Asia-Pacific Journal of Operational Research 15 (1998) 63-74

MULTIOBJECTIVE DUALITY USING FRITZ JOHN CONDITIONS

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A Mond-Weir type dual is formulated for a multiobjective fractional programming problem and duality theorems are established that relate to efficient solutions of the primal and dual problems. The strong and converse duality theorems are proved using Fritz John type necessary conditions, which avoid the need of a constrain qualification.

Keywords. Multiobjective programming, fractional programming, efficiency, duality, generalized convexity.

1. Introduction

Egudo (1989) considered a nonlinear multiobjective programming problem and used the concept of efficiency of the duality relations for Wolfe and Mond-Weir type dual problems under certain convexity assumptions. Mukherjee (1991) extended these results for a Mond-Weir type dual to fractional multiobjective programs. The strong duality theorems in Egudo (1989) and Mukherjee (1991) require a constraint qualification for an equivalent scalar nonlinear program. Moreover, they did not establish any converse duality theorem.

In the present paper, we consider the following multiobjective fractional programming problem:

$$(VFP) \quad \text{Minimize} \quad F(x) = \left[\frac{f_1(x)}{h_1(x)}, \frac{f_2(x)}{h_2(x)}, \ldots, \frac{f_k(x)}{h_k(x)}\right]$$
$$\text{subject to} \quad g(x) \leqq 0, \qquad (1)$$

where $f = (f_1, f_2, ..., f_k)^t : R^n \to R^k$, $h = (h_1, h_2, ..., h_k)^t : R^n \to R^k$ and $g = (g_1, g_2, ..., g_m)^t : R^n \to R^m$ are differentiable functions. We assume that the first $p(p \leqq k)$ components of h are nonlinear and the rest are linear. Also, for each $x \in X = \{x \in R^n : g(x) \leqq 0\}$,

$$f_j(x) \geqq 0, \quad j = 1, 2, ..., p, \quad \text{and}$$
$$h_j(x) > 0, \quad j = 1, 2, ..., k.$$

Received May 1995.

We formulate the following Mond-Weir type dual:

(VMD) Maximize $v = (v_1, v_2, ..., v_k)$

$$\text{subject to } \nabla\left(\sum_{j=1}^{k} u_j(f_j(y) - v_j h_j(y)) + \sum_{i=1}^{m} w_i g_i(y)\right) = 0 \quad (2)$$

$$f_j(y) - v_j h_j(y) \geqq 0, \quad j = 1, 2, ..., k \quad (3)$$

$$w^t g(y) \geqq 0 \quad (4)$$

$$v_p \geqq 0, \ (u, w) \geq 0, \quad (5)$$

where $v_p = (v_1, v_2, ..., v_p)$, and establish duality relations between the efficient solutions of (VFP) and those of (VMD).

It may be noted that unlike Egudo (1989) and Mukherjee (1991), we do not include the constraint $\sum_{j=1}^{k} u_j = 1$ in the dual problem (VMD). Though this constraint plays an important role to prove duality relations for Wolfe type dual to any multiobjective program, it is not needed to study Mond-Weir type duality. In fact, by omitting it from the dual we are able to establish the strong duality theorem without needing a constraint qualification. Moreoever, its presence in the dual creates difficulties to establish a converse duality theorem as the converse duality theorem proved here could not be obtained with its presence in the dual problem (see Remark 3 after the converse duality theorem).

Remark 1. In Mukherjee (1991) the dual constraint corresponding to (2) has not been stated correctly. It subtracts a scalar from a vector quantity.

The following convention of vectors in R^n will be followed throughout this paper : $x \leqq y \Leftrightarrow x_i \leqq y_i$, $i = 1, 2, ..., n$; $x \leq y \Leftrightarrow x \leqq y$ and $x \neq y$; $x < y \Leftrightarrow x_i < y_i$, $i = 1, 2, ..., n$. The index sets $K = \{1, 2, ..., k\}$, $M = \{1, 2, ..., m\}$ and $P = \{1, 2, ..., p\}$. Also, the set $Q = K - P$ and for each $r \in K$, $K_r = K - \{r\}$. The symbol $\sum_i (\sum_j)$ means the sum over all $i \in M (j \in K)$. For other notations and definitions we refer to Mangasarian (1969).

