J. Appl. Math. & Informatics Vol. **26***(2008), No. 5 - 6, pp. 991 - 1010*
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SUFFICIENCY AND DUALITY IN MINMAX FRACTIONAL SUBSET PROGRAMMING INVOLVING GENERALIZED TYPE-I FUNCTIONS

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Abstract. A discrete minmax fractional subset programming problem is considered. Various parametric and parameter-free global sufficient optimality conditions and duality results are discussed under generalized $(\mathcal{F}, \alpha, \rho, \theta)$-V-type-I n-set functions.

AMS Mathematics Subject Classification : 90C30, 90C32, 90C46, 90C47
Key words and phrases : Minmax fractional subset programming; Generalized $(\mathcal{F}, \alpha, \rho, \theta)$-V-type-I n-set functions; Sufficient optimality conditions; Duality

1. Introduction

The notion of duality for generalized linear fractional programming problem with point-functions was initiated by Von Neumann [10] in the context of an economic equilibrium problem. Recently, various optimality conditions, duality results, and computational algorithms for several classes of generalized fractional programs have been appeared in the related literature. A fairly extensive list of references pertaining to different aspects of generalized fractional programming problems is given in [14, 15].

In this paper, we consider the following discrete minmax subset programming problem:

$$\textbf{(P)}\ \text{Minimize} \max_{1\le i\le k} \frac{F_i(T)}{G_i(T)}$$

$$\text{subject to } H_j(T) \le 0,\ j \in M,\ T \in \mathcal{A}^n,$$

where $\mathcal{A}^n$ is the n-fold product of σ-algebra $\mathcal{A}$ of subsets of a given set X, F_i, G_i, $i \in K = \{1, 2, \dots, k\}$ and H_j, $j \in M = \{1, 2, \dots, m\}$ are real-valued

Received July 20, 2007. Revised January 9, 2008. Accepted January 13, 2008.
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functions defined on $\mathcal{A}^n$, and for each $i \in K$, $F_i(T) \geq 0$ and $G_i(T) > 0$ for all $T \in \mathcal{A}^n$ such that $H_j(T) \leq 0$, $j \in M$.

Zalmai [11] established necessary and sufficient optimality conditions and various duality results for (P). A Lagrangian-type dual problem was constructed for (P) in [12] via a Gordan-type transposition theorem and appropriate duality results were proved. Preda [9] extended the concept of ρ-convexity [11] to (F, ρ)-convexity and obtained duality theorems. Bhatia and Kumar [3] derived sufficient optimality conditions and duality results for different combinations of the problem function by using ρ-convexity. In [2], Bector and Singh discussed optimality and duality theorems involving generalized b-vexity assumptions, however, the Lagrangian-type dual problem was discussed in [5]. Lai and Liu [6] presented parameter-free necessary and sufficient optimality conditions for (P). They constructed also two parameter-free dual models and discussed duality results. Zalmai [14] obtained nonparametric sufficient optimality conditions and duality results for minmax programming problems under generalized $(\mathcal{F}, \rho, \theta)$-convexity assumptions. A number of parametric and parameter-free sufficient optimality conditions and duality results were discussed in [15] under generalized $(\mathcal{F}, \alpha, \rho, \theta)$-V-convexity [13]. Recently, Ahmad and Sharma [1] derived sufficient optimality conditions for a multiobjective subset programming problem under $(\mathcal{F}, \alpha, \rho, d)$-type I functions.

In this paper, motivated by Zalmai [14, 15] and Mishra [7], we present parametric and parameter-free sufficient optimality conditions for (P) under generalized $(\mathcal{F}, \alpha, \rho, \theta)$-V-type-I n-set functions. Moreover, appropriate duality theorems are proved for parametric and parameter-free dual models of (P).

2. Notations and preliminaries

Let $(X, \mathcal{A}, \mu)$ be a finite atomless measure space with $L_1(X, \mathcal{A}, \mu)$ separable, and let d be the pseudometric on $\mathcal{A}^n$ defined by

$$d(T, Y) = \left[\sum_{p \in N} \mu^2 (T_p \triangle Y_p) \right]^{\frac{1}{2}}, \ T_p,\ Y_p \in \mathcal{A},\ p \in N = \{1, 2, \ldots, n\},$$

where $\triangle$ denotes the symmetric difference; thus, $(\mathcal{A}^n, d)$ is a pseudometric space. For $h \in L_1(X, \mathcal{A}, \mu)$ and $S \in \mathcal{A}$ with characteristic function $\chi_S \in L_\infty(X, \mathcal{A}, \mu)$, the integral $\int_S h \, d\mu$ will be denoted by $\langle h, \chi_S \rangle$.

Corley [4] introduced the notion of differentiability for n-set functions as:

A function $F : \mathcal{A} \to R$ is said to be differentiable at T^*, if there exists $DF(T^*) \in L_1(X, \mathcal{A}, \mu)$, the derivative of F at T^*, such that for each $T \in \mathcal{A}$,

$$F(T) = F(T^*) + \langle DF(T^*),\ \chi_T - \chi_{T^*} \rangle + V_F(T, T^*),$$

where $V_F(T, T^*)$ is $o(d(T, T^*))$, that is, $\lim\limits_{d(T,T^*) \to 0} \dfrac{V_F(T, T^*)}{d(T, T^*)} = 0$.

A function $G : \mathcal{A}^n \to R$ is said to have a partial derivative at $T^* = (T_1^*, T_2^*, \ldots, T_n^*) \in \mathcal{A}^n$ with respect to its pth argument, if the function

$$F(T_p) = G(T_1^*, \ldots, T_{p-1}^*, T_p, T_{p+1}^*, \ldots, T_n^*)$$

has derivative $DF(T_p^*),\ p \in N$; in that case, the pth partial derivative of G at T^* is defined to be $D_pG(T^*) = DF(T_p^*),\ p \in N$.

A function $G : \mathcal{A}^n \to R$ is said to be differentiable at T^*, if all the partial derivatives $DG_p(T^*),\ p \in N$, exist and

$$G(T) = G(T^*) + \sum_{p \in N} \langle DG_p(T^*),\ \chi_{T_p} - \chi_{T_p^*} \rangle + W_G(T, T^*),$$

where $W_G(T, T^*)$ is $o(d(T, T^*))$ for all $T \in \mathcal{A}^n$.

It was shown in [8] that for any triplet $(T, Y, \lambda) \in \mathcal{A} \times \mathcal{A} \times [0, 1]$, there exist sequences $\{T_k\}$ and $\{Y_k\}$ in $\mathcal{A}$ such that

$$\chi_{T_k} \overset{w^*}{\to} \lambda \chi_{T \setminus Y} \ \textit{and}\ \chi_{Y_k} \overset{w^*}{\to} (1 - \lambda) \chi_{Y \setminus T} \tag{1}$$

imply

$$\chi_{T_k \cup Y_k \cup (T \cap Y)} \overset{w^*}{\to} \lambda \chi_T + (1 - \lambda) \chi_Y, \tag{2}$$

where $\overset{w^*}{\to}$ denotes weak* convergence of elements in $L_\infty(X, \mathcal{A}, \mu)$, and $T \setminus Y$ is the complement of T relative to Y. The sequence $\{V_k(\lambda)\} = \{T_k \cup Y_k \cup (T \cap Y)\}$ satisfying (1) and (2) is called the *Morris sequence* associated with (T, Y, λ).

