v)q\big]e\Big].$$

In order to prove strong and converse duality theorems, we need the following Fritz John type necessary optimality conditions proposed by Suneja et al. [18] instead of using Fritz John type necessary optimality conditions [2]. The necessary conditions in [2] are for scalar programming problem and cannot be applied for multiobjective case as used by Mishra [14]. □

Suneja et al. [18] considered the following multiobjective programming problem:

(**VP**) K-minimize $f(x)$
subject to $-g(x) \in Q,\ x \in C,$

where $C \subset R^n,\ f : R^n \to R^k,\ g : R^n \to R^m,\ K$ and Q are closed convex cones with nonempty interiors in R^k and R^m, respectively. They derived the following Fritz John type necessary conditions for a feasible point of (VP) to be a weakly efficient solution.

Lemma 1. *If x° is a weakly efficient solution of (VP), then there exist $\hat{\alpha} \in K_+^*,\ \hat{\beta} \in Q_+^*$ not both zero such that*

$$\Big(\hat{\alpha}^T\nabla f(x^\circ) + \hat{\beta}^T\nabla g(x^\circ)\Big)(x - x^\circ) \geqq 0, \quad \text{for all } x \in C,$$
$$\hat{\beta}^T g(x^\circ) = 0,$$

where $K_+^* = \Big\{z \in R^k | z^T x \geqq 0,\ \text{for all } x \in K\Big\}$ is the positive dual cone of K.

Remark 5. Observe that for the problems (WP) and (WD), the cone considered is $K = R_+^k = K_+^*$. Also, since $Q^* = -Q_+^*$, the above lemma holds with $\hat{\alpha} \in R_+^k$ and $\hat{\beta} \in -Q^*$ for the problem of the form

Minimize $f(x)$
subject to $-g(x) \in Q, \quad x \in C.$

Thus the above lemma holds with $\hat{\alpha} \in R_+^k$ and $\hat{\beta} \in Q^*$ for the problem of the form

Minimize $f(x)$
subject to $g(x) \in Q, \quad x \in C.$

Theorem 2 (Strong duality). *Let $(\bar{x},\bar{y},\bar{\lambda},\bar{p})$ be a weakly efficient solution of (WP), and $\lambda = \bar{\lambda}$ fixed in (WD). Let*

(i) the matrix $\nabla_{yy}(\bar{\lambda}^T K)(\bar{x},\bar{y})$ be nonsingular,
(ii) the set $\{\nabla_y K_1(\bar{x},\bar{y}), \nabla_y K_2(\bar{x},\bar{y}), \ldots, \nabla_y K_k(\bar{x},\bar{y})\}$ be linearly independent,
(iii) $\nabla_y(\nabla_{yy}(\bar{\lambda}^T K)(\bar{x},\bar{y})\bar{p})\bar{p} \notin$ Span $\{\nabla_y K_1(\bar{x},\bar{y}), \nabla_y K_2(\bar{x},\bar{y}), \ldots, \nabla_y K_k(\bar{x},\bar{y})\} \setminus \{0\}$; and
(iv) $\bar{p} \neq 0$ implies $\nabla_y(\nabla_{yy}(\bar{\lambda}^T K)(\bar{x},\bar{y})\bar{p})\bar{p} \neq 0.$

Then $(\bar{x},\bar{y},\bar{\lambda},\bar{q} = 0)$ is feasible for (WD), and the two objectives have the same values. Also, if the hypotheses of weak duality (Theorem 1) are satisfied for all feasible solutions of (WP) and (WD), then $(\bar{x},\bar{y},\bar{\lambda},\bar{q} = 0)$ is a weakly efficient solution of (WD).