Definition 1. A point $\bar{x} \in X$ is said to be an efficient solution of (VFP) if there exists no $x \in X$ such that $F(x) \leq F(\bar{x})$.

2. Weak Duality Theorems

Let Z be the set of all feasible solutions of the dual problem (VMD).

Theorem 1. Let $x \in X$ and $(y, u, v, w) \in Z$. If for each $(y, u, v, w) \in Z$, either

(a) $\sum_j u_j(f_j - v_j h_j) + \sum_i w_i g_i$ is strictly convex at y or

(b) $u > 0$ and $\sum_j u_j(f_j - v_j h_j) + \sum_i w_i g_i$ is convex at y,

then $F(x) \nleq v$.

Proof. Suppose to the contrary that $F(x) \leq v$ for some $x \in X$ and $(y, u, v, w) \in Z$. That is, there exists an $r \in K$ such that

$$\frac{f_r(x)}{h_r(x)} < v_r$$

and

$$\frac{f_j(x)}{h_j(x)} \leqq v_j, \quad \text{for all} \quad j \in K_r.$$

Or

$$f_r(x) - v_r h_r(x) < 0$$

and

$$f_j(x) - v_j h_j(x) \leqq 0, \quad \text{for all} \quad j \in K_r.$$

Therefore

$$\sum_j u_j(f_j(x) - v_j h_j(x)) \leqq 0. \tag{6}$$

Using (1), (3), (4) and (5), we get

$$\sum_j u_j(f_j(x) - v_j h_j(x)) + \sum_i w_i g_i(x) \leqq \sum_j u_j(f_j(y) - v_j h_j(y)) + \sum_i w_i g_i(y). \tag{7}$$

Now by hypothesis (a),

$$\nabla \left(\sum_j u_j(f_j(y) - v_j h_j(y)) + \sum_i w_i g_i(y) \right) (x - y) < 0, \tag{8}$$

which contradicts (2).

Under hypothesis (b) inequality (6) holds as strict inequality. Therefore (7) also holds as strict inequality, which with convexity assumption yields (8), again contradicting (2). □

The following weak duality theorem holds under weaker convexity assumptions. It is also given in Mukherjee (1991). Since a positive linear combination of pseudoconvex functions is not necessarily pseudoconvex, his proof under hypothesis (a) seems to be invalid. We provide here a correct proof for this part.

Theorem 2. Let $x \in X$ and $(y,u,v,w) \in Z$. If for each $(y,u,v,w) \in Z$, $\sum_i w_i g_i$ is quasiconvex at y and any of the following holds:

(a) $u > 0$ and $f_j - v_j h_j$ is pseudoconvex at y for each $j \in K$.
(b) $u > 0$ and $\sum_j u_j(f_j - v_j h_j)$ is pseudoconvex at y,
(c) $\sum_j u_j(f_j - v_j h_j)$ is strictly pseudoconvex at y,

then $F(x) \not\leq v$.

Proof. For every $x \in X$ and $(y,u,v,w) \in Z$, $w^t g(x) \leqq w^t g(y)$. Using quasiconvexity of $w^t g$, we get

$$w^t \nabla g(y)(x-y) \leqq 0.$$

Therefore, from equation (2)

$$\nabla\left(\sum_j u_j(f_j(y) - v_j h_j(y))\right)(x-y) \geqq 0. \tag{9}$$

Now suppose contrary to the result that $F(x) \leq v$ for some $x \in X$ and $(y,u,v,w) \in Z$. That is, there exists an $r \in K$ such that

$$f_r(x) - v_r h_r(x) < 0$$

and

$$f_j(x) - v_j h_j(x) \leqq 0, \quad \text{for all } \; j \in K_r.$$

Using (3), we get

$$f_r(x) - v_r h_r(x) < f_r(y) - v_r h_r(y)$$

and

$$f_j(x) - v_j h_j(x) \leqq f_j(y) - v_j h_j(y), \quad \text{for all } \; j \in K_r.$$