Definition 1. A function $F : \mathcal{A}^n \to R$ is said to be (strictly) convex if for every $(T, Y, \lambda) \in \mathcal{A}^n \times \mathcal{A}^n \times [0, 1]$, there exists a Morris sequence $\{V_k(\lambda)\}$ in $\mathcal{A}^n$ such that

$$\limsup_{k \to \infty} F(V_k(\lambda))\ (<) \leq\ \lambda F(T) + (1 - \lambda) F(Y).$$

It was shown in [4, 8] that if a differentiable function $F : \mathcal{A}^n \to R$ is (strictly) convex, then

$$F(T)\ (>) \geq\ F(Y) + \sum_{p \in N} \langle D_p F(Y),\ \chi_{T_p} - \chi_{Y_p} \rangle\ \forall\ T, Y \in \mathcal{A}^n.$$

Definition 2. A function $\mathcal{F}(T, T^*; \cdot) : L_1^n(X, \mathcal{A}, \mu) \to R$ is said to be sublinear with respect to its third argument, if for fixed $T, T^* \in \mathcal{A}^n$, and for every $f, g \in L_1^n(X, \mathcal{A}, \mu)$ and $c \in R_+ \equiv [0, \infty)$,

$$\mathcal{F}(T, T^*; f + g) \leq \mathcal{F}(T, T^*; f) + \mathcal{F}(T, T^*; g)$$

and

$$\mathcal{F}(T, T^*; cf)\ =\ c\,\mathcal{F}(T, T^*; f).$$

Consider the following generalized subset programming problem:

(P1) Minimize $\max_{1 \leq i \leq k} F_i(T)$

subject to $T \in X_\circ$,

where $X_\circ = \{T \in \mathcal{A}^n \mid H_j(T) \leq 0,\ j \in M\}$ denote the set of all feasible solutions of (P).

The following definitions [7] are needed in the sequel:

Let $\mathcal{F}(T, T^*; \cdot) : L_1^n(X, \mathcal{A}, \mu) \to R$ be a sublinear functional, $\theta : \mathcal{A}^n \times \mathcal{A}^n \to \mathcal{A}^n \times \mathcal{A}^n$ be a function such that $T \neq T^* \Rightarrow \theta(T, T^*) \neq (0, 0)$ and let the functions $F : \mathcal{A}^n \to R^k$ and $H : \mathcal{A}^n \to R^m$ with components F_i, $i \in K$, and H_j, $j \in M$, respectively, be differentiable at $T^* \in \mathcal{A}^n$.

Definition 3. (F, H) is said to be $(\mathcal{F}, \alpha, \rho, \theta)$-V-type-I at $T^* \in \mathcal{A}^n$, if there exist vectors $\alpha = (\alpha_1^1, \alpha_2^1, \ldots, \alpha_k^1, \alpha_1^2, \alpha_2^2, \ldots, \alpha_m^2)$ and $\rho = (\rho_1^1, \rho_2^1, \ldots, \rho_k^1, \rho_1^2, \rho_2^2, \ldots, \rho_m^2) \in R^{k+m}$, where $\alpha_i^1, \alpha_j^2 : \mathcal{A}^n \times \mathcal{A}^n \to R_+ \setminus \{0\}$, and $\rho_i^1, \rho_j^2 \in R$ for $i \in K$, $j \in M$, such that for each $T \in X_\circ$, and for all $i \in K$, $j \in M$

$$F_i(T) - F_i(T^*) \geq \mathcal{F}(T, T^*; \alpha_i^1(T, T^*) DF_i(T^*)) + \rho_i^1 d^2(\theta(T, T^*)),$$
$$-H_j(T^*) \geq \mathcal{F}(T, T^*; \alpha_j^2(T, T^*) DH_j(T^*)) + \rho_j^2 d^2(\theta(T, T^*)).$$

Definition 4. (F, H) is said to be $(\mathcal{F}, \bar{\alpha}, \bar{\rho}, \theta)$-V-pseudoquasi type-I at $T^* \in \mathcal{A}^n$, if there exist vectors $\bar{\alpha} = (\bar{\alpha}_1^1, \bar{\alpha}_2^1, \ldots, \bar{\alpha}_k^1, \bar{\alpha}_1^2, \bar{\alpha}_2^2, \ldots, \bar{\alpha}_m^2)$ and $\bar{\rho} = (\bar{\rho}^1, \bar{\rho}^2) \in R^2$, where $\bar{\alpha}_i^1, \bar{\alpha}_j^2 : \mathcal{A}^n \times \mathcal{A}^n \to R_+ \setminus \{0\}$ for $i \in K$, $j \in M$, such that for each $T \in X_\circ$,

$$\mathcal{F}\left(T, T^*; \sum_{i \in K} DF_i(T^*)\right) \geq -\bar{\rho}^1 d^2(\theta(T, T^*)) \Rightarrow \sum_{i \in K} \bar{\alpha}_i^1 F_i(T) \geq \sum_{i \in K} \bar{\alpha}_i^1 F_i(T^*),$$
$$-\sum_{j \in M} \bar{\alpha}_j^2 H_j(T^*) \leq 0 \Rightarrow \mathcal{F}\left(T, T^*; \sum_{j \in M} DH_j(T^*)\right) \leq -\bar{\rho}^2 d^2(\theta(T, T^*)).$$

If in the above definition, the first inequality is satisfied as

$$\mathcal{F}\left(T, T^*; \sum_{i \in K} DF_i(T^*)\right) \geq -\bar{\rho}^1 d^2(\theta(T, T^*)) \Rightarrow \sum_{i \in K} \bar{\alpha}_i^1 F_i(T) > \sum_{i \in K} \bar{\alpha}_i^1 F_i(T^*),$$

then we say that (F, H) is $(\mathcal{F}, \bar{\alpha}, \bar{\rho}, \theta)$-V-strictly pseudoquasi type-I at T^*.

Definition 5. (F, H) is said to be $(\mathcal{F}, \tilde{\alpha}, \tilde{\rho}, \theta)$-V-quasi strictlypseudo type-I at $T^* \in \mathcal{A}^n$, if there exist vectors $\tilde{\alpha} = (\tilde{\alpha}_1^1, \tilde{\alpha}_2^1, \ldots, \tilde{\alpha}_k^1, \tilde{\alpha}_1^2, \tilde{\alpha}_2^2, \ldots, \tilde{\alpha}_m^2)$, and $\tilde{\rho} = (\tilde{\rho}^1, \tilde{\rho}^2) \in R^2$, where $\tilde{\alpha}_i^1, \tilde{\alpha}_j^2 : \mathcal{A}^n \times \mathcal{A}^n \to R_+ \setminus \{0\}$, for $i \in K$, $j \in M$, such that for each $T \in X_\circ$,

$$\sum_{i \in K} \tilde{\alpha}_i^1 F_i(T) \leq \sum_{i \in K} \tilde{\alpha}_i^1 F_i(T^*) \Rightarrow \mathcal{F}\left(T, T^*; \sum_{i \in K} DF_i(T^*)\right) \leq -\tilde{\rho}^1 d^2(\theta(T, T^*)),$$
$$\mathcal{F}\left(T, T^*; \sum_{j \in M} DH_j(T^*)\right) \geq -\tilde{\rho}^2 d^2(\theta(T, T^*)) \Rightarrow -\sum_{j \in M} \tilde{\alpha}_j^2 H_j(T^*) > 0.$$

If in the above definition, the first inequality is satisfied as

$$\sum_{i \in K} \tilde{\alpha}_i^1 F_i(T) < \sum_{i \in K} \tilde{\alpha}_i^1 F_i(T^*) \Rightarrow \mathcal{F}\left(T, T^*; \sum_{i \in K} DF_i(T^*)\right) \leq -\tilde{\rho}^1 d^2(\theta(T, T^*)),$$

then we say that (F, H) is $(\mathcal{F}, \tilde{\alpha}, \tilde{\rho}, \theta)$-V-prestrictquasi strictlypseudo type-I at T^*.

We next recall a set of necessary optimality conditions and other related results which form the basis for our discussion of sufficiency criteria for (P).

