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OPTIMALITY CONDITIONS AND MIXED DUALITY IN NONDIFFERENTIABLE PROGRAMMING

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ABSTRACT. We establish the Kuhn-Tucker sufficient optimality conditions for a class of nondifferentiable programming in the framework of generalized (F, ρ)-convex functions. A mixed type dual is presented for nondifferentiable programming problem and various duality theorems are derived. This mixed dual formulation unifies the two existing symmetric dual formulation in the literature.

1. INTRODUCTION

Consider the nonlinear programming problem:

Primal (P)

$$\text{Minimize } \psi(x) = f(x) + (x^t Bx)^{1/2}$$
$$\text{Subject to } x \in X = \{x \in S : g(x) \leq 0\}, \tag{1}$$

where S is an open subset of $R^n, f : S \to R$ and $g : S \to R^m$ are differentiable functions and B is an $(n \times n)$ positive semi-definite matrix. If B=0, then ψ is differentiable and (P) is the usual nonlinear programming problem. Special cases of (P), with f not differentiable have appeared in [2, 6, 10, 11]. An application of a special case of (P) to the problem of minimizing cost function which includes costs directly proportional to Euclidean distances appears in [4].

The formulation of stochastic linear programming leads to a deterministic nonlinear programming problem [11], where the nonlinearity occurs in the objective function as the sum of square roots of positive semidefinite quadratic forms. However, it is difficult to solve directly this problem because of the nondifferentiability of the terms in objective function. Then it is very useful to establish a dual to the nonolinear problem of which a solution may be easily obtained. A solution of the dual problem helps to obtain a solution of the primal problem. Mond [7] obtained a set of necessary and sufficient optimality conditions for (P) involving convex functions. A Wolfe type dual to (P) is then formulated and duality theorems are established under convexity assumptions. Chandra, Craven and Mond [3] proposed a Mond-Weir type dual of (P) and discussed duality results under pseudoconvexity and quasiconvexity assumptions.

In this paper, we discuss sufficient optimality conditions for (P) under generalized (F, ρ)-convexity assumptions. A mixed type dual is formulated for (P) on the lines of Bector, Chandra and Abha [1] and various duality theorems are established.

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2. Notations and Preliminary Results

Let $\bar{x}$ satisfy (1). Define the set

$$Z(\bar{x}) = \Big\{ z : z^t \nabla g_i(\bar{x}) \leq 0 \ (i \in I),\ z^t \nabla f(\bar{x}) + z^t B\bar{x}/(\bar{x}^t B\bar{x})^{\frac{1}{2}} < 0 \ \text{if } B\bar{x} \neq 0,\ z^t \nabla f(\bar{x}) + (z^t Bz)^{\frac{1}{2}} < 0 \text{ if } B\bar{x} = 0 \Big\},$$

where $I = \{i : g_i(\bar{x}) = 0, i \in M = \{1, 2, \cdots, m\}\}$.

For readers convenience, we write the following definitions of the generalized (F, ρ) convexity from [9]:

Definition 2.1. A functional $F : S \times S \times R^n \longrightarrow R$ is sublinear if for any $x, \bar{x} \in S$,

(i) $F(x, \bar{x};\ a + b) \leq F(x, \bar{x};\ a) + F(x, \bar{x};\ b)$ for any $a, b \in R^n$,

and

(ii) $F(x, \bar{x};\ \alpha a) = \alpha F(x, \bar{x};\ a)$ for any $\alpha \in R, \alpha \geq 0$, and $a \in R^n$.

From (ii) it follows that $F(x, \bar{x};\ 0) = 0$.

Let F be sublinear functional and the numerical function $\phi : S \rightarrow R$ be differentiable at $\bar{x} \in S$ and $\rho \in R$. Let $d(.,.) : S \times S \longrightarrow R$. Assume $d(x, x) = 0$ for all x.

Definition 2.2. The function ϕ is said to be (F, ρ)-quasiconvex at $\bar{x} \in S$, if for all $x \in S$,

$$\phi(x) \leq \phi(\bar{x}) \Rightarrow F(x, \bar{x}; \nabla\phi(\bar{x})) \leq -\rho d^2(x, \bar{x}).$$

Definition 2.3. The function ϕ is said to be (F, ρ)- pseudoconvex at $\bar{x} \in S$, if for all $x \in S$,

$$F(x, \bar{x}; \nabla\phi(\bar{x})) \geq -\rho d^2(x, \bar{x}) \Rightarrow \phi(x) \geq \phi(\bar{x}).$$

Note that, the above definitions are slightly different from those in [9], since we do not assume $d(.,.)$ to be a pseudometric.