Since a pseudoconvex function is also quasiconvex, by hypothesis (a), pseudoconvexity of $f_r - v_r h_r$ and quasiconvexity of $f_j - v_j h_j (j \in K_r)$ imply

$$\nabla(f_r(y) - v_r h_r(y))(x-y) < 0$$

and

$$\nabla(f_j(y) - v_j h_j(y))(x-y) \leqq 0, \quad \text{for all } \; j \in K_r.$$

Also, since $u > 0$, the above inequalities yield

$$\nabla\left(\sum_j u_j(f_j(y) - v_j h_j(y))\right)(x - y) < 0, \tag{10}$$

which contradicts (9).

The proof for hypotheses (b) and (c) is given in Mukherjee (1991). □

Next we prove a weak duality theorem between (VFP) and (VMD) under ρ-convexity.

Definition 2. (Vial, 1982 and 1985). A differentiable function $f : R^n \to R$ is said to be ρ-convex if and only if for all $x, y \in R^n$,

$$f(x) - f(y) \geqq \nabla f(y)(x - y) + \rho\|x - y\|^2 \quad \text{for some} \quad \rho \in R.$$

If $\rho > 0$, then f is strictly convex; if $\rho = 0$, then f is convex and if $\rho < 0$, then f is weakly convex.

Theorem 3. Let $x \in X$ and $(y, u, v, w) \in Z$. If for each $(y, u, v, w) \in Z$, $f_j - v_j h_j$, $j = 1, 2, ..., k$ are ρ_j-convex, g_i, $i = 1, 2, ..., m$ are σ_i-convex and either

(a) $\sum_j u_j \rho_j + \sum_i w_i \sigma_i > 0$ or

(b) $u > 0$ and $\sum_j u_j \rho_j + \sum_i w_i \sigma_i \geqq 0$,

then $F(x) \nleq v$.

Proof. Suppose to the contrary that $F(x) \leq v$ for some $x \in X$ and $(y, u, v, w) \in Z$. That is, there exists an $r \in K$ such that

$$f_r(x) - v_r h_r(x) < 0 \tag{11}$$

and

$$f_j(x) - v_j h_j(x) \leqq 0, \quad \text{for all} \quad j \in K_r. \tag{12}$$

From (1), (3), (4), (5), (11) and (12), we get

$$\sum_j u_j(f_j(x) - v_j h_j(x)) - \sum_j u_j(f_j(y) - v_j h_j(y)) + \sum_i w_i g_i(x) - \sum_i w_i g_i(y) \leqq 0. \tag{13}$$

Since $f_j - v_j h_j$, $j = 1, 2, ..., k$ are ρ_j-convex and g_i, $i = 1, 2, ..., m$ are σ_i-convex, (13) implies

$$\nabla \left(\sum_j u_j (f_j(y) - v_j h_j(y)) + \sum_i w_i g_i(y) \right) (x - y)$$

$$\leqq - \left(\sum_j u_j \rho_j + \sum_i w_i \sigma_i \right) \|x - y\|^2. \tag{14}$$

Also, from (3) and (11)

$$f_r(x) - v_r h_r(x) < f_r(y) - v_r h_r(y),$$

which gives

$$x \neq y. \tag{15}$$

Now (14), (15) and hypothesis (a) imply

$$\nabla \left(\sum_j u_j (f_j(y) - v_j h_j(y)) + \sum_i w_i g_i(y) \right) (x - y) < 0, \tag{16}$$

contradicting (2).

The proof for hypothesis (b) is given in given in Mukherjee (1991). □

Corollary 1. Let a weak duality theorem hold between (VFP) and (VMD). If $\bar{x} \in X$ and $(\bar{y}, \bar{u}, \bar{v}, \bar{w}) \in Z$ such that $F(\bar{x}) = \bar{v}$, then $\bar{x}$ is efficient for (VFP) and $(\bar{y}, \bar{u}, \bar{v}, \bar{w})$ is efficient for (VMD).