Theorem 2.1 [11]**.** *Assume that F_i, G_i, $i \in K$, and H_j, $j \in M$, are differentiable at $T^* \in \mathcal{A}^n$, and that for each $i \in K$, there exists $\hat{T}^i \in \mathcal{A}^n$ such that*

$$H_j(T^*) + \sum_{p\in N} \langle D_p H_j(T^*),\ \chi_{\hat{T}^i_p} - \chi_{T^*_p} \rangle < 0,\ j \in M. \tag{3}$$

If T^* is an optimal solution of (P), then there exist $\lambda^* \in R$, $\mu^* \in U = \left\{\mu \in R^k_+ : \sum_{i\in K} \mu_i = 1\right\}$, and $\nu^* \in R^m_+$ such that

$$\sum_{p\in N} \left\langle \sum_{i\in K} \mu^*_i [D_p F_i(T^*) - \lambda^* D_p G_i(T^*)] + \sum_{j\in M} \nu^*_j D_p H_j(T^*),\ \chi_{T_p} - \chi_{T^*_p} \right\rangle \geq 0,$$

$\forall\ T \in \mathcal{A}^n$,

$$\mu^*_i [F_i(T^*) - \lambda^* G_i(T^*)] = 0,\ i \in K,\ \ \nu^*_j H_j(T^*) = 0,\ j \in M.$$

For brevity, we shall henceforth refer to an $T^* \in X_\circ$ satisfying (3) as a regular feasible solution of (P).

It is easily seen that one obtains the following parameter-free version of Theorem 2.1 by eliminating the parameter λ^* and redefining the multipliers associated with the inequality constraints.

Theorem 2.2. *Assume that F_i, G_i, $i \in K$, and H_j, $j \in M$, are differentiable at $T^* \in \mathcal{A}^n$. If T^* is a regular optimal solution of (P), then there exist $\mu^* \in U$ and $\nu^* \in R^m_+$ such that*

$$\sum_{p\in N} \Bigg\langle \sum_{i\in K} \mu^*_i [\Gamma(T^*, \mu^*) D_p F_i(T^*) - \Theta(T^*, \mu^*) D_p G_i(T^*)] + \sum_{j\in M} \nu^*_j D_p H_j(T^*),$$

$$\chi_{T_p} - \chi_{T^*_p} \Bigg\rangle \geq 0,\ \forall\ T \in \mathcal{A}^n,$$

$$\mu^*_i [\Gamma(T^*, \mu^*) F_i(T^*) - \Theta(T^*, \mu^*) G_i(T^*)] = 0,\ i \in K,$$

$$\psi(T^*) \equiv \max_{1\leq i\leq k} \frac{F_i(T^*)}{G_i(T^*)} = \frac{\Theta(T^*, \mu^*)}{\Gamma(T^*, \mu^*)},\quad \nu^*_j H_j(T^*) = 0,\ j \in M,$$

where $\Theta(T^*, \mu^*) = \sum_{i\in K} \mu^*_i F_i(T^*)$ and $\Gamma(T^*, \mu^*) = \sum_{i\in K} \mu^*_i G_i(T^*)$.

Finally, we state following lemma that provides an alternative expression for the objective function of (P).

Lemma 2.1 [11]**.** *For each $T \in \mathcal{A}^n$, one has*

$$\max_{1\leq i\leq k} \frac{F_i(T)}{G_i(T)} = \max_{\mu\in U} \frac{\sum_{i\in K} \mu_i F_i(T)}{\sum_{i\in K} \mu_i G_i(T)}.$$

In overall treatment of sufficiency and duality theorems, it is assumed that the functions F_i, G_i, $i \in K$, and H_j, $j \in M$ are differentiable on $\mathcal{A}^n$.

3. Parametric sufficient optimality conditions

In this section, we establish the parametric sufficient optimality conditions for (P). For stating optimality Theorems 3.2 and 3.3, we use the real-valued functions $\mathcal{B}_i(\cdot, \lambda^*, \mu^*)$ and $\mathcal{C}_j(\cdot, \nu^*)$ defined, for fixed λ^*, μ^*, and ν^* on $\mathcal{A}^n$, and for all $i \in K$, $j \in M$ by

$$\mathcal{B}_i(T, \lambda^*, \mu^*) = \mu_i^*[F_i(T) - \lambda^* G_i(T)], \quad \text{and} \quad \mathcal{C}_j(T, \nu^*) = \nu_j^* H_j(T).$$

Theorem 3.1. *Let $T^* \in X_\circ$ and let there exist $\mu^* \in U$, $\nu^* \in R_+^m$ and $\lambda^* \in R_+$ such that*

$$\mathcal{F}\left(T, T^*; \sum_{i\in K} \mu_i^*[DF_i(T^*) - \lambda^* DG_i(T^*)] + \sum_{j\in M} \nu_j^* DH_j(T^*)\right) \geq 0, \ \forall\ T \in \mathcal{A}^n, \tag{1}$$

$$\mu_i^*[F_i(T^*) - \lambda^* G_i(T^*)] = 0, \ i \in K, \tag{2}$$

$$\nu_j^* H_j(T^*) = 0, \ j \in M. \tag{3}$$

If

(*i*) $[(F_1(\cdot) - \lambda^* G_1(\cdot), \ldots, F_k(\cdot) - \lambda^* G_k(\cdot)), (H_1(\cdot), \ldots, H_m(\cdot))]$ *is* $(\mathcal{F}, \alpha, \rho, \theta)$-*V-type-I at* T^*,

(*ii*) $\alpha_1^1 = \alpha_2^1 = \ldots = \alpha_k^1 = \alpha_1^2 = \alpha_2^2 = \ldots = \alpha_m^2 = \delta$, *and*

(*iii*) $\sum_{i\in K} \mu_i^* \rho_i^1 + \sum_{j\in M} \nu_j^* \rho_j^2 \geq 0$,

then T^* is an optimal solution of (P).

Proof. The inequality (1) along with the sublinearity of $\mathcal{F}$ implies

$$\mathcal{F}\left(T, T^*; \sum_{i\in K} \mu_i^*[DF_i(T^*) - \lambda^* DG_i(T^*)]\right) + \mathcal{F}\left(T, T^*; \sum_{j\in M} \nu_j^* DH_j(T^*)\right) \geq 0. \tag{4}$$

By hypothesis (i), we get

$$\begin{gathered}(F_i(T) - \lambda^* G_i(T)) - (F_i(T^*) - \lambda^* G_i(T^*)) \\ \geq \mathcal{F}(T, T^*; \alpha_i^1(T, T^*)[DF_i(T^*) - \lambda^* DG_i(T^*)]) + \rho_i^1 d^2(\theta(T, T^*)), i \in K, \\ -H_j(T^*) \geq \mathcal{F}(T, T^*; \alpha_j^2(T, T^*) DH_j(T^*)) + \rho_j^2 d^2(\theta(T, T^*)), j \in M.\end{gathered}$$

On multiplying the first inequality by $\mu^* \geq 0$, second by $\nu^* \geq 0$, and using (2), (3) and (ii), we obtain

$$\begin{gathered}\mu_i^*(F_i(T) - \lambda^* G_i(T)) \geq \mathcal{F}(T, T^*; \delta(T, T^*)\mu_i^*[DF_i(T^*) - \lambda^* DG_i(T^*)]) \\ +\mu_i^* \rho_i^1 d^2(\theta(T, T^*)), \ i \in K, \\ 0 \geq \mathcal{F}(T, T^*; \delta(T, T^*)\nu_j^* DH_j(T^*)) + \nu_j^* \rho_j^2 d^2(\theta(T, T^*)), \ j \in M,\end{gathered}$$

which on being summarized yield

$$\sum_{i\in K}\mu_i^*(F_i(T)-\lambda^* G_i(T)) \geq \mathcal{F}\left(T,T^*;\delta(T,T^*)\sum_{i\in K}\mu_i^*[DF_i(T^*)-\lambda^* DG_i(T^*)]\right)$$

$$+\mathcal{F}\left(T,T^*;\delta(T,T^*)\sum_{j\in M}\nu_j^* DH_j(T^*)\right)+\left(\sum_{i\in K}\mu_i^*\rho_i^1+\sum_{j\in M}\nu_j^*\rho_j^2\right)d^2(\theta(T,T^*)).$$