Proof. Since $(\bar{x},\bar{y},\bar{\lambda},\bar{p})$ is a weakly efficient solution of (WP), by Lemma 1, there exist $\alpha \in R^k,\ \beta \in (C_2^*)^* = C_2,\ \omega \in R^k$ and $\mu \in R$ such that

$$\Big[\nabla_x(\alpha^T K)(\bar{x},\bar{y}) + \nabla_{xy}(\bar{\lambda}^T K)(\bar{x},\bar{y})(\beta - (\alpha^T e)\bar{y}) + \nabla_x\{\nabla_{yy}(\bar{\lambda}^T K)(\bar{x},\bar{y})\bar{p}\}\Big(\beta - (\alpha^T e)\Big(\bar{y} + \frac{1}{2}\bar{p}\Big)\Big)\Big](x - \bar{x}) \geqq 0, \quad \forall x \in C_1, \tag{11}$$

$$\nabla_y K(\bar{x},\bar{y})(\alpha - (\alpha^T e)\bar{\lambda}) + \nabla_{yy}(\bar{\lambda}^T K)(\bar{x},\bar{y})(\beta - (\alpha^T e)\bar{y}) - (\alpha^T e)\nabla_{yy}(\bar{\lambda}^T K)(\bar{x},\bar{y})\bar{p} + \nabla_y\{\nabla_{yy}(\bar{\lambda}^T K)(\bar{x},\bar{y})\bar{p}\}\Big(\beta - (\alpha^T e)\Big(\bar{y} + \frac{1}{2}\bar{p}\Big)\Big) = 0, \tag{12}$$

and Eqs. (13)–(16) in [19] along with $(\alpha, \omega) \geqq 0$ and $(\alpha, \beta, \omega, \mu) \neq 0$ hold. Following the proof of Theorem 2 in [19], we obtain

$$\beta = (\alpha^T e)(\bar{y} + \bar{p}), \tag{13}$$

and

$$\alpha^T e > 0. \tag{14}$$

Substituting (13) and (14) in (12), we get

$$\nabla_y \{\nabla_{yy}(\bar{\lambda}^T K)(\bar{x}, \bar{y})\bar{p}\}\bar{p} = \frac{-2}{(\alpha^T e)} \left[\nabla_y K(\bar{x}, \bar{y})(\alpha - (\alpha^T e)\bar{\lambda})\right]. \tag{15}$$

Now, we prove that $\bar{p} = 0$. In case, $\bar{p} \neq 0$, then (iv) implies

$$\nabla_y (\nabla_{yy}(\bar{\lambda}^T K)(\bar{x}, \bar{y})\bar{p})\bar{p} \neq 0,$$

which together with (15) shows that

$$\nabla_y (\nabla_{yy}(\bar{\lambda}^T K)(\bar{x}, \bar{y})\bar{p})\bar{p} \in Span\{\nabla_y K_1(\bar{x}, \bar{y}), \nabla_y K_2(\bar{x}, \bar{y}), \ldots, \nabla_y K_k(\bar{x}, \bar{y})\} \setminus \{0\},$$

a contradiction to assumption (iii). Hence, $\bar{p} = 0$. Also, when $\bar{p} = 0$, Eq. (15) yields

$$\nabla_y K(\bar{x}, \bar{y})(\alpha - (\alpha^T e)\bar{\lambda}) = 0,$$

which on applying hypothesis (ii) gives

$$\alpha = (\alpha^T e)\bar{\lambda}. \tag{16}$$

Using (13), (14) and (16) along with $\bar{p} = 0$, it follows from (11) that

$$\left[\nabla_x (\bar{\lambda}^T K)(\bar{x}, \bar{y})\right](x - \bar{x}) = \left[\nabla_x (\alpha^T K)(\bar{x}, \bar{y}) / \alpha^T e\right](x - \bar{x}) \geqq 0, \quad \forall x \in C_1. \tag{17}$$

Let $x \in C_1$. Then $x + \bar{x} \in C_1$ as C_1 is a closed convex cone, and so (17) shows that for every $x \in C_1$

$$\bar{x}^T \nabla_x (\bar{\lambda}^T K)(\bar{x}, \bar{y}) \geqslant 0,$$

which implies

$$-\nabla_x (\bar{\lambda}^T K)(\bar{x}, \bar{y}) \in C_1^*. \tag{18}$$

Also, from (13), $\bar{p} = 0$ and (14), we have

$$\bar{y} = \frac{\beta}{\alpha^T e} \in C_2. \tag{19}$$

Thus, from (18) and (19), it follows that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{q} = 0)$ is a feasible solution of (WD). Also, by letting $x = 0$ and $x = 2\bar{x}$, simultaneously in (17), we get

$$\bar{x}^T \nabla_x (\bar{\lambda}^T K)(\bar{x}, \bar{y}) = 0. \tag{20}$$

Further, on the similar lines of the proof of Theorem 2 in [19], we obtain

$$\bar{y}^T \nabla_y (\bar{\lambda}^T K)(\bar{x}, \bar{y}) = 0. \tag{21}$$

Eqs. (20) and (21) together imply the equality of objective values. By using weak duality (Theorem 1), it can be seen that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{q} = 0)$ is a weakly efficient solution of (WD). □