Proof. Suppose to the contrary that $\bar{x}$ is not efficient for (VFP), then there exists an $x^* \in X$ such that

$$F(x^*) \leq F(\bar{x}) = \bar{v},$$

a contradiction to the weak duality theorem. Hence $\bar{x}$ is an efficient solution for (VFP). Similarly it follows that $(\bar{y}, \bar{u}, \bar{v}, \bar{w})$ is efficient for (VMD). □

Remark 2. Unlike Egudo (1989) and Mukherjee (1991), the above result holds for different vectors $\bar{x}$ and $\bar{y}$.

3. Strong and Converse Duality Theorems

The following Fritz John type necessary conditions will be used to prove the strong and converse duality theorems. This avoids the need of a constraint qualification (Egudo, 1989 and Mukherjee, 1991).

Lemma 1. (Gulati and Islam, 1989). Let $\bar{x} \in X$ be an efficient solution for the problem

$$\begin{aligned} &\text{Maximize} \quad f(x) = (f_1(x), f_2(x), ..., f_k(x)) \\ &\text{subject to} \quad x \in X^0 = \{x \in R^n : g(x) \leqq 0, h(x) = 0\} \end{aligned}$$

and let $h : R^n \to R^\ell$ have continuous first order partial derivatives at $\bar{x}$. Then there exist $\bar{\alpha} \in R^k$, $\bar{\beta} \in R^m$ and $\bar{\gamma} \in R^\ell$ such that

$$\bar{\alpha}^t \nabla f(\bar{x}) - \bar{\beta}^t \nabla g(\bar{x}) + \bar{\gamma}^t \nabla h(\bar{x}) = 0$$
$$\bar{\beta}^t g(\bar{x}) = 0$$
$$\bar{\alpha} \geqq 0,\ \bar{\beta} \geqq 0,\ (\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \neq 0.$$

Theorem 4. (Strong Duality). Let $\bar{x}$ be an efficient solution for (VFP). Then there exist $\bar{u}, \bar{v} \in R^k$ and $\bar{w} \in R^m$ such that $(\bar{x}, \bar{u}, \bar{v}, \bar{w})$ is feasible for (VMD). Also, if weak duality holds between (VFP) and (VMD), then $(\bar{x}, \bar{u}, \bar{v}, \bar{w})$ is efficient for (VMD).

Proof. Since $\bar{x}$ is efficient for (VFP), by Lemma 1 there exist $\bar{\tau} \in R^k$ and $\bar{w} \in R^m$ such that

$$\sum_j \bar{\tau}_j \nabla \frac{f_j(\bar{x})}{h_j(\bar{x})} + \sum_i \bar{w}_i \nabla g_i(\bar{x}) = 0 \tag{17}$$

$$\bar{w}^t g(\bar{x}) = 0 \tag{18}$$

$$(\bar{\tau}, \bar{w}) \geq 0. \tag{19}$$

Or

$$\sum_j \frac{\bar{\tau}_j}{h_j(\bar{x})} \left(\nabla f_j(\bar{x}) - \frac{f_j(\bar{x})}{h_j(\bar{x})} \nabla h_j(\bar{x}) \right) + \sum_i \bar{w}_i \nabla g_i(\bar{x}) = 0 \tag{20}$$

$$\bar{w}^t g(\bar{x}) = 0 \tag{21}$$

$$(\bar{\tau}, \bar{w}) \geq 0. \tag{22}$$

Setting $\bar{u}_j = \dfrac{\bar{\tau}_j}{h_j(\bar{x})}$ and $\bar{v}_j = \dfrac{f_j(\bar{x})}{h_j(\bar{x})}$ for all $j \in K$, we get

$$\nabla \left(\sum_j \bar{u}_j (f_j(\bar{x}) - \bar{v}_j h_j(\bar{x})) + \sum_i \bar{w}_i g_i(\bar{x}) \right) = 0 \tag{23}$$

$$f_j(\bar{x}) - \bar{v}_j h_j(\bar{x}) = 0 \quad \text{for all } \ j \in K \tag{24}$$

$$\bar{w}^t g(\bar{x}) \geqq 0,\ \bar{v}_p \geqq 0,\ (\bar{u}, \bar{w}) \geq 0. \tag{25}$$