This inequality in view of (iii), (4), $\delta(T,T^*)>0$, and the sublinearity of $\mathcal{F}$ gives

$$\sum_{i\in K}\frac{\mu_i^*}{\delta(T,T^*)}(F_i(T)-\lambda^* G_i(T))\geq 0.$$

As $\delta(T,T^*)>0$, the above inequality reduces to

$$\sum_{i\in K}\mu_i^*(F_i(T)-\lambda^* G_i(T))\geq 0. \tag{5}$$

Now from Lemma 2.1, we have

$$\psi(T)\equiv\max_{1\leq i\leq k}\frac{F_i(T)}{G_i(T)}=\max_{\mu\in U}\frac{\sum_{i\in K}\mu_i F_i(T)}{\sum_{i\in K}\mu_i G_i(T)}\geq\frac{\sum_{i\in K}\mu_i^* F_i(T)}{\sum_{i\in K}\mu_i^* G_i(T)}\geq\lambda^*,\ (by\ (5)). \tag{6}$$

Hence, in view of (2) and (6), we conclude that T^* is an optimal solution of (P). □

Theorem 3.2. *Let $T^*\in X_\circ$ and let there exist $\mu^*\in U$, $\nu^*\in R_+^m$ and $\lambda^*\in R_+$ satisfying (1) to (3). If*

(i) $[(\mathcal{B}_1(\cdot,\lambda^*,\mu^*),\ldots,\mathcal{B}_k(\cdot,\lambda^*,\mu^*)),(\mathcal{C}_1(\cdot,\nu^*),\ldots,\mathcal{C}_m(\cdot,\nu^*))]$ is $(\mathcal{F},\bar{\alpha},\bar{\rho},\theta)$-V-pseudoquasi type-I at T^*, and

(ii) $\bar{\rho}^1+\bar{\rho}^2\geq 0$,

then T^* is an optimal solution of (P).

Proof. The inequality (3) and $\bar{\alpha}_j^2(T,T^*)>0, j\in M$ imply

$$\sum_{j\in M}\bar{\alpha}_j^2(T,T^*)\nu_j^* H_j(T^*)=0. \tag{7}$$

From (7) and hypothesis (i), we obtain

$$\mathcal{F}\left(T,T^*;\sum_{j\in M}\nu_j^* DH_j(T^*)\right)+\bar{\rho}^2 d^2(\theta(T,T^*))\leq 0. \tag{8}$$

The inequality (8) along with (4) and hypothesis (ii) yields

$$\mathcal{F}\left(T,T^*;\sum_{i\in K}\mu_i^*[DF_i(T^*)-\lambda^* DG_i(T^*)]\right)+\bar{\rho}^1 d^2(\theta(T,T^*))\geq 0,$$

which because of hypothesis (i) gives

$$\sum_{i\in K}\bar{\alpha}_i^1(T,T^*)\mu_i^*(F_i(T)-\lambda^*G_i(T))\geq\sum_{i\in K}\bar{\alpha}_i^1(T,T^*)\mu_i^*(F_i(T^*)-\lambda^*G_i(T^*)).$$

From the above inequality and (2), we have

$$\sum_{i\in K}\bar{\alpha}_i^1(T,T^*)\mu_i^*(F_i(T)-\lambda^*G_i(T))\geq 0. \tag{9}$$

By virtue of Lemma 2.1, we have

$$\begin{aligned}\psi(T)\equiv\max_{1\leq i\leq k}\frac{F_i(T)}{G_i(T)}&=\max_{1\leq i\leq k}\frac{\bar{\alpha}_i^1(T,T^*)F_i(T)}{\bar{\alpha}_i^1(T,T^*)G_i(T)}\ (as\ \bar{\alpha}_i^1(T,T^*)>0,\ i\in K)\\ &=\max_{\mu\in U}\frac{\sum_{i\in K}\mu_i\bar{\alpha}_i^1(T,T^*)F_i(T)}{\sum_{i\in K}\mu_i\bar{\alpha}_i^1(T,T^*)G_i(T)}\\ &\geq\frac{\sum_{i\in K}\mu_i^*\bar{\alpha}_i^1(T,T^*)F_i(T)}{\sum_{i\in K}\mu_i^*\bar{\alpha}_i^1(T,T^*)G_i(T)}\geq\lambda^*,\ (by\ (9)).\end{aligned} \tag{10}$$

Hence, in view of (2) and (10), we conclude that T^* is an optimal solution of (P). □

The proof of the next theorem is analogues to Theorem 3.2, and hence being omitted.

Theorem 3.3. *Let $T^*\in X_\circ$ and let there exist $\mu^*\in U$, $\nu^*\in R_+^m$ and $\lambda^*\in R_+$ satisfying (1) to (3). If*

(i) $[(\mathcal{B}_1(\cdot,\lambda^*,\mu^*),\ldots,\mathcal{B}_k(\cdot,\lambda^*,\mu^*)),(\mathcal{C}_1(\cdot,\nu^*),\ldots,\mathcal{C}_m(\cdot,\nu^*))]$ is $(\mathcal{F},\tilde{\alpha},\tilde{\rho},\theta)$-V-prestrict quasi strictlypseudo type-I at T^*, and

(ii) $\tilde{\rho}^1+\tilde{\rho}^2\geq 0$,

then T^* is an optimal solution of (P).

In order to prove next sufficient optimality theorems, we introduce some additional notations. Let $\{I_\circ,I_1,\ldots,I_r\}$ be a partition of the index set M, thus $I_\beta\subseteq M$ for each $\beta\in\{0,1,\ldots,r\}$, $I_\beta\cap I_\gamma=\emptyset$, if $\beta\neq\gamma$ and $\cup_{\beta=0}^r I_\beta=M$. For fixed λ^*, μ^* and ν^*, we define the real-valued functions on $\mathcal{A}^n$ as:

$$\mathcal{E}_i(T,\lambda^*,\mu^*,\nu^*)=\mu_i^*\left[F_i(T)-\lambda^*G_i(T)+\sum_{j\in I_\circ}\nu_j^*H_j(T)\right],\ i\in K,$$

and

$$\mathcal{L}_\beta(T,\nu^*)=\sum_{j\in I_\beta}\nu_j^*H_j(T),\ \beta=1,2,\ldots,r.$$

Theorem 3.4. *Let $T^*\in X_\circ$ and let there exist $\mu^*\in U$, $\nu^*\in R_+^m$ and $\lambda^*\in R_+$ satisfying (1) to (3). If*

(i) $[(\mathcal{E}_1(\cdot,\lambda^*,\mu^*,\nu^*),\ldots,\mathcal{E}_k(\cdot,\lambda^*,\mu^*,\nu^*)),(\mathcal{L}_1(\cdot,\nu^*),\ldots,\mathcal{L}_r(\cdot,\nu^*))]$ is $(\mathcal{F},\bar{\alpha},\bar{\rho},\theta)$-V-pseudoquasi type-I at T^*, and

(ii) $\bar{\rho}^1+\sum_{\beta=1}^{r}\bar{\rho}_\beta^2\geq 0$,

then T^* is an optimal solution of (P).