Remark 6. In view of Remark 3(i), (WP) and (WD) reduce to symmetric dual pair of Gupta and Kailey [11]. In this case, strong duality (Theorem 2.1) in [11] can be put as special case of our strong duality (Theorem 2) by invoking weak duality (Theorem 1) of Yang et al. [19].

Since in symmetric duality, the dual of the dual is the primal problem. The converse duality theorem can easily be proved by assuming the dual as the primal problem in strong duality theorem. Therefore, it is merely stated and its proof is being omitted.

Theorem 3 (Converse duality). *Let $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{q})$ be a weakly efficient solution of (WD), and $\lambda = \bar{\lambda}$ fixed in (WP). Let*

(i) the matrix $\nabla_{xx}(\bar{\lambda}^T K)(\bar{u}, \bar{v})$ be nonsingular,
(ii) the set $\{\nabla_x K_1(\bar{u}, \bar{v}), \nabla_x K_2(\bar{u}, \bar{v}), \ldots, \nabla_x K_k(\bar{u}, \bar{v})\}$ be linearly independent,
(iii) $\nabla_x (\nabla_{xx}(\bar{\lambda}^T K)(\bar{u}, \bar{v})\bar{q})\bar{q} \notin Span\{\nabla_x K_1(\bar{u}, \bar{v}), \nabla_x K_2(\bar{u}, \bar{v}), \ldots, \nabla_x K_k(\bar{u}, \bar{v})\} \setminus \{0\}$; and
(iv) $\bar{q} \neq 0$ implies $\nabla_x (\nabla_{xx}(\bar{\lambda}^T K)(\bar{u}, \bar{v})\bar{q})\bar{q} \neq 0$.

Then $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{p} = 0)$ is feasible for (WP), and the two objectives have the same values. Also, if the hypotheses of weak duality (Theorem 1) are satisfied for all feasible solutions of (WP) and (WD), then $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{p} = 0)$ is a weakly efficient solution of (WP).

4. Wolfe type mixed integer second order symmetric duality

In this section, we constrain some of the primal and dual variables to belong to the arbitrary sets of integers U and V as in Balas [1] and Gulati and Ahmad [7]. Suppose that the first n_1 $(0 \leqq n_1 \leqq n)$ components of x belong to U and the first m_1 $(0 \leqq m_1 \leqq m)$ components of y

belong to V. So we write $(x,y)=(x^1,x^2,y^1,y^2)$, where $x^1=(x_1,x_2,\ldots,x_{n_1})\in U$, and $y^1=(y_1,y_2,\ldots,y_{m_1})\in V$, x^2 and y^2 being the remaining components of x and y such that $x^2\in C_1$ and $y^2\in C_2$.

In the sequel, we require the following notion of separability [1,7] of a vector function:

Definition 5. Let $s^1,s^2,\ldots,s^r$ be elements of an arbitrary vector space. A vector function $H(s^1,s^2,\ldots,s^r)$ is called additively separable with respect to s^1, if there exist vector functions $M(s^1)$ (independent of $s^2,s^3,\ldots,s^r$) and $N(s^2,s^3,\ldots,s^r)$ (independent of s^1) such that

$$H(s^1,s^2,\ldots,s^r)=M(s^1)+N(s^2,s^3,\ldots,s^r).$$

Partitioning the vector variables x and y, we now formulate the following pair of multiobjective mixed integer second order symmetric dual programs over arbitrary cones:

