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Duality in nondifferentiable minimax fractional programming with generalized convexity

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Abstract

A Mond–Weir type dual for a class of nondifferentiable minimax fractional programming problem is considered. Appropriate duality results are proved involving (F, α, ρ, d)-pseudoconvex functions.

Keywords: Nondifferentiable minimax programming; Fractional programming; Duality; Generalized convexity

1. Introduction

Fractional programming is an interesting subject appeared in many types of optimization problems. For example, it can be used in engineering and economics to minimize a ratio of functions between a given period of time and a utilized resource in order to measure the efficiency or productivity of a system. In these types of problems the objective function is usually given as a ratio of functions in fractional programming form (see Stancu-Minasian [16]).

Optimization problems with minimax type functions arise in the design of electronic circuits, however minimax fractional problems appear in the formulation of discrete and continuous rational approximation problems with respect to the Chebyshev norm [3], in continuous rational games [14], in multiobjective programming [15], in engineering design as well as in some portfolio selection problems discussed by Bajona-xandri and Martinez-legaz [2].

Minimax mathematical programming has been of much interest in the recent past [1,4,5,11,13,18–20]. Schmitendorf [13] established necessary and sufficient optimality conditions for minimax problem. Tanimoto [17] applied these optimality conditions to define a dual problem and derived duality theorems, which were extended for the fractional analogue of generalized minimax problem by Yadav and Mukherjee [19].

Motivated by various concepts of generalized convexity, Liang et al. [8,9] introduced a unified formulation of generalized convexity, which was called (F, α, ρ, d)-convexity and obtained some corresponding optimality

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conditions and duality results for the single objective fractional problems and multiobjective problems. Recently, Liang and Shi [10] obtained sufficient conditions and duality theorems for minimax fractional problem under (F,α,ρ,d)-convexity. Lai et al. [7] derived necessary and sufficient conditions for nondifferentiable minimax fractional problem with generalized convexity and applied these optimality conditions to construct one parametric dual model and also discussed duality theorems. Lai and Lee [6] obtained duality theorems for two parametric-free dual models of nondifferentiable minimax fractional problem involving generalized convexity assumptions. Recently, Mishra et al. [12] established duality results for one parametric and two parametric-free dual models of nondifferentiable minimax fractional programming problem under generalized univexity.

The optimization problem considered in this paper is the nondifferentiable minimax fractional programming problem which consists of minimizing the supremum of ratio of functions involving square roots. Taking motivation from the work of Liang et al. [8], Lai et al. [7] and Lai and Lee [6], we consider a Mond–Weir type dual model for this problem and establish weak, strong and strict converse duality theorems with (F,α,ρ,d)-pseudoconvexity. This work improves and generalizes some existing results on minimax fractional programming.

2. Notations and preliminary results

Let R^n be the n-dimensional Euclidean space and X an open set in R^n.

Definition 2.1. A functional $F: X\times X\times R^n \to R$ is said to be sublinear if $\forall x, \bar{x}\in X$

(i) $F(x,\bar{x};a_1+a_2) \leqslant F(x,\bar{x};a_1)+F(x,\bar{x};a_2)\ \forall a_1,a_2\in R^n$,
(ii) $F(x,\bar{x};\beta a)=\beta F(x,\bar{x};a)\ \forall\beta\in R_+$ and $\forall a\in R^n$.

By (ii), it is clear that $F(x,\bar{x},0)=0$.

Definition 2.2 [8,9]. Given an open set $X\subset R^n$, a number $\rho\in R$, and two functions $\alpha: X\times X\to R_+\backslash\{0\}$ and $d(\cdot,\cdot): X\times X\to R$, a differentiable function ζ over X is said to be (F,α,ρ,d)-convex at $\bar{x}$, if for any $x\in X$, $F: X\times X\times R^n\to R$ is sublinear, and $\zeta(x)$ satisfies the following condition:

$$\zeta(x)-\zeta(\bar{x}) \geqslant F(x,\bar{x};\alpha(x,\bar{x})\nabla\zeta(\bar{x}))+\rho d^2(x,\bar{x}).$$