Proof. From (3) and $\bar{\alpha}_j^2(T,T^*) > 0, j \in M$, we have $\sum_{j\in I_\beta} \bar{\alpha}_j^2(T,T^*)\nu_j^* H_j(T^*) = 0$, $\beta = 1,2,\ldots,r$, which along with hypothesis (i) gives

$$\mathcal{F}\left(T,T^*;\sum_{j\in I_\beta}\nu_j^* DH_j(T^*)\right) \leq -\bar{\rho}_\beta^2\, d^2(\theta(T,T^*)),\ \beta = 1,2,\ldots,r. \tag{11}$$

By (1) and the sublinearity of $\mathcal{F}$, we obtain

$$\mathcal{F}\left(T,T^*;\sum_{i\in K}\mu_i^*[DF_i(T^*) - \lambda^* DG_i(T^*)] + \sum_{j\in I_\circ}\nu_j^* DH_j(T^*)\right)$$
$$+\sum_{\beta=1}^{r}\mathcal{F}\left(T,T^*;\sum_{j\in I_\beta}\nu_j^* DH_j(T^*)\right) \geq 0,$$
$$\mathcal{F}\left(T,T^*;\sum_{i\in K}\mu_i^*[DF_i(T^*) - \lambda^* DG_i(T^*)] + \sum_{j\in I_\circ}\nu_j^* DH_j(T^*)\right)$$
$$-\sum_{\beta=1}^{r}\bar{\rho}_\beta^2\, d^2(\theta(T,T^*)) \geq 0, (by\ (11)).$$

As hypothesis (ii) holds, and $\sum_{i\in K}\mu_i^* = 1$, the above inequality becomes

$$\mathcal{F}\left(T,T^*;\sum_{i\in K}\mu_i^*[DF_i(T^*) - \lambda^* DG_i(T^*) + \sum_{j\in I_\circ}\nu_j^* DH_j(T^*)]\right)$$
$$+\bar{\rho}^1 d^2(\theta(T,T^*)) \geq 0. \tag{12}$$

Inequality (12) together with hypothesis (i) implies

$$\sum_{i\in K}\bar{\alpha}_i^1(T,T^*)\mathcal{E}_i(T,\lambda^*,\mu^*,\nu^*) \geq \sum_{i\in K}\bar{\alpha}_i^1(T,T^*)\mathcal{E}_i(T^*,\lambda^*,\mu^*,\nu^*),$$

which in view of (2) and (3) yields

$$\sum_{i\in K}\bar{\alpha}_i^1(T,T^*)\mu_i^*\left[F_i(T) - \lambda^* G_i(T) + \sum_{j\in I_\circ}\nu_j^* H_j(T)\right] \geq 0.$$

Since $T \in X_\circ$ and $\nu^* \in R_+^m$, we get

$$\sum_{i\in K}\bar{\alpha}_i^1(T,T^*)\mu_i^*(F_i(T) - \lambda^* G_i(T)) \geq 0,$$

which is identical to (9). Therefore, following the proof of Theorem 3.2, we conclude that T^* is an optimal solution of (P). □

Theorem 3.5. *Let $T^* \in X_\circ$ and let there exist $\mu^* \in U$, $\nu^* \in R_+^m$ and $\lambda^* \in R_+$ satisfying (1) to (3). If*

(i) $[(\mathcal{E}_1(\cdot,\lambda^*,\mu^*,\nu^*),\ldots,\mathcal{E}_k(\cdot,\lambda^*,\mu^*,\nu^*)),(\mathcal{L}_1(\cdot,\nu^*)),\ldots,\mathcal{L}_r(\cdot,\nu^*))]$ *is* $(\mathcal{F},\tilde{\alpha},\tilde{\rho},\theta)$*-V-prestrictquasi strictlypseudo type-I at* T^**, and*

(ii) $\tilde{\rho}^1 + \sum_{\beta=1}^{r} \tilde{\rho}_\beta^2 \geq 0$,

then T^* is an optimal solution of (P).

Proof. The proof follows on the similar lines of Theorem 3.4. □

4. Parameter-free sufficient optimality conditions

In this section, we discuss parameter-free versions of the parametric sufficient optimality conditions for (P) obtained in Section 3. For stating optimality Theorems 4.2 - 4.5, we use the functions $\mathcal{C}_j(\cdot,\nu^*)$ and $\mathcal{L}_\beta(\cdot,\nu^*)$ defined in Section 3, and the real-valued functions $\Lambda_i(\cdot,T^*,\mu^*)$ and $\Pi_i(\cdot,T^*,\mu^*,\nu^*)$ defined, for fixed T^*, μ^*, and ν^* on $\mathcal{A}^n$, and for all $i \in K$, by

$$\Lambda_i(T,T^*,\mu^*) = \mu_i^*[\Gamma(T^*,\mu^*)F_i(T) - \Theta(T^*,\mu^*)G_i(T)],$$

and

$$\Pi_i(T,T^*,\mu^*,\nu^*) = \mu_i^*[\Gamma(T^*,\mu^*)F_i(T) - \Theta(T^*,\mu^*)G_i(T) + \sum_{j\in I_\circ} \nu_j^* H_j(T)].$$

Theorem 4.1. *Let $T^* \in X_\circ$ and let there exist $\mu^* \in U$ and $\nu^* \in R_+^m$ such that*

$$\mathcal{F}\left(T,T^*;\sum_{i\in K}\mu_i^*[\Gamma(T^*,\mu^*)DF_i(T^*) - \Theta(T^*,\mu^*)DG_i(T^*)] + \sum_{j\in M}\nu_j^* DH_j(T^*)\right)$$

$$\geq 0,\ \forall\ T \in \mathcal{A}^n, \tag{1}$$

$$\mu_i^*[\Gamma(T^*,\mu^*)F_i(T^*) - \Theta(T^*,\mu^*)G_i(T^*)] = 0,\ i \in K, \tag{2}$$

$$\psi(T^*) \equiv \max_{1\leq i\leq k}\frac{F_i(T^*)}{G_i(T^*)} = \frac{\Theta(T^*,\mu^*)}{\Gamma(T^*,\mu^*)}, \tag{3}$$

$$\nu_j^* H_j(T^*) = 0,\ j \in M. \tag{4}$$

If

(i) $[(\Gamma(T^*,\mu^*)F_1(\cdot) - \Theta(T^*,\mu^*)G_1(\cdot),\ldots,\Gamma(T^*,\mu^*)F_k(\cdot) - \Theta(T^*,\mu^*)G_k(\cdot)),(H_1(\cdot),\ldots,H_m(\cdot))]$ *is* $(\mathcal{F},\alpha,\rho,\theta)$*-V-type-I at* T^**,*

(ii) $\alpha_1^1 = \alpha_2^1 = \ldots = \alpha_k^1 = \alpha_1^2 = \alpha_2^2 = \ldots = \alpha_m^2 = \delta$*, and*

(iii) $\sum_{i\in K}\mu_i^*\rho_i^1 + \sum_{j\in M}\nu_j^*\rho_j^2 \geq 0$,

then T^* is an optimal solution of (P).