$$(\textbf{WSP})\quad \max_{x^1}\ \min_{x^2,y}\quad K(x,y)-\left[(y^2)^T\nabla_{y^2}(\lambda^TK)(x,y)\right]e-\left[(y^2)^T\nabla_{y^2y^2}(\lambda^TK)(x,y)p\right]e-\tfrac{1}{2}\left[p^T\nabla_{y^2y^2}(\lambda^TK)(x,y)p\right]e$$
$$\text{subject to}\quad \nabla_{y^2}(\lambda^TK)(x,y)+\nabla_{y^2y^2}(\lambda^TK)(x,y)p\in C_2^*,$$
$$x^1\in U,\quad y^1\in V,\quad x^2\in C_1,$$
$$\lambda>0,\quad \lambda^Te=1.$$

$$(\textbf{WSD})\quad \min_{v^1}\ \max_{u,v^2}\quad K(u,v)-\left[(u^2)^T\nabla_{x^2}(\lambda^TK)(u,v)\right]e-\left[(u^2)^T\nabla_{x^2x^2}(\lambda^TK)(u,v)q\right]e-\tfrac{1}{2}\left[q^T\nabla_{x^2x^2}(\lambda^TK)(u,v)q\right]e$$
$$\text{subject to}\quad -\nabla_{x^2}(\lambda^TK)(u,v)-\nabla_{x^2x^2}(\lambda^TK)(u,v)q\in C_1^*,$$
$$u^1\in U,\quad v^1\in V,\quad v^2\in C_2,$$
$$\lambda>0,\quad \lambda^Te=1,$$

where $p\in R^{m-m_1}$, $q\in R^{n-n_1}$, $\lambda\in R^k$, and $e=(1,1,\ldots,1)^T\in R^k$.

Remark 7

(*i*) If $k=1$, $C_1=R^n_+$ and $C_2=R^m_+$, then (WSP) and (WSD) reduce to Wolfe type mixed integer symmetric dual programs of Gulati and Ahmad [7].
(*ii*) If we replace $\lambda>0$ with $\lambda\geqslant 0$ and put $p=0=q$, above pair reduces to the pair of Wolfe type multiobjective mixed integer symmetric dual programs discussed by Kim and Song [13].
(*iii*) Let $U=\emptyset$ and $V=\emptyset$. Then the above mixed integer second order symmetric dual programs are reduced to second order symmetric dual programs of the previous section.

In [15], Mishra and Wang established symmetric duality results for the following pair of multiobjective minimax mixed integer symmetric dual programs:

$$(\textbf{MSP1})\quad \max_{x^1}\ \min_{x^2,y}\quad K(x,y)-\left[(y^2)^T\nabla_{y^2}(\lambda^TK)(x,y)\right]e-\left[(y^2)^T\nabla_{y^2y^2}(\lambda^TK)(x,y)p_2\right]-\tfrac{1}{2}p_2^T\nabla_{y^2y^2}(\lambda^TK)(x,y)p_2$$
$$\text{subject to}\quad \nabla_{y^2}(\lambda^TK)(x,y)\in C_2^*,\quad \nabla_{y^2y^2}(\lambda^TK)(x,y)p_2\in C_2^*,$$
$$x^1\in U,\quad y^1\in V,\quad x^2\in C_1,\quad y^2\in C_2,\quad \lambda\geqslant 0,\quad \lambda^Te=1.$$

$$(\textbf{MSD1})\quad \min_{v^1}\ \max_{u,v^2}\quad K(u,v)-\left[(u^2)^T\nabla_{x^2}(\lambda^TK)(u,v)\right]e-\left[(u^2)^T\nabla_{x^2x^2}(\lambda^TK)(u,v)p_1\right]e-\tfrac{1}{2}p_1^T\nabla_{x^2x^2}(\lambda^TK)(u,v)p_1$$
$$\text{subject to}\quad -\nabla_{x^2}(\lambda^TK)(u,v)\in C_1^*,\quad -\nabla_{x^2x^2}(\lambda^TK)(u,v)p_1\in C_1^*,$$
$$u^1\in U,\quad v^1\in V,\quad u^2\in C_1,\quad v^2\in C_2,\quad \lambda\geqslant 0,\quad \lambda^Te=1.$$

It may be noted that in defining the functions and in problems formulation, Mishra and Wang [15] have done the same mistakes as done in [14] (see Remark 1 and (ii), (iii) of Remark 2). Also, in proving duality results they erroneously used the assumptions $\eta_1(x^2,u^2)\in C_1$ and $\eta_2(v^2,y^2)\in C_2$ as in [14] (see Remark 4).