The following result from [7] is needed in the sequel.

Theorem 2.1 (Necessary Conditions). *Let $\bar{x}$ be an optimal solution of (P) and let $Z(\bar{x})$ be empty. Then there exist $u \in R^m$ and $w \in R^n$ such that*

$$\nabla f(\bar{x}) + \nabla u^t g(\bar{x}) + Bw = 0 \tag{2}$$

$$u^t g(\bar{x}) = 0 \tag{3}$$

$$(\bar{x}^t B\bar{x})^{\frac{1}{2}} = \bar{x}^t Bw \tag{4}$$

$$w^t Bw \leq 1 \tag{5}$$

$$u \geq 0. \tag{6}$$

3. Optimality Conditions

In this section, we derive sufficient conditions for optimality of (P) under the assumption of generalized (F, ρ)-convexity.

Theorem 3.1 (Sufficient Conditions). *Assume that $(\bar{x}, u, w)$ satisfies relations (2)-(6). If $f(.) + (.)^t Bw$ is (F, ρ)-pseudoconvex at $\bar{x} \in X$, $u^t g(.)$ is (F, σ)-quasiconvex at $\bar{x} \in X$, and $\rho + \sigma \geq 0$, then $\bar{x}$ is an optimal solution of* (P).

Proof. Let x be any feasible solution of (P). From (1), (3) and (6), we have

$$u^t g(x) \leq 0 = u^t g(\bar{x}).$$

Using the (F,σ)-quasiconvexity of $u^t g(.)$ at $\bar{x}$, we get

$$F(x,\bar{x};\nabla u^t g(\bar{x})) \leq -\sigma d^2(x,\bar{x}). \tag{7}$$

Therefore from (2) and (7)

$$F(x,\bar{x};\nabla f(\bar{x}) + Bw) \geq \sigma d^2(x,\bar{x}).$$

Since $\rho + \sigma \geq 0$, we have

$$F(x,\bar{x};\nabla f(\bar{x}) + Bw) \geq -\rho d^2(x,\bar{x}).$$

The (F,ρ)-pseudoconvexity of $f(.) + (.)^t Bw$ at $\bar{x}$, we get

$$f(x) + x^t Bw \geq f(\bar{x}) + \bar{x}^t Bw.$$

Hence $\bar{x}$ is an optimal solution of (P). □

4. MIXED TYPE DUALITY

For $M = \{1,2,\cdots,m\}, J \subseteq M$, let $K = M \setminus J$ be the set of indices j which are in M but not in J. Let $g(x)$ be partitioned as $g(x) = (g_J(x), g_K(x))^t$. We now introduce the following mixed type dual for the problem (P):

Dual(MD) Maximize $G(y,u,w) = f(y) + u_J^t g_J(y) + y^t Bw$

Subject to

$$\nabla f(y) + \nabla u^t g(y) + Bw = 0 \tag{7}$$

$$u_K^t g_K(y) \geq 0 \tag{8}$$

$$w^t Bw \leq 1 \tag{9}$$

$$u = (u_J, u_K) \geq 0, \tag{10}$$

where $y, w \in R^n$ and $u \in R^m$ and $\nabla f(y)$ denotes the gradient vector of f at y.

Let Y denote the set of all feasible solutions of the dual problem (MD).

Theorem 4.1 (Weak Duality). *Let $x \in X$ and $(y,u,w) \in Y$. Let $\phi(.) = f(.) + u_J^t g_J(.) + (.)^t Bw$ be (F,ρ)-pseudoconvex and $u_K^t g_K(.)$ be (F,σ)-quasiconvex at y over X with $\rho + \sigma \geq 0$. Then*

$$\text{infimum (P)} \geq \text{supremum (MD)}.$$

Proof. Since $x \in X$ and $(y,u,w) \in Y$, we have

$$u_K^t g_K(x) \leq u_K^t g_K(y).$$

The (F,σ)-quasiconvexity of $u_K^t g_K(.)$, at y gives

$$F(x,y;\nabla u_K^t g_K(y)) \leq -\sigma d^2(x,y). \tag{12}$$

The sublinearity of F and (8) imply that

$$\begin{aligned} 0 &= F(x,y;\nabla f(y) + \nabla u^t g(y) + Bw) \\ &\leq F(x,y;\nabla f(y) + \nabla u_J^t g_J(y) + Bw) + F(x,y;\nabla u_K^t g_K(y)) \end{aligned}$$

$$\leq F(x,y;\nabla f(y)+\nabla u_J^t g_J(y)+Bw)-\sigma d^2(x,y) \qquad \text{(using (12))}.$$