Definition 2.3. Given an open set $X\subset R^n$, a number $\rho\in R$, and two functions $\alpha: X\times X\to R_+\backslash\{0\}$ and $d(\cdot,\cdot): X\times X\to R$, a differentiable function ζ over X is said to be (F,α,ρ,d)-pseudoconvex at $\bar{x}$, if for any $x\in X$, there exists a sublinear functional $F: X\times X\times R^n\to R$ such that

$$\zeta(x)<\zeta(\bar{x}) \Rightarrow F(x,\bar{x};\alpha(x,\bar{x})\nabla\zeta(\bar{x}))<-\rho d^2(x,\bar{x}).$$

Further, ζ is said to be strictly (F,α,ρ,d)-pseudoconvex at $\bar{x}$, if for any $x\in X$, there exists a sublinear functional $F: X\times X\times R^n\to R$ such that

$$F(x,\bar{x};\alpha(x,\bar{x})\nabla\zeta(\bar{x})) \geqslant -\rho d^2(x,\bar{x}) \Rightarrow \zeta(x)>\zeta(\bar{x}).$$

We now consider the following nondifferentiable minimax fractional programming problem:

$$(P)\quad \min_{x\in R^n}\sup_{y\in Y}\frac{f(x,y)+(x^tBx)^{1/2}}{h(x,y)-(x^tDx)^{1/2}},$$
$$\text{subject to}\quad g(x)\leqslant 0,\ \ x\in X,$$

where Y is a compact subset of R^m, $f, h: R^n\times R^m\to R$, are C^1 on $R^n\times R^m$ and $g: R^n\to R^p$ is C^1 on R^n. B and D are $n\times n$ positive semidefinite matrices.

Let $S = \{x \in X : g(x) \leqslant 0\}$ denote the set of all feasible solutions of (P). For each $(x, y) \in R^n \times R^m$, we define

$$\phi(x,y) = \frac{f(x,y) + (x^tBx)^{1/2}}{h(x,y) - (x^tDx)^{1/2}},$$

such that for each $(x,y) \in S \times Y$, $f(x,y) + (x^tBx)^{1/2} \geqslant 0$ and $h(x,y) - (x^tDx)^{1/2} > 0$. For each $x \in S$, we define $J(x) = \{j \in J : g_j(x) = 0\}$, where $J = \{1, 2, \ldots, p\}$

$$Y(x) = \left\{ y \in Y : \frac{f(x,y) + (x^tBx)^{1/2}}{h(x,y) - (x^tDx)^{1/2}} = \sup_{z \in Y} \frac{f(x,z) + (x^tBx)^{1/2}}{h(x,z) - (x^tDx)^{1/2}} \right\},$$

$$K(x) = \Bigg\{ (s,t,\tilde{y}) \in N \times R^s_+ \times R^{ms} : 1 \leqslant s \leqslant n+1, t = (t_1, t_2, \ldots, t_s) \in R^s_+ \text{ with } \sum_{i=1}^{s} t_i = 1, \tilde{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_s) \text{ with } \bar{y}_i \in Y(x) (i = 1, 2, \ldots, s) \Bigg\}.$$

Since f and h are continuously differentiable and Y is compact in R^m, it follows that for each $x^* \in S$, $Y(x^*) \neq \emptyset$, and for any $\bar{y}_i \in Y(x^*)$, we have a positive constant

$$k_0 = \phi(x^*, \bar{y}_i) = \frac{f(x^*, \bar{y}_i) + (x^{*t}Bx^*)^{1/2}}{h(x^*, \bar{y}_i) - (x^{*t}Dx^*)^{1/2}}.$$

We shall need the following generalized Schwartz inequality.

Let B be a positive semidefinite matrix of order n. Then for all x, $w \in R^n$,

$$x^tBw \leqslant (x^tBx)^{\frac{1}{2}}(w^tBw)^{\frac{1}{2}}. \tag{1}$$

We observe that equality holds if $Bx = \lambda Bw$ for some $\lambda \geqslant 0$. Evidently, if $(w^tBw)^{\frac{1}{2}} \leqslant 1$, we have

$$x^tBw \leqslant (x^tBx)^{\frac{1}{2}}.$$

If the functions f, g and h in problem (P) are continuously differentiable with respect to $x \in R^n$, then Lai et al. [7] derived the following necessary conditions for optimality of (P). In what follows ∇ stands for gradient vector with respect to x.