Theorem 4 (Symmetric duality). *Let $(\bar{x},\bar{y},\bar{\lambda},\bar{p})$ be a weakly efficient solution of (WSP). Also, let*

(*i*) $K_i(x,y)$, $i=1,2,\ldots,k$ *be thrice differentiable in* x^2 and y^2,
(*ii*) $K_i(x,y)$, $i=1,2,\ldots,k$ *be additively separable with respect to* x^1 or y^1,
(*iii*) $\lambda^TK(\cdot,v)$ *be second order invex in x with respect to* η_1,
(*iv*) $-\lambda^TK(x,\cdot)$ *be second order invex in y with respect to* η_2,
(*v*) $\eta_1(x^2,u^2)+u^2\in C_1$, *for all* $x^2,u^2\in C_1$,
(*vi*) $\eta_2(v^2,y^2)+y^2\in C_2$, *for all* $v^2,y^2\in C_2$,
(*vii*) *the matrix* $\nabla_{y^2y^2}(\bar{\lambda}^TK)(\bar{x},\bar{y})$ *be nonsingular*,
(*viii*) *the set* $\{\nabla_{y^2}K_1(\bar{x},\bar{y}),\nabla_{y^2}K_2(\bar{x},\bar{y}),\ldots,\nabla_{y^2}K_k(\bar{x},\bar{y})\}$ *be linearly independent*,
(*ix*) $\nabla_{y^2}\left(\nabla_{y^2y^2}(\bar{\lambda}^TK)(\bar{x},\bar{y})\bar{p}\right)\bar{p}\notin Span\{\nabla_{y^2}K_1(\bar{x},\bar{y}),\nabla_{y^2}K_2(\bar{x},\bar{y}),\ldots,\nabla_{y^2}K_k(\bar{x},\bar{y})\}\setminus\{0\}$; *and*
(*x*) $\bar{p}\neq 0$ *implies* $\nabla_{y^2}\left(\nabla_{y^2y^2}(\bar{\lambda}^TK)(\bar{x},\bar{y})\bar{p}\right)\bar{p}\neq 0$.

Then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{q} = 0)$ is a weakly efficient solution of (WSD), and the two objectives have the same values.

Proof. Let

$$g = \max_{x^1} \min_{x^2, y} \left[K(x,y) - \left[(y^2)^T \nabla_{y^2}(\lambda^T K)(x,y)\right]e - \left[(y^2)^T \nabla_{y^2y^2}(\lambda^T K)(x,y)p\right]e - \frac{1}{2}\left[p^T \nabla_{y^2y^2}(\lambda^T K)(x,y)p\right]e : (x,y,\lambda,p) \in G \right]$$

and

$$h = \min_{v^1} \max_{u, v^2} \left[K(u,v) - \left[(u^2)^T \nabla_{x^2}(\lambda^T K)(u,v)\right]e - \left[(u^2)^T \nabla_{x^2x^2}(\lambda^T K)(u,v)q\right]e - \frac{1}{2}\left[q^T \nabla_{x^2x^2}(\lambda^T K)(u,v)q\right]e : (u,v,\lambda,q) \in H \right],$$

where G and H are feasible regions of (WSP) and (WSD), respectively.

As $K_i(x,y),\ i = 1,2,\ldots,k$ is taken to be additively separable with respect to x^1 or y^1 (say with respect to x^1), it follows that

$$K_i(x,y) = K_i^1(x^1) + K_i^2(x^2,y), \quad i = 1,2,\ldots,k.$$

Therefore $\nabla_{y^2}(\lambda^T K)(x,y) = \nabla_{y^2}(\lambda^T K^2)(x^2,y),\ \nabla_{y^2y^2}(\lambda^T K)(x,y) = \nabla_{y^2y^2}(\lambda^T K^2)(x^2,y)$, and g can be written as