Since $\rho+\sigma\geq 0$, we have

$$F(x,y;\nabla f(y)+\nabla u_J^t g_J(y)+Bw)\geq -\rho d^2(x,y).$$

The (F,ρ)-pseudoconvexity of $\phi(.)=f(.)+u_J^t g_J(.)+(.)^t Bw$ at y implies

$$f(x)+u_J^t g_J(x)+x^t Bw\geq f(y)+u_J^t g_J(y)+y^t Bw. \tag{13}$$

On the other hand let $x^*=B^{1/2}x$ and $w^*=B^{1/2}w$. From the Schwartz inequality and $w^{*t}w^*=w^t Bw\leq 1$,

$$x^t Bw=x^{*t}w^*\leq \|x^*\|\|w^*\|\leq \|x^*\|=(x^t Bx)^{1/2}. \tag{14}$$

From (1), (11), (13) and (14), we obtain

$$f(x)+(x^t Bx)^{1/2}\geq f(y)+u_J^t g_J(y)+y^t Bw,$$

and hence

$$\text{infimum (P)}\geq \text{supremum (MD)}.$$

□

Theorem 4.2 (Strong Duality). *Let $\bar{x}$ be a local or global optimal solution of (P) and let $Z(\bar{x})$ be empty. Then there exist $\bar{u}\in R^m$ and $\bar{w}\in R^n$ such that $(\bar{x},\bar{u},\bar{w})\in Y$ and the objective function values of* (P) *and* (MD) *are equal. Also, if the assumptions of weak duality theorem hold, then $\bar{x}$ and $(\bar{x},\bar{u},\bar{w})$ are global optimal solutions of* (P) *and* (MD) *respectively.*

Proof. By Theorem 2.1, there exist $\bar{u}\in R^m$ and $\bar{w}\in R^n$ such that

$$\nabla f(\bar{x})+\nabla\bar{u}^t g(\bar{x})+B\bar{w}=0 \tag{15}$$

$$\bar{u}^t g(\bar{x})=0 \tag{16}$$

$$(\bar{x}^t B\bar{x})^{1/2}=\bar{x}^t B\bar{w} \tag{17}$$

$$\bar{w}^t B\bar{w}\leq 1 \tag{18}$$

$$\bar{u}\geq 0. \tag{19}$$

From equation (16),

$$\bar{u}_K^t g_K(\bar{x})=0 \text{ and } \bar{u}_J^t g_J(\bar{x})=0. \tag{20}$$

Relations (15), (18), (19) and (20) imply that $(\bar{x},\bar{u},\bar{w})$ is feasible for (MD). Also, from (17) and (20) we obtain that the both objective function values of (P) and (MD) are equal. Now optimality of (P) and (MD) follow from weak duality theorem. □

Theorem 4.3 (Converse Duality). *Let $(\bar{y},\bar{u},\bar{w})$ be an optimal solution of* (MD). *Let*

(i) *the $(n\times n)$ Hessian matrix $[\nabla^2 f(\bar{y})+\nabla^2\bar{u}^t g(\bar{y})]$ be positive or negative definite, and*

(ii) $\nabla\bar{u}_K^t g_K(\bar{y})\neq 0$.

Then $\bar{y}$ is an optimal solution for (P) *and the two optimal values of* (P) *and* (MD) *are equal.*

Proof. Since $(\bar{y}, \bar{u}, \bar{w})$ is an optimal solution for (MD), there exist $\alpha \in R, \beta \in R^n, \gamma \in R, \xi \in R$ and $\eta \in R^m$ satisfying the following Fritz John conditions [5]:

$$\alpha[\nabla f(\bar{y}) + \nabla \bar{u}_J^t g_J(\bar{y}) + B\bar{w}] - [\nabla^2 f(\bar{y}) + \nabla^2 \bar{u}^t g(\bar{y})]\beta + \gamma \nabla \bar{u}_K^t g_K(\bar{y}) = 0 \quad (21)$$