Theorem 2.1 (Necessary conditions). *If x^* is a solution of problem (P) satisfying $x^{*t}Bx^* > 0$, $x^{*t}Dx^* > 0$, and $\nabla g_j(x^*)$, $j \in J(x^*)$ are linearly independent, then there exist $(s, t^*, \bar{y}) \in K(x^*)$, $k_0 \in R_+$, w, $v \in R^n$, and $\mu^* \in R^p_+$ such that*

$$\sum_{i=1}^{s} t_i^* \{\nabla f(x^*, \bar{y}_i) + Bw - k_0(\nabla h(x^*, \bar{y}_i) - Dv)\} + \nabla \sum_{j=1}^{p} \mu_j^* g_j(x^*) = 0, \tag{2}$$

$$f(x^*, \bar{y}_i) + (x^{*t}Bx^*)^{\frac{1}{2}} - k_0(h(x^*, \bar{y}_i) - (x^{*t}Dx^*)^{\frac{1}{2}}) = 0, \quad i = 1, 2, \ldots, s, \tag{3}$$

$$\sum_{j=1}^{p} \mu_j^* g_j(x^*) = 0, \tag{4}$$

$$t_i^* \geqslant 0 \ (i = 1, 2, \ldots, s), \quad \sum_{i=1}^{s} t_i^* = 1, \tag{5}$$

$$\begin{cases} w^tBw \leqslant 1, \ v^tDv \leqslant 1, \\ (x^{*t}Bx^*)^{1/2} = x^{*t}Bw, \\ (x^{*t}Dx^*)^{1/2} = x^{*t}Dv. \end{cases} \tag{6}$$

Remark. In the above theorem, both matrices B and D are positive semidefinite at the solution x^*. If one of $(x^{*t}Bx^*)$ and $(x^{*t}Dx^*)$ is zero, or both B and D are singular, then for $(s, t^*, \bar{y}) \in K(x^*)$, we can take a set $Z_{\bar{y}}(x^*)$ as defined in [6] by

$Z_{\bar{y}}(x^*) = \{z \in R^n : z^t \nabla g_j(x^*) \leqslant 0,\ j \in J(x^*)$ satisfying one of the following conditions$\}$:

(i) $x^{*t}Bx^* > 0,\ x^{*t}Dx^* = 0$

$$\Rightarrow z^t \left(\sum_{i=1}^{s} t_i^* \left\{ \nabla f(x^*, \bar{y}_i) + \frac{Bx^*}{(x^{*t}Bx^*)^{\frac{1}{2}}} - k_0 \nabla h(x^*, \bar{y}_i) \right\} \right) + (z^t(k_0^2 D)z)^{\frac{1}{2}} < 0,$$

(ii) $x^{*t}Bx^* = 0,\ x^{*t}Dx^* > 0$

$$\Rightarrow z^t \left(\sum_{i=1}^{s} t_i^* \left\{ \nabla f(x^*, \bar{y}_i) - k_0 \left(\nabla h(x^*, \bar{y}_i) - \frac{Dx^*}{(x^{*t}Dx^*)^{\frac{1}{2}}} \right) \right\} \right) + (z^t B z)^{\frac{1}{2}} < 0,$$

(iii) $x^{*t}Bx^* = 0,\ x^{*t}Dx^* = 0$

$$\Rightarrow z^t \left(\sum_{i=1}^{s} t_i^* \{ \nabla f(x^*, \bar{y}_i) - k_0 \nabla h(x^*, \bar{y}_i) \} \right) + (z^t(k_0^2 D)z)^{\frac{1}{2}} + (z^t B z)^{\frac{1}{2}} < 0.$$

If we insert the condition $Z_{\bar{y}}(x^*) = \emptyset$ in Theorem 2.1, then the result of Theorem 2.1 still holds.

3. Duality model

In this section, we first restate Theorem 2.1, in the form of following theorem, which can be proved on the lines of Theorem 4 in [6].