Proof. The inequality (1) along with the sublinearity of $\mathcal{F}$ implies

$$\mathcal{F}\left(T,T^*;\sum_{i\in K}\mu_i^*[\Gamma(T^*,\mu^*)DF_i(T^*) - \Theta(T^*,\mu^*)DG_i(T^*)]\right)$$

$$+\mathcal{F}\left(T, T^*; \sum_{j\in M} \nu_j^* DH_j(T^*)\right) \geq 0. \tag{5}$$

By hypothesis (i), we get

$$(\Gamma(T^*,\mu^*)F_i(T) - \Theta(T^*,\mu^*)G_i(T)) - (\Gamma(T^*,\mu^*)F_i(T^*) - \Theta(T^*,\mu^*)G_i(T^*))$$
$$\geq \mathcal{F}(T,T^*;\alpha_i^1(T,T^*)[\Gamma(T^*,\mu^*)DF_i(T^*) - \Theta(T^*,\mu^*)DG_i(T^*)]) + \rho_i^1 d^2(\theta(T,T^*)),$$
$$-H_j(T^*) \geq \mathcal{F}(T,T^*;\alpha_j^2(T,T^*)DH_j(T^*)) + \rho_j^2 d^2(\theta(T,T^*)), i\in K, j\in M.$$

On multiplying the first inequality by $\mu^* \geq 0$, second by $\nu^* \geq 0$, and using (2), (4) and (ii), we obtain

$$\mu_i^*(\Gamma(T^*,\mu^*)F_i(T) - \Theta(T^*,\mu^*)G_i(T)) \geq \mathcal{F}(T,T^*;\delta(T,T^*)\mu_i^*[\Gamma(T^*,\mu^*)DF_i(T^*)$$
$$-\Theta(T^*,\mu^*)DG_i(T^*)]) + \mu_i^*\rho_i^1 d^2(\theta(T,T^*)),\ i\in K,$$
$$0 \geq \mathcal{F}(T,T^*;\delta(T,T^*)\nu_j^* DH_j(T^*)) + \nu_j^*\rho_j^2 d^2(\theta(T,T^*)),\ j\in M.$$

Taking summation over $i \in K$ and $j \in M$, respectively and then adding to obtain

$$\sum_{i\in K}\mu_i^*(\Gamma(T^*,\mu^*)F_i(T) - \Theta(T^*,\mu^*)G_i(T))$$
$$\geq \mathcal{F}(T,T^*;\delta(T,T^*)\sum_{i\in K}\mu_i^*[\Gamma(T^*,\mu^*)DF_i(T^*) - \Theta(T^*,\mu^*)DG_i(T^*)])$$
$$+\mathcal{F}(T,T^*;\delta(T,T^*)\sum_{j\in M}\nu_j^* DH_j(T^*)) + (\sum_{i\in K}\mu_i^*\rho_i^1 + \sum_{j\in M}\nu_j^*\rho_j^2)d^2(\theta(T,T^*)),$$

which in view of (5), (iii), $\delta(T,T^*) > 0$ and the sublinearity of $\mathcal{F}$ gives

$$\sum_{i\in K}\frac{\mu_i^*}{\delta(T,T^*)}(\Gamma(T^*,\mu^*)F_i(T) - \Theta(T^*,\mu^*)G_i(T)) \geq 0.$$

Since $\delta(T,T^*) > 0$, the above inequality becomes

$$\sum_{i\in K}\mu_i^*(\Gamma(T^*,\mu^*)F_i(T) - \Theta(T^*,\mu^*)G_i(T)) \geq 0. \tag{6}$$

From Lemma 2.1, we get

$$\psi(T) \equiv \max_{1\leq i\leq k}\frac{F_i(T)}{G_i(T)} = \max_{\mu\in U}\frac{\sum_{i\in K}\mu_i F_i(T)}{\sum_{i\in K}\mu_i G_i(T)}$$
$$\geq \frac{\sum_{i\in K}\mu_i^* F_i(T)}{\sum_{i\in K}\mu_i^* G_i(T)} \geq \frac{\Theta(T^*,\mu^*)}{\Gamma(T^*,\mu^*)} = \psi(T^*),\ (by\ (6)\ and\ (3)). \tag{7}$$

Hence T^* is an optimal solution of (P). □

Theorem 4.2. *Let $T^* \in X_\circ$ and let there exist $\mu^* \in U$ and $\nu^* \in R_+^m$ satisfying (1) to (4). If*

(i) $[(\Lambda_1(\cdot,T^*,\mu^*),\dots,\Lambda_k(\cdot,T^*,\mu^*)),(\mathcal{C}_1(\cdot,\nu^*),\dots,\mathcal{C}_m(\cdot,\nu^*))]$ *is* $(\mathcal{F},\bar{\alpha},\bar{\rho},\theta)$-*V-pseudoquasi type-I at T^*, and*

(ii) $\bar{\rho}^1 + \bar{\rho}^2 \geq 0,$

then T^* is an optimal solution of (P).

Proof. Following the proof of Theorem 3.2, we get

$$\mathcal{F}\left(T, T^*; \sum_{j\in M} \nu_j^* DH_j(T^*)\right) + \bar{\rho}^2 d^2(\theta(T, T^*)) \leq 0,$$

which by the virtue of (5) and hypothesis (ii) implies

$$\mathcal{F}(T, T^*; \sum_{i\in K} \mu_i^*[\Gamma(T^*, \mu^*) DF_i(T^*) - \Theta(T^*, \mu^*) DG_i(T^*)]) + \bar{\rho}^1 d^2(\theta(T, T^*)) \geq 0.$$

This inequality together with hypothesis (i) gives

$$\sum_{i\in K} \bar{\alpha}_i^1(T, T^*)\mu_i^*(\Gamma(T^*, \mu^*)F_i(T) - \Theta(T^*, \mu^*)G_i(T))$$
$$\geq \sum_{i\in K} \bar{\alpha}_i^1(T, T^*)\mu_i^*(\Gamma(T^*, \mu^*)F_i(T^*) - \Theta(T^*, \mu^*)G_i(T^*)),$$

or

$$\sum_{i\in K} \bar{\alpha}_i^1(T, T^*)\mu_i^*(\Gamma(T^*, \mu^*)F_i(T) - \Theta(T^*, \mu^*)G_i(T)) \geq 0, \ (by\ (2)). \tag{8}$$

By Lemma 2.1, (3) and (8), it follows that

$$\psi(T) \equiv \max_{1\leq i\leq k} \frac{F_i(T)}{G_i(T)} = \max_{1\leq i\leq k} \frac{\bar{\alpha}_i^1(T, T^*)F_i(T)}{\bar{\alpha}_i^1(T, T^*)G_i(T)} \ (as\ \bar{\alpha}_i^1(T, T^*) > 0,\ i \in K)$$
$$= \max_{\mu\in U} \frac{\sum_{i\in K} \mu_i \bar{\alpha}_i^1(T, T^*)F_i(T)}{\sum_{i\in K} \mu_i \bar{\alpha}_i^1(T, T^*)G_i(T)}$$
$$\geq \frac{\sum_{i\in K} \mu_i^* \bar{\alpha}_i^1(T, T^*)F_i(T)}{\sum_{i\in K} \mu_i^* \bar{\alpha}_i^1(T, T^*)G_i(T)} \geq \frac{\Theta(T^*, \mu^*)}{\Gamma(T^*, \mu^*)} = \psi(T^*).$$

Hence $\psi(T) \geq \psi(T^*)$, which shows that T^* is an optimal solution of (P). □

Theorem 4.3. *Let $T^* \in X_\circ$ and let there exist $\mu^* \in U$ and $\nu^* \in R_+^m$ satisfying (1) to (4). If*

(i) $[(\Lambda_1(\cdot, T^*, \mu^*), \ldots, \Lambda_k(\cdot, T^*, \mu^*)), (\mathcal{C}_1(\cdot, \nu^*), \ldots, \mathcal{C}_m(\cdot, \nu^*))]$ *is* $(\mathcal{F}, \tilde{\alpha}, \tilde{\rho}, \theta)$*-V-prestrictquasi strictlypseudo type-I at T^*, and*

(ii) $\tilde{\rho}^1 + \tilde{\rho}^2 \geq 0$,

then T^* is an optimal solution of (P).