Therefore $(\bar{x}, \bar{u}, \bar{v}, \bar{w})$ is feasible for (VMD). Now efficiency of $(\bar{x}, \bar{u}, \bar{v}, \bar{w})$ for (VMD) follows from Corollary 1. □

Theorem 5. (Converse Duality). Let $(\bar{y}, \bar{u}, \bar{v}, \bar{w})$ be an efficient solution for (VMD) and f, h and g have continuous second partial derivatives at $\bar{y}$. If,

(i) $\bar{u} > 0$,
(ii) $\nabla(f_j(\bar{y}) - \bar{v}_j h_j(\bar{y}))$, $j = 2, ..., k$ are linearly independent, and
(iii) the $n \times n$ Hessian matrix $\nabla^2 \left(\sum_j \bar{u}_j (f_j(\bar{y}) - \bar{v}_j h_j(\bar{y})) + \sum_i \bar{w}_i g_i(\bar{y}) \right)$ is positive or negative definite,

then $\bar{y} \in X$ and $F(\bar{y}) = \bar{v}$. Also, if a weak duality theorem holds, then $\bar{y}$ is an efficient solution for (VFP).

Proof. Since $(\bar{y}, \bar{u}, \bar{v}, \bar{w})$ is an efficient solution for (VMD), by Lemma 1 there exist $\bar{\alpha} \in R^k$, $\bar{\beta} \in R^n$, $\bar{\gamma} \in R^k$, $\bar{\xi} \in R$, $\bar{\eta} \in R^k$, $\bar{\zeta} \in R^p$ and $\bar{\nu} \in R^m$ such that

$$\bar{\beta}^t \nabla^2 \left(\sum_j \bar{u}_j (f_j(\bar{y}) - \bar{v}_j h_j(\bar{y})) + \sum_i \bar{w}_i g_i(\bar{y}) \right) + \sum_j \bar{\gamma}_j (\nabla f_j(\bar{y}) - \bar{v}_j \nabla h_j(\bar{y})) + \bar{\xi} \nabla \bar{w}^t g(\bar{y}) = 0 \tag{26}$$

$$(\nabla f_j(\bar{y}) - \bar{v}_j \nabla h_j(\bar{y}))\bar{\beta} + \bar{\eta}_j = 0, \ j \in K \tag{27}$$

$$\bar{\alpha}_j - \bar{u}_j \nabla h_j(\bar{y})\bar{\beta} - \bar{\gamma}_j h_j(\bar{y}) + \bar{\zeta}_j = 0, \ j \in P \tag{28}$$

$$\bar{\alpha}_j - \bar{u}_j \nabla h_j(\bar{y})\bar{\beta} - \bar{\gamma}_j h_j(\bar{y}) = 0, \ j \in Q \tag{29}$$

$$\nabla g_i(\bar{y})\bar{\beta} + \bar{\xi} g_i(\bar{y}) + \bar{\nu}_i = 0, \ i \in M \tag{30}$$

$$\bar{\gamma}_j (f_j(\bar{y}) - \bar{v}_j h_j(\bar{y})) = 0, \ j \in K \tag{31}$$

$$\bar{\xi} \bar{w}^t g(\bar{y}) = 0 \tag{32}$$

$$\bar{\eta}^t \bar{u} = 0 \tag{33}$$

$$\bar{\zeta}^t \bar{v}_p = 0 \tag{34}$$

$$\bar{\nu}^t \bar{w} = 0 \tag{35}$$

$$(\bar{\alpha}, \bar{\gamma}, \bar{\xi}, \bar{\eta}, \bar{\zeta}, \bar{\nu}) \geqq 0, \ (\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\xi}, \bar{\eta}, \bar{\zeta}, \bar{\nu}) \neq 0. \tag{36}$$