Theorem 3.1 (Necessary conditions). *If x^* is a solution of problem (P). Assuming $Z_{\bar{y}}(x^*)$ to be empty, there exist $(s, t^*, \bar{y}) \in K(x^*)$, w, $v \in R^n$ and $\mu^* \in R_+^p$ satisfying*

$$\nabla \left(\frac{\sum_{i=1}^{s} t_i^* \{ f(x^*, \bar{y}_i) + x^{*t}Bw \} + \sum_{j=1}^{p} \mu_j^* g_j(x^*)}{\sum_{i=1}^{s} t_i^* \{ h(x^*, \bar{y}_i) - x^{*t}Dv \}} \right) = 0, \tag{7}$$

$$\sum_{j=1}^{p} \mu_j^* g_j(x^*) = 0, \tag{8}$$

$$t_i^* \in R_+^s (i = 1, 2, \ldots, s), \quad \sum_{i=1}^{s} t_i^* = 1, \tag{9}$$

$$w^t B w \leqslant 1, \quad (x^{*t}Bx^*)^{1/2} = x^{*t}Bw, \tag{10}$$

$$v^t D v \leqslant 1, \quad (x^{*t}Dx^*)^{1/2} = x^{*t}Dv. \tag{11}$$

Theorem 3.1 can be employed to construct the following Mond–Weir type dual as follows:

$$(D) \qquad \max_{(s, t^*, \bar{y}) \in K(z)} \ \sup_{(z, \mu, v, w) \in H(s, t^*, \bar{y})} \frac{\sum_{i=1}^{s} t_i^* \{ f(z, \bar{y}_i) + z^t B w \} + \sum_{j=1}^{p} \mu_j g_j(z)}{\sum_{i=1}^{s} t_i^* \{ h(z, \bar{y}_i) - z^t D v \}},$$

where $H(s, t^*, \bar{y})$ denote the set of all $(z, \mu, w, v) \in R^n \times R_+^p \times R^n \times R^n$ satisfying

$$\nabla \left(\frac{\sum_{i=1}^{s} t_i^* \{ f(z, \bar{y}_i) + z^t B w \} + \sum_{j=1}^{p} \mu_j g_j(z)}{\sum_{i=1}^{s} t_i^* \{ h(z, \bar{y}_i) - z^t D v \}} \right) = 0, \tag{12}$$

$$\begin{cases} w^t B w \leqslant 1, & (z^t B z)^{1/2} = z^t B w, \\ v^t D v \leqslant 1, & (z^t D z)^{1/2} = z^t D v. \end{cases} \tag{13}$$

If the set $H(s,t^*,\bar{y})$ is empty, we define supremum over it to be $-\infty$. For convenience, we use the notation:

$$\psi(\cdot) = \left[\sum_{i=1}^{s} t_i^*\{h(z,\bar{y}_i) - z^t Dv\right]\left[\sum_{i=1}^{s} t_i^*\{f(\cdot,\bar{y}_i) + (\cdot)^t Bw\} + \sum_{j=1}^{p} \mu_j g_j(\cdot)\right]$$
$$- \left[\sum_{i=1}^{s} t_i^*\{f(z,\bar{y}_i) + z^t Bw\} + \sum_{j=1}^{p} \mu_j g_j(z)\right]\left[\sum_{i=1}^{s} t_i^*\{h(\cdot,\bar{y}_i) - (\cdot)^t Dv\}\right].$$

Suppose that

$$\sum_{i=1}^{s} t_i^*\{f(z,\bar{y}_i) + z^t Bw\} + \sum_{j=1}^{p} \mu_j g_j(z) \geqslant 0 \quad \text{and} \quad \sum_{i=1}^{s} t_i^*\{h(z,\bar{y}_i) - (z)^t Dv\} > 0,$$

for all $(s,t^*,\bar{y}) \in K(z)$, $(z,\mu,w,v) \in H(s,t^*,\bar{y})$. The following weak duality theorem can be proved.

Theorem 3.2 (Weak duality). *Suppose that x and $(z,\mu,v,w,s,t,\bar{y})$ are respectively the feasible solutions for (P) and (D). Also assume that $\psi(\cdot)$ is (F,α,ρ,d)-pseudoconvex at z and the inequality*

$$\frac{\rho}{\alpha(x,z)} \geqslant 0$$

holds, then

$$\sup_{y\in Y} \frac{f(x,y) + (x^t Bx)^{1/2}}{h(x,y) - (x^t Dx)^{1/2}} \geqslant \frac{\sum_{i=1}^{s} t_i\{f(z,\bar{y}_i) + z^t Bw\} + \sum_{j=1}^{p} \mu_j g_j(z)}{\sum_{i=1}^{s} t_i\{h(z,\bar{y}_i) - z^t Dv\}}.$$