Proof. The proof follows on the similar lines of Theorem 4.2. □

Theorem 4.4. *Let $T^* \in X_\circ$ and let there exist $\mu^* \in U$ and $\nu^* \in R_+^m$ satisfying (1) to (4). If*

(i) $[(\Pi_1(\cdot, T^*, \mu^*, \nu^*), \ldots, \Pi_k(\cdot, T^*, \mu^*, \nu^*)), (\mathcal{L}_1(\cdot, \nu^*)), \ldots, \mathcal{L}_r(\cdot, \nu^*))]$ *is* $(\mathcal{F}, \bar{\alpha}, \bar{\rho}, \theta)$*-V-pseudoquasi type-I at T^*, and*

(ii) $\bar{\rho}^1 + \sum_{\beta=1}^{r} \bar{\rho}_\beta^2 \geq 0$,

then T^* is an optimal solution of (P).

Proof. Following Theorem 3.4, one can get the inequality

$$\mathcal{F}\left(T, T^*; \sum_{j \in I_\beta} \nu_j^* DH_j(T^*)\right) \leq -\bar{\rho}_\beta^2 \, d^2(\theta(T, T^*)), \ \beta = 1, 2, \ldots, r. \tag{9}$$

The inequality (1) and the sublinearity of $\mathcal{F}$ give

$$\mathcal{F}\left(T, T^*; \sum_{i \in K} \mu_i^* [\Gamma(T^*, \mu^*) DF_i(T^*) - \Theta(T^*, \mu^*) DG_i(T^*)] + \sum_{j \in I_\circ} \nu_j^* DH_j(T^*)\right)$$
$$+ \sum_{\beta=1}^{r} \mathcal{F}\left(T, T^*; \sum_{j \in I_\beta} \nu_j^* DH_j(T^*)\right) \geq 0,$$

or

$$\mathcal{F}\left(T, T^*; \sum_{i \in K} \mu_i^* [\Gamma(T^*, \mu^*) DF_i(T^*) - \Theta(T^*, \mu^*) DG_i(T^*)] + \sum_{j \in I_\circ} \nu_j^* DH_j(T^*)\right)$$
$$- \sum_{\beta=1}^{r} \bar{\rho}_\beta^2 \, d^2(\theta(T, T^*)) \geq 0, (by\ (9)),$$

which in view of hypothesis (ii), and $\sum_{i \in K} \mu_i^* = 1$ yields that

$$\mathcal{F}(T, T^*; \sum_{i \in K} \mu_i^* [\Gamma(T^*, \mu^*) DF_i(T^*) - \Theta(T^*, \mu^*) DG_i(T^*) + \sum_{j \in I_\circ} \nu_j^* DH_j(T^*)])$$
$$+ \bar{\rho}^1 d^2(\theta(T, T^*)) \geq 0. \tag{10}$$

The inequality (10) along with hypothesis (i) implies

$$\sum_{i \in K} \bar{\alpha}_i^1(T, T^*) \Pi_i(T, T^*, \mu^*, \nu^*) \geq \sum_{i \in K} \bar{\alpha}_i^1(T, T^*) \Pi_i(T^*, T^*, \mu^*, \nu^*),$$

which by (2) and (4) gives

$$\sum_{i \in K} \bar{\alpha}_i^1(T, T^*) \mu_i^* [\Gamma(T^*, \mu^*) F_i(T) - \Theta(T^*, \mu^*) G_i(T) + \sum_{j \in I_\circ} \nu_j^* H_j(T)]) \geq 0.$$

Since $T \in X_\circ$ and $\nu^* \in R_+^m$, we get

$$\sum_{i \in K} \bar{\alpha}_i^1(T, T^*) \mu_i^* (\Gamma(T^*, \mu^*) F_i(T) - \Theta(T^*, \mu^*) G_i(T)) \geq 0.$$

Now, following the proof of Theorem 4.2, we obtain the required condition that T^* is an optimal solution of (P). □

Theorem 4.5. *Let $T^* \in X_\circ$ and let there exist $\mu^* \in U$ and $\nu^* \in R_+^m$ satisfying (1) to (4). If*

(*i*) $[(\Pi_1(\cdot, T^*, \mu^*, \nu^*), \ldots, \Pi_k(\cdot, T^*, \mu^*, \nu^*)), (\mathcal{L}_1(\cdot, \nu^*)), \ldots, \mathcal{L}_r(\cdot, \nu^*))]$ *is* $(\mathcal{F}, \tilde{\alpha}, \tilde{\rho}, \theta)$*-V-prestrictquasi strictlypseudo type-I at T^*, and* (*ii*) $\tilde{\rho}^1 + \sum_{\beta=1}^{r} \tilde{\rho}_\beta^2 \geq 0,$

then T^* *is an optimal solution of (P).*

Proof. The proof is analogues to that of Theorem 4.4. □

5. Duality model I

In this section, duality theorems are proved for the following parametric dual problem:

(DI) Maximize λ

subject to

$$\mathcal{F}\left(T, Y; \sum_{i\in K} \mu_i[DF_i(Y) - \lambda DG_i(Y)] + \sum_{j\in M} \nu_j DH_j(Y)\right) \geq 0, \ \forall\, T \in \mathcal{A}^n, \quad (1)$$

$$\mu_i(F_i(Y) - \lambda G_i(Y)) \geq 0, \ i \in K, \quad (2)$$

$$\nu_j H_j(Y) \geq 0, \ j \in M, \quad (3)$$

$$Y \in \mathcal{A}^n, \ \lambda \in R_+, \ \mu \in U, \ \nu \in R_+^m,$$

In order to prove duality theorems, we use the functions $\mathcal{B}_i(\cdot, \lambda, \mu)$ and $\mathcal{C}_j(\cdot, \nu)$ introduced in Section 3.

Theorem 5.1 (Weak Duality)**.** *Let* T *and* (Y, λ, μ, ν) *be the feasible solutions of (P) and (DI), respectively. If*

(*i*) $[(\mathcal{B}_1(\cdot, \lambda, \mu), \ldots, \mathcal{B}_k(\cdot, \lambda, \mu)), (\mathcal{C}_1(\cdot, \nu), \ldots, \mathcal{C}_m(\cdot, \nu))]$ *is* $(\mathcal{F}, \bar{\alpha}, \bar{\rho}, \theta)$*-V-pseudo quasi type-I at* Y*, and*

(*ii*) $\bar{\rho}^1 + \bar{\rho}^2 \geq 0,$

then $\psi(T) \geq \lambda$.

Proof. From (3) and $\bar{\alpha}_j^2(T, Y) > 0, j \in M$, we have

$$-\sum_{j\in M} \bar{\alpha}_j^2(T, Y)\nu_j H_j(Y) \leq 0. \quad (4)$$

The inequality (4) and hypothesis (i) give

$$\mathcal{F}(T, Y; \sum_{j\in M} \nu_j DH_j(Y)) + \bar{\rho}^2 d^2(\theta(T, Y)) \leq 0. \quad (5)$$

The inequality (5) along with (1), the sublinearity of $\mathcal{F}$ and hypothesis (ii) implies

$$\mathcal{F}(T, Y; \sum_{i\in K} \mu_i[DF_i(Y) - \lambda DG_i(Y)]) + \bar{\rho}^1 d^2(\theta(T, Y)) \geq 0,$$

which in view of hypothesis (i) gives

$$\sum_{i\in K} \bar{\alpha}_i^1(T, Y)\mu_i(F_i(T) - \lambda G_i(T)) \geq \sum_{i\in K} \bar{\alpha}_i^1(T, Y)\mu_i(F_i(Y) - \lambda G_i(Y)).$$

From the above inequality and (2), we obtain

$$\sum_{i\in K} \bar{\alpha}_i^1(T,Y)\mu_i(F_i(T)-\lambda G_i(T)) \geq 0. \tag{6}$$