$$\begin{aligned} g &= \max_{x^1} \min_{x^2, y} \Big[K^1(x^1) + K^2(x^2,y) - \left[(y^2)^T \nabla_{y^2}(\lambda^T K^2)(x^2,y)\right]e - \left[(y^2)^T \nabla_{y^2y^2}(\lambda^T K^2)(x^2,y)p\right]e \\ &\quad - \frac{1}{2}\left[p^T \nabla_{y^2y^2}(\lambda^T K^2)(x^2,y)p\right]e : \nabla_{y^2}(\lambda^T K^2)(x^2,y) + \nabla_{y^2y^2}(\lambda^T K^2)(x^2,y)p \in C_2^*,\ x^2 \in C_1,\ x^1 \in U,\ y^1 \in V,\ \lambda > 0,\ \lambda^T e = 1 \Big] \\ &= \max_{x^1} \min_{y^1} \min_{x^2, y^2} \Big[K^1(x^1) + K^2(x^2,y) - \left[(y^2)^T \nabla_{y^2}(\lambda^T K^2)(x^2,y)\right]e - \left[(y^2)^T \nabla_{y^2y^2}(\lambda^T K^2)(x^2,y)p\right]e \\ &\quad - \frac{1}{2}\left[p^T \nabla_{y^2y^2}(\lambda^T K^2)(x^2,y)p\right]e : \nabla_{y^2}(\lambda^T K^2)(x^2,y) + \nabla_{y^2y^2}(\lambda^T K^2)(x^2,y)p \in C_2^*,\ x^2 \in C_1,\ x^1 \in U,\ y^1 \in V,\ \lambda > 0,\ \lambda^T e = 1 \Big]. \end{aligned}$$

Or

$$g = \max_{x^1} \min_{y^1} \left[K^1(x^1) + \phi(y^1) : x^1 \in U,\ y^1 \in V \right],$$

where

$$\begin{aligned} \phi(y^1) = \min_{x^2, y^2} \Big[& K^2(x^2,y) - \left[(y^2)^T \nabla_{y^2}(\lambda^T K^2)(x^2,y)\right]e - \left[(y^2)^T \nabla_{y^2y^2}(\lambda^T K^2)(x^2,y)p\right]e - \frac{1}{2}\left[p^T \nabla_{y^2y^2}(\lambda^T K^2)(x^2,y)p\right]e : \\ & \nabla_{y^2}(\lambda^T K^2)(x^2,y) + \nabla_{y^2y^2}(\lambda^T K^2)(x^2,y)p \in C_2^*,\ x^2 \in C_1,\ x^1 \in U,\ y^1 \in V,\ \lambda > 0,\ \lambda^T e = 1 \Big]. \end{aligned} \tag{22}$$

Similarly, h can be written as

$$h = \min_{v^1} \max_{u^1} \left[K^1(u^1) + \psi(v^1) : u^1 \in U,\ v^1 \in V \right],$$

where

$$\begin{aligned} \psi(v^1) = \max_{u^2, v^2} \Big[& K^2(u^2,v) - \left[(u^2)^T \nabla_{x^2}(\lambda^T K^2)(u^2,v)\right]e - \left[(u^2)^T \nabla_{x^2x^2}(\lambda^T K^2)(u^2,v)q\right]e - \frac{1}{2}\left[q^T \nabla_{x^2x^2}(\lambda^T K^2)(u^2,v)q\right]e : \\ & -\nabla_{x^2}(\lambda^T K^2)(u^2,v) - \nabla_{x^2x^2}(\lambda^T K^2)(u^2,v)q \in C_1^*,\ v^2 \in C_2,\ u^1 \in U,\ v^1 \in V,\ \lambda > 0,\ \lambda^T e = 1 \Big]. \end{aligned} \tag{23}$$

For any given y^1, (22) and (23) are Wolfe type multiobjective second order symmetric dual programs and hence, in view of the hypotheses made here, Theorems 1and 2 of Section 3 become applicable. Therefore, for $y^1 = \bar{y}^1,\ \phi(\bar{y}^1) = \psi(\bar{y}^1)$. The remaining part of the theorem can be proved on the lines of Gulati and Ahmad [7]. □

5. Conclusion

The main purpose behind the present study is to draw one's attention towards the mistakes and shortcomings occured in the work of Mishra [14] and Mishra and Wang [15]. On the other hand, we have presented the Wolfe type multiobjective second order symmetric/minimax mixed integer symmetric dual programs over arbitrary cones and proved appropriate duality theorems under second order invexity assumptions. Following Kim and Kim [12], the work can be further generalized to nondifferentiable case where the objectives involve support functions of compact convex sets.

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