Proof. By means of a contradiction, suppose that

$$\sup_{y\in Y} \frac{f(x,y) + (x^t Bx)^{1/2}}{h(x,y) - (x^t Dx)^{1/2}} < \frac{\sum_{i=1}^{s} t_i\{f(z,\bar{y}_i) + z^t Bw\} + \sum_{j=1}^{p} \mu_j g_j(z)}{\sum_{i=1}^{s} t_i\{h(z,\bar{y}_i) - z^t Dv\}},$$

for all $y \in Y$. If we replace y by $\bar{y}_i$ in the above inequality and sum up after multiplying by t_i, then we have

$$\left[\sum_{i=1}^{s} t_i\{f(x,\bar{y}_i) + (x^t Bx)^{1/2}\}\right]\left[\sum_{i=1}^{s} t_i\{h(z,\bar{y}_i) - z^t Dv\}\right]$$
$$< \left[\sum_{i=1}^{s} t_i\{f(z,\bar{y}_i) + z^t Bw\} + \sum_{j=1}^{p} \mu_j g_j(z)\right]\left[\sum_{i=1}^{s} t_i\{h(x,\bar{y}_i) - (x^t Dx)^{1/2}\}\right].$$

Using the generalized Schwartz inequality and (13), we get

$$\phi(x) \leqslant \left[\sum_{i=1}^{s} t_i\{h(z,\bar{y}_i) - z^t Dv\}\right]\left[\sum_{i=1}^{s} t_i\{f(x,\bar{y}_i) + (x^t Bx)^{\frac{1}{2}}\} + \sum_{j=1}^{p} \mu_j g_j(x)\right]$$
$$- \left[\sum_{i=1}^{s} t_i\{f(z,\bar{y}_i) + z^t Bw\} + \sum_{j=1}^{p} \mu_j g_j(z)\right]\left[\sum_{i=1}^{s} t_i\{h(x,\bar{y}_i) - (x^t Dx)^{\frac{1}{2}}\}\right]$$
$$< \sum_{j=1}^{p} \mu_j g_j(x) \times \sum_{i=1}^{s} t_i\{h(z,\bar{y}_i) - z^t Dv\}.$$

Since $\sum_{i=1}^{s} t_i\{h(z,\bar{y}_i) - z^t Dv\} > 0$ and $\sum_{j=1}^{p} \mu_j g_j(x) \leqslant 0$, it follows that

$$\psi(x) < 0 = \psi(z).$$

As $\psi(\cdot)$ is (F,α,ρ,d)-pseudoconvex at z. Therefore, $F(x,z;\alpha(x,z)\nabla\psi(z)) < -\rho d^2(x,z)$, that is

$$F\left(x,z;\alpha(x,z)\left\{\left[\sum_{i=1}^{s} t_i\{h(z,\bar{y}_i)-z^tDv\right]\nabla\left[\sum_{i=1}^{s} t_i\{f(z,\bar{y}_i)+z^tBw\}+\sum_{j=1}^{p}\mu_j g_j(z)\right]\right.\right.$$
$$\left.\left.-\left[\sum_{i=1}^{s} t_i\{f(z,\bar{y}_i)+z^tBw\}+\sum_{j=1}^{p}\mu_j g_j(z)\right]\nabla\left[\sum_{i=1}^{s} t_i\{h(z,\bar{y}_i)-z^tDv\}\right]\right\}\right) < -\rho d^2(x,z).$$

On multiplying the above inequality by $\frac{1}{\alpha(x,z)\left[\sum_{i=1}^{s} t_i\{h(z,\bar{y}_i)-z^tDv\}\right]^2}$ and using the sublinearity of F, we have

$$F\left(x,z;\nabla\left(\frac{\sum_{i=1}^{s} t_i\{f(z,\bar{y}_i)+z^tBw\}+\sum_{j=1}^{p}\mu_j g_j(z)}{\sum_{i=1}^{s} t_i\{h(z,\bar{y}_i)-z^tDv\}}\right)\right) < -\frac{\rho d^2(x,z)}{\alpha(x,z)\left[\sum_{i=1}^{s} t_i\{h(z,\bar{y}_i)-z^tDv\}\right]^2}.$$