Now $\bar{u} > 0$, $\bar{\eta}^t \bar{u} = 0 \Rightarrow \bar{\eta} = 0$. Therefore from (27),

$$(\nabla f_j(\bar{y}) - \bar{v}_j \nabla h_j(\bar{y}))\bar{\beta} = 0 \quad \text{for all} \ \ j \in K, \tag{37}$$

which with equation (2) gives

$$\nabla \left(\sum_i \bar{w}_i g_i(\bar{y}) \right) \bar{\beta} = 0. \tag{38}$$

On multiplying (26) by $\bar{\beta}$ from the right and using (37) and (38), we get

$$\bar{\beta}^t \nabla^2 \left(\sum_j \bar{u}_j (f_j(\bar{y}) - \bar{v}_j h_j(\bar{y})) + \sum_i \bar{w}_i g_i(\bar{y}) \right) \bar{\beta} = 0, \tag{39}$$

which by hypothesis (iii) of the theorem implies $\bar{\beta} = 0$. Therefore equations (2) and (26) imply

$$\sum_j [(\bar{\gamma}_j - \bar{\xi}\bar{u}_j)\nabla(f_j(\bar{y}) - \bar{v}_j h_j(\bar{y}))] = 0. \tag{40}$$

This with hypothesis (ii) yields

$$\bar{\xi}\bar{u}_j = \bar{\gamma}_j \quad \text{for all} \quad j \in K. \tag{41}$$

Now suppose $\bar{\xi} = 0$. Then from equations (30) and (41), $\bar{\nu}_i = 0$ for all $i \in M$ and $\bar{\gamma}_j = 0$ for all $j \in K$ respectively. Therefore equations (28) and (29) give $\bar{\alpha}_j = 0$ for all $j \in Q$ and $\bar{\alpha}_j + \bar{\zeta}_j = 0$ for all $j \in P$. Since $\bar{\alpha}_j$ and $\bar{\zeta}_j$ are non-negative, we get $\bar{\alpha}_j = 0 = \bar{\zeta}_j$ for all $j \in P$. Thus

$$(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\xi}, \bar{\eta}, \bar{\zeta}, \bar{\nu}) = 0,$$

a contradiction to (36). Hence $\bar{\xi} > 0$. Now from equation (30),

$$g_i(\bar{y}) = -\frac{\bar{\nu}_i}{\bar{\xi}} \leqq 0 \quad \text{for all} \quad i \in M.$$

That is, $\bar{y}$ is feasible for (VFP). Moreover, equation (31) along with $\bar{\gamma}_j = \bar{\xi}\bar{u}_j > 0$ for all $j \in K$ yields

$$\frac{f_j(\bar{y})}{h_j(\bar{y})} = \bar{v}_j \quad \text{for all} \quad j \in K.$$

Hence by Corollary 1, $\bar{y}$ is efficient for (VFP). □

Remark 3. As stated in Section 1, unlike Egudo (1989) and Mukherjee (1991) we do not include the constraint $\sum_j u_j = 1$ in the dual problem (VMD). If it is added in (VMD), then we need one more Lagrangian multiplier $\bar{\tau} \in R$ corresponding to this constraint and therefore equation (27) is obtained as

$$(\nabla f_j(\bar{y}) - \bar{v}_j \nabla h_j(\bar{y}))\bar{\beta} + \bar{\eta}_j + \bar{\tau} = 0 \quad \text{for all} \quad j \in K.$$

As shown above, we get $\bar{\eta} = 0$. But we are unable to show $\bar{\tau} = 0$ and therefore we do not obtain equation (37), which plays a key role to derive the converse duality theorem.

Thereom 6. (Strict Converse Duality). Let $\bar{x}$ and $(\bar{y}, \bar{u}, \bar{v}, \bar{w})$ be efficient solutions for problems (VFP) and (VMD) respectively such that

$$F(\bar{x}) = \bar{v}. \tag{42}$$

If $\sum_j \bar{u}_j(f_j - \bar{v}_j h_j) + \sum_i \bar{w}_i g_i$ is strictly pseudoconvex at $\bar{y}$, then $\bar{y} = \bar{x}$, that is, $\bar{y}$ is an efficient solution of (VFP).