$$\alpha g_J(\bar{y}) - \nabla g_J(\bar{y})\beta + \eta_J = 0 \quad (22)$$

$$- \nabla g_K(\bar{y})\beta + \gamma g_K(\bar{y}) + \eta_K = 0 \quad (23)$$

$$\alpha B\bar{y} - B\beta - 2\xi B\bar{w} = 0 \quad (24)$$

$$\gamma \bar{u}_K^t g_K(\bar{y}) = 0 \quad (25)$$

$$\xi(1 - \bar{w}^t B\bar{w}) = 0 \quad (26)$$

$$\eta_K^t \bar{u}_K = 0 \quad (27)$$

$$\eta_J^t \bar{u}_J = 0 \quad (28)$$

$$(\alpha, \gamma, \xi, \eta) \geq 0 \quad (29)$$

$$(\alpha, \beta, \gamma, \xi, \eta) \neq 0. \quad (30)$$

The first constraint of the dual problem (MD) and the equation (21) yield

$$(\gamma - \alpha) \nabla \bar{u}_K^t g_K(\bar{y}) - [\nabla^2 f(\bar{y}) + \nabla^2 \bar{u}^t g(\bar{y})]\beta = 0. \quad (31)$$

On multiplying (23) by $\bar{u}_K$ and using (25) and (27), we get

$$\nabla \bar{u}_K^t g_K(\bar{y})\beta = 0 \quad (32)$$

On multiplying (31) by β^t from the left and using (32), we obtain

$$\beta^t[\nabla^2 f(\bar{y}) + \nabla^2 \bar{u}^t g(\bar{y})]\beta = 0.$$

Since $[\nabla^2 f(\bar{y}) + \nabla^2 \bar{u}^t g(\bar{y})]$ is assumed to be positive or negative definite, the above equation implies $\beta = 0$. Therefore (31) gives

$$(\gamma - \alpha) \nabla \bar{u}_K^t g_K(\bar{y}) = 0.$$

Since $\nabla \bar{u}_K^t g_K(\bar{y}) \neq 0$, we get

$$\gamma = \alpha. \quad (33)$$

Now suppose $\alpha = 0$. Then $\gamma = 0$. Then equation (22) and (23) imply $\eta_J = 0$ and $\eta_K = 0$ respectively. Equation (24) together with(26) gives $\xi = 0$, a contradiction to (30). Thus $\alpha > 0$. This gives $\gamma > 0$.

Since $\beta = 0$, equations (22) and (23) lead to

$$g_J(\bar{y}) = -\frac{\eta_J}{\alpha} \leq 0, \quad (34)$$

and

$$g_K(\bar{y}) = -\frac{\eta_K}{\gamma} \leq 0.$$

That is, $\bar{y}$ is feasible for (P). Also, (24) gives

$$B\bar{y} = 2\xi B\bar{w}/\alpha. \quad (35)$$

and hence $\bar{y}^t B\bar{w} = (\bar{y}B\bar{y})^{1/2}(\bar{w}^t B\bar{w})^{1/2}$. In case $\xi > 0$, (26) gives $\bar{w}^t B\bar{w} = 1$ and so $\bar{y}^t B\bar{w} = (\bar{y}B\bar{y})^{1/2}$. In case $\xi = 0$, (35) gives $B\bar{y} = 0$ and so $\bar{y}^t B\bar{w} = (\bar{y}B\bar{y})^{1/2} = 0$. Thus, in either case $\bar{y}^t B\bar{w} = (\bar{y}B\bar{y})^{1/2}$.

On the other hand (28) and (34) gives

$$\bar{u}_J^t g_J(\bar{y}) = 0.$$

Hence $G(\bar{y}, \bar{u}, \bar{w}) = f(\bar{y}) + \bar{y}^t B\bar{w} = f(\bar{y}) + (\bar{y}^t B\bar{y})^{1/2}$. The result, then follows from weak duality Theorem 4.1. □

5. Special Cases

(i) Let $J = \phi$, then (MD) becomes the Mond-Weir type dual obtained by Chandra Craven and Mond [3].

(ii) Let $K = \phi$, then (MD) reduces to the following problem:

$$\text{Maximize } G(y, u, w) = f(y) + u^t g(y) + y^t Bw$$

Subject to

$$\nabla f(y) + \nabla u^t g(y) + Bw = 0$$
$$w^t Bw \leq 1$$
$$u \geq 0.$$

This problem may be equivalently written as:

$$\text{Maximize } G(y, u, w) = f(y) + u^t g(y) - y^t[\nabla f(y) + \nabla u^t g(y)]$$

Subject to

$$\nabla f(y) + \nabla u^t g(y) + Bw = 0$$
$$w^t Bw \leq 1$$
$$u \geq 0,$$

which is the Wolfe type dual considered by Mond [7].

(iii) For B = 0, $K = \phi$, the dual (MD) reduces to the Wolfe dual. Similarly for B = 0, $J = \phi$, we get the Mond Weir dual [8].

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