Using the fact that $\frac{\rho}{\alpha(x,z)} \geqslant 0$, we have

$$F\left(x,z;\nabla\left(\frac{\sum_{i=1}^{s} t_i\{f(z,\bar{y}_i)+z^tBw\}+\sum_{j=1}^{p}\mu_j g_j(z)}{\sum_{i=1}^{s} t_i\{h(z,\bar{y}_i)-z^tDv\}}\right)\right) < 0. \tag{14}$$

In the light of (12), the inequality (14) contradicts $F(x,z;0)=0$. □

Theorem 3.3 (Strong duality). *Suppose that $\bar{x}$ is optimal for (P) and $\nabla g_j(\bar{x})$, $j\in J(\bar{x})$ is linearly independent. Then there exist $(\bar{s},\bar{t},\bar{y}^\star)\in K(\bar{x})$ and $(\bar{x},\bar{\mu},\bar{v},\bar{w})\in H(\bar{s},\bar{t},\bar{y}^\star)$ such that $(\bar{x},\bar{\mu},\bar{v},\bar{w},\bar{s},\bar{t},\bar{y}^\star)$ is feasible for (D). Further, if the weak duality (Theorem 3.2) holds for all feasible $(z,\mu,v,w,s,t,\bar{y})$ of (D), then $(\bar{x},\bar{\mu},\bar{v},\bar{w},\bar{s},\bar{t},\bar{y}^\star)$ is optimal for (D) and the two objectives have the same extreme values.*

Proof. Since $\bar{x}$ is an optimal solution for (P) and $\nabla g_j(\bar{x})$, $j\in J(\bar{x})$ is linearly independent, then by Theorem 3.1, there exist $(\bar{s},\bar{t},\bar{y}^\star)\in K(\bar{x})$ and $(\bar{x},\bar{\mu},\bar{v},\bar{w})\in H(\bar{s},\bar{t},\bar{y}^\star)$ such that $(\bar{x},\bar{\mu},\bar{v},\bar{w},\bar{s},\bar{t},\bar{y}^\star)$ is feasible for (D) and the two objective values are equal. The optimality of $(\bar{x},\bar{\mu},\bar{v},\bar{w},\bar{s},\bar{t},\bar{y}^\star)$ for (P) thus follows from weak duality (Theorem 3.2). □

Theorem 3.4 (Strict converse duality). *Let $\bar{x}$ and $(\bar{z},\bar{\mu},\bar{v},\bar{w},\bar{s},\bar{t},\bar{y}^*)$ be optimal solutions for (P) and (D) respectively. Also suppose that $\psi(\cdot)$ is strictly (F,α,ρ,d)-pseudoconvex at $\bar{z}$, for all $(\bar{s},\bar{t},\bar{y}^*)\in K(\bar{x})$, $(\bar{z},\bar{\mu},\bar{w},\bar{v})\in H(\bar{s},\bar{t},\bar{y}^*)$, and the inequality*

$$\frac{\rho}{\alpha(\bar{x},\bar{z})} \geqslant 0$$

holds, and $\nabla g_j(\bar{x})$, $j\in J(\bar{x})$ is linearly independent. Then $\bar{z}=\bar{x}$; that is, $\bar{z}$ is optimal for (P).

Proof. We shall assume that $\bar{x}\neq\bar{z}$ and exhibit a contradiction. Since $(\bar{z},\bar{\mu},\bar{v},\bar{w},\bar{s},\bar{t},\bar{y}^*)$ is feasible for (D), it follows that

$$\nabla\left(\frac{\sum_{i=1}^{\bar{s}}\bar{t}_i\{f(\bar{z},\bar{y}_i^*)+\bar{z}^tB\bar{w}\}+\sum_{j=1}^{p}\bar{\mu}_j g_j(\bar{z})}{\sum_{i=1}^{\bar{s}}\bar{t}_i\{h(\bar{z},\bar{y}_i^*)-\bar{z}^tD\bar{v}\}}\right)=0.$$