Proof. $F(\bar{x}) = \bar{v}$ implies $f_j(\bar{x}) - \bar{v}_j h_j(\bar{x}) = 0$ for all $j \in K$. Therefore

$$\sum_j \bar{u}_j(f_j(\bar{x}) - \bar{v}_j h_j(\bar{x})) = 0. \tag{43}$$

Now suppose that $\bar{x} \neq \bar{y}$. From (2), we have

$$\left[\nabla\left(\sum_j \bar{u}_j(f_j(\bar{y}) - \bar{v}_j h_j(\bar{y})) + \sum_i \bar{w}_i g_i(\bar{y})\right)\right](\bar{x} - \bar{y}) = 0.$$

Strict pseudoconvexity of $\sum_j \bar{u}_j(f_j - \bar{v}_j h_j) + \sum_i \bar{w}_i g_i$ implies

$$\sum_j \bar{u}_j(f_j(\bar{x}) - \bar{v}_j h_j(\bar{x})) - \sum_j \bar{u}_j(f_j(\bar{y}) - \bar{v}_j h_j(\bar{y})) + \sum_i \bar{w}_i g_i(\bar{x}) - \sum_i \bar{w}_i g_i(\bar{y}) > 0.$$

Using (1), (3), (4) and (5), we get

$$\sum_j \bar{u}_j(f_j(\bar{x}) - \bar{v}_j h_j(\bar{x})) > 0$$

a contradiction to (43). Hence $\bar{x} = \bar{y}$. □

4. PARTICULAR CASES

(b) Let $h_j(x) = 1$ for each $j \in K$ and $x \in X$, then we get the following primal problem:

$$(P_1) \quad \text{Minimize } (f_1(x), f_2(x), ..., f_k(x)) \quad \text{subject to } g(x) \leqq 0.$$

In this case (VMD) becomes

$$
\begin{aligned}
(D_1) \quad & \text{Maximize} \ (v_1, v_2, \ldots, v_k) \\
& \text{subject to} \ \nabla\left(\sum_j u_j f_j(y) + \sum_i w_i g_i(y)\right) = 0 \\
& f_j(y) - v_j \geqq 0, \ j = 1, 2, \ldots, k \\
& w^t g(y) \geqq 0 \\
& (u, w) \geq 0.
\end{aligned}
$$

The dual (D_1) is equivalent to

$$
\begin{aligned}
& \text{Maximize} \ (f_1(y), f_2(y), \ldots, f_k(y)) \\
& \text{subject to} \ \nabla\left(\sum_j u_j f_j(y) + \sum_i w_i g_i(y)\right) = 0 \\
& w^t g(y) \geqq 0, \ (u, w) \geq 0
\end{aligned}
$$

which is the Mond-Weir type dual obtained by Weir and Mond (1986).

(b) Let $k = 1$, then (VFP) becomes the nonlinear fractional programming problem:

$$
\begin{aligned}
(P_2) \quad & \text{Minimize} \ \frac{f_1(x)}{h_1(x)} \\
& \text{subject to} \ g(x) \leqq 0
\end{aligned}
$$

and the dual (VMD) reduces to

$$
\begin{aligned}
(D_2) \quad & \text{Maximize} \ v_1 \\
& \text{subject to} \ \nabla(u_1(f_1(y) - v_1 h_1(y)) + w^t g(y) = 0 \\
& f_1(y) - v_1 h_1(y) \geq 0, \ w^t g(y) \geqq 0 \\
& v_1 \geqq 0, \ (u_1, w) \geq 0
\end{aligned}
$$

which is a new dual to (P_2). If $h_1(x)$ is linear, then the dual variable v_1 in (D_2) is unrestricted in sign.

ACKNOWLEDGEMENT

The second author is thankful to the University Grants Commission, India for financial support.

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