The above inequality along with the sublinearity of F and $\frac{\rho}{\alpha(\bar{x},\bar{z})}\geqslant 0$ implies

$$F\left(\bar{x},\bar{z};\nabla\left(\frac{\sum_{i=1}^{\bar{s}}\bar{t}_i\{f(\bar{z},\bar{y}_i^*)+\bar{z}^tB\bar{w}\}+\sum_{j=1}^{p}\bar{\mu}_j g_j(\bar{z})}{\sum_{i=1}^{\bar{s}}\bar{t}_i\{h(\bar{z},\bar{y}_i^*)-\bar{z}^tD\bar{v}\}}\right)\right)=0\geqslant-\frac{\rho d^2(\bar{x},\bar{z})}{\alpha(\bar{x},\bar{z})},$$

which together with the sublinearity of F and $\alpha(\bar{x},\bar{z})>0$ yields

$$F\left(\bar{x},\bar{z};\alpha(\bar{x},\bar{z})\nabla\left(\frac{\sum_{i=1}^{\bar{s}}\bar{t}_i\{f(\bar{z},\bar{y}_i^*)+\bar{z}^tB\bar{w}\}+\sum_{j=1}^{p}\bar{\mu}_j g_j(\bar{z})}{\sum_{i=1}^{\bar{s}}\bar{t}_i\{h(\bar{z},\bar{y}_i^*)-\bar{z}^tD\bar{v}\}}\right)\right)\geqslant-\rho d^2(\bar{x},\bar{z}).$$

Using strict (F,α,ρ,d)-pseudoconvexity of $\psi(\cdot)$, we obtain

$$\psi(\bar{x}) > \psi(\bar{z}).$$

Since $\psi(\bar{z}) = 0$, then we have $\psi(\bar{x}) > 0$, that is

$$\left[\sum_{i=1}^{\bar{s}} \bar{t}_i\{h(\bar{z},\bar{y}_i^*) - \bar{z}^t D\bar{v}\}\right]\left[\sum_{i=1}^{\bar{s}} \bar{t}_i\{f(\bar{x},\bar{y}_i^*) + (\bar{x}^t B\bar{w})\} + \sum_{j=1}^{p} \bar{\mu}_j g_j(\bar{x})\right]$$
$$> \left[\sum_{i=1}^{\bar{s}} \bar{t}_i\{f(\bar{z},\bar{y}_i^*) + \bar{z}^t B\bar{w}\} + \sum_{j=1}^{p} \bar{\mu}_j g_j(\bar{z})\right]\left[\sum_{i=1}^{\bar{s}} \bar{t}_i\{h(\bar{x},\bar{y}_i^*) - (\bar{x}^t D\bar{v})\}\right]. \tag{15}$$

The relations (1), (13), (15) and $\sum_{j=1}^{p} \bar{\mu}_j g_j(\bar{x}) \leqslant 0$ imply

$$\sup_{y\in Y} \frac{f(\bar{x},y) + (\bar{x}^t B\bar{x})^{1/2}}{h(\bar{x},y) - (\bar{x}^t D\bar{x})^{1/2}} > \frac{\sum_{i=1}^{\bar{s}} \bar{t}_i\{f(\bar{z},\bar{y}_i^*) + \bar{z}^t B\bar{w}\} + \sum_{j=1}^{p} \bar{\mu}_j g_j(\bar{z})}{\sum_{i=1}^{\bar{s}} \bar{t}_i\{h(\bar{z},\bar{y}_i^*) - \bar{z}^t D\bar{v}\}}. \tag{16}$$

Since $\bar{x}$ is optimal for (P) and $g_j(\bar{x})$, $j \in J(\bar{x})$ is linearly independent, by strong duality (Theorem 3.3), there exist $(\bar{s},\bar{t},\bar{y}^*) \in K(\bar{x})$ and $(\bar{x},\bar{\mu},\bar{v},\bar{w}) \in H(\bar{s},\bar{t},\bar{y}^*)$ so that $(\bar{x},\bar{\mu},\bar{v},\bar{w},\bar{s},\bar{t},\bar{y}^*)$ turns to be an optimal solution of (D) and

$$\sup_{y\in Y} \frac{f(\bar{x},y) + (\bar{x}^t B\bar{x})^{1/2}}{h(\bar{x},y) - (\bar{x}^t D\bar{x})^{1/2}} = \frac{\sum_{i=1}^{\bar{s}} \bar{t}_i\{f(\bar{z},\bar{y}_i^*) + \bar{z}^t B\bar{w}\} + \sum_{j=1}^{p} \bar{\mu}_j g_j(\bar{z})}{\sum_{i=1}^{\bar{s}} \bar{t}_i\{h(\bar{z},\bar{y}_i^*) - \bar{z}^t D\bar{v}\}},$$

which contradicts the fact of (16). Hence $\bar{x} = \bar{z}$. □

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