By virtue of Lemma 2.1, we have

$$\begin{aligned}\psi(T) \equiv \max_{1\leq i\leq k} \frac{F_i(T)}{G_i(T)} &= \max_{1\leq i\leq k} \frac{\bar{\alpha}_i^1(T,Y)F_i(T)}{\bar{\alpha}_i^1(T,Y)G_i(T)} \ (as\ \bar{\alpha}_i^1(T,Y)>0,\ i\in K)\\ &= \max_{a\in U} \frac{\sum_{i\in K} a_i\bar{\alpha}_i^1(T,Y)F_i(T)}{\sum_{i\in K} a_i\bar{\alpha}_i^1(T,Y)G_i(T)}\\ &\geq \frac{\sum_{i\in K} \mu_i\bar{\alpha}_i^1(T,Y)F_i(T)}{\sum_{i\in K} \mu_i\bar{\alpha}_i^1(T,Y)G_i(T)} \geq \lambda \ (by\ (6)).\end{aligned}$$

Hence $\psi(T) \geq \lambda$. □

Theorem 5.2 (Weak Duality)**.** *Let T and (Y,λ,μ,ν) be the feasible solutions of (P) and (DI), respectively. If*

(i) $[(\mathcal{B}_1(\cdot,\lambda,\mu),\ldots,\mathcal{B}_k(\cdot,\lambda,\mu)),(\mathcal{C}_1(\cdot,\nu),\ldots,\mathcal{C}_m(\cdot,\nu))]$ is $(\mathcal{F},\tilde{\alpha},\tilde{\rho},\theta)$ - V - prestrict quasistrictlypseudo type-I at Y, and

(ii) $\tilde{\rho}^1+\tilde{\rho}^2 \geq 0$,

then $\psi(T)\geq\lambda$.

Proof. The proof follows on the similar lines of Theorem 5.1. □

Theorem 5.3 (Strong Duality)**.** *Let T^* be a regular optimal solution of (P), let $\mathcal{F}(T,T^*;DF(T^*)) = \sum_{p\in N}\langle D_pF(T^*),\ \chi_{T_p}-\chi_{T_p^*}\rangle$ for any differentiable function $F:\mathcal{A}^n\to R$ and $T\in\mathcal{A}^n$, and assume that any of the weak duality theorems (Theorem 5.1 or 5.2) holds for all feasible solutions of (DI). Then there exist $\lambda^*\in R_+,\ \mu^*\in U$, and $\nu^*\in R_+^m$ such that $(T^*,\lambda^*,\mu^*,\nu^*)$ is an optimal solution of (DI) and the objective values of (P) and (DI) are equal.*

Proof. By Theorem 2.1, there exist $\lambda^*\in R_+,\ \mu^*\in U$, and $\nu^*\in R_+^m$ such that $(T^*,\lambda^*,\mu^*,\nu^*)$ is a feasible solution of (DI). Since $\psi(T^*)=\lambda^*$, it follows from the weak duality theorem (Theorem 5.1 or 5.2) that $(T^*,\lambda^*,\mu^*,\nu^*)$ is an optimal solution of (DI). □

Theorem 5.4 (Strict Converse Duality)**.** *Let T^* be a regular optimal solution of (P) and let $(Y^*,\lambda^*,\mu^*,\nu^*)$ be an optimal solution of (DI) such that*

(i) $[(\mathcal{B}_1(\cdot,\lambda^*,\mu^*),\ldots,\mathcal{B}_k(\cdot,\lambda^*,\mu^*)),(\mathcal{C}_1(\cdot,\nu^*),\ldots,\mathcal{C}_m(\cdot,\nu^*))]$ is $(\mathcal{F},\bar{\alpha},\bar{\rho},\theta)$-$V$-strictlypseudoquasi type-I at Y^*, and

(ii) $\bar{\rho}^1+\bar{\rho}^2\geq 0$.

Also, suppose that, for any differentiable function $F:\mathcal{A}^n\to R$, $\mathcal{F}(T,T^*;DF(T^*)) = \sum_{p\in N}\langle D_pF(T^*),\ \chi_{T_p}-\chi_{T_p^*}\rangle,\ T\in\mathcal{A}^n$. Then $Y^*=T^*$, that is, Y^* is an optimal solution of (P), and $\psi(T^*)=\lambda^*$.

Proof. We assume $Y^*\neq T^*$ and exhibit a contradiction. Now, following the proof of weak duality (Theorem 5.1), we get inequality (6) as strict inequality so that we obtain $\psi(T^*) > \lambda^*$, which contradicts the fact that $\psi(T^*) = \lambda^*$. Hence $Y^* = T^*$. □

6. Duality model II

In this section, we present more general parametric dual model by making use of partitioning scheme introduced in Section 3.

(DII) Maximize λ
subject to

$$\mathcal{F}\left(T, Y; \sum_{i\in K}\mu_i[DF_i(Y) - \lambda DG_i(Y)] + \sum_{j\in M}\nu_j DH_j(Y)\right) \geq 0,\ \forall\ T \in \mathcal{A}^n, \quad (1)$$

$$\mu_i(F_i(Y) - \lambda G_i(Y) + \sum_{j\in I_\circ}\nu_j H_j(Y)) \geq 0,\ i \in K, \quad (2)$$

$$\sum_{j\in I_\beta}\nu_j H_j(Y) \geq 0,\ \beta = 1, 2, \ldots, r, \quad (3)$$

$$Y \in \mathcal{A}^n,\ \lambda \in R_+,\ \mu \in U,\ \nu \in R^m_+.$$

We use the functions $\mathcal{E}_i(\cdot, \lambda, \mu, \nu)$ and $\mathcal{C}_j(\cdot, \nu)$ introduced in Section 3.

Theorem 6.1 (Weak Duality). *Let T and (Y, λ, μ, ν) be the feasible solutions of (P) and (DII), respectively. If*

(i) $[(\mathcal{E}_1(\cdot, \lambda, \mu, \nu), \ldots, \mathcal{E}_k(\cdot, \lambda, \mu, \nu)), (\mathcal{L}_1(\cdot, \nu), \ldots, \mathcal{L}_r(\cdot, \nu))]$ is $(\mathcal{F}, \bar{\alpha}, \bar{\rho}, \theta)$-V-pseudo quasi type-I at Y, and

(ii) $\bar{\rho}^1 + \sum_{\beta=1}^{r}\bar{\rho}^2_\beta \geq 0$,

then $\psi(T) \geq \lambda$.

Proof. From (3) and $\bar{\alpha}^2_j(T, Y) > 0, j \in M$, we have $-\sum_{j\in I_\beta}\bar{\alpha}^2_j(T, Y)\nu_j H_j(Y) \leq 0$, $\beta = 1, 2, \ldots, r$, which along with hypothesis (i) gives

$$\mathcal{F}(T, Y; \sum_{j\in I_\beta}\nu_j DH_j(Y)) \leq -\bar{\rho}^2_\beta\, d^2(\theta(T, Y)),\ \beta = 1, 2, \ldots, r. \quad (4)$$

By (1) and the sublinearity of $\mathcal{F}$, we obtain

$$\mathcal{F}(T, Y; \sum_{i\in K}\mu_i[DF_i(Y) - \lambda DG_i(Y)] + \sum_{j\in I_\circ}\nu_j DH_j(Y)) + \sum_{\beta=1}^{r}\mathcal{F}(T, Y; \sum_{j\in I_\beta}\nu_j DH_j(Y)) \geq 0,$$

or

$$\mathcal{F}(T, Y; \sum_{i\in K}\mu_i[DF_i(Y) - \lambda DG_i(Y)] + \sum_{j\in I_\circ}\nu_j DH_j(Y)) - \sum_{\beta=1}^{r}\bar{\rho}^2_\beta d^2(\theta(T, Y)) \geq 0, (by\ (4)).$$

As hypothesis (ii) holds, and $\sum_{i\in K}\mu_i = 1$, the above inequality becomes

$$\mathcal{F}(T,Y;\sum_{i\in K}\mu_i[DF_i(Y)-\lambda DG_i(Y)+\sum_{j\in I_\circ}\nu_j DH_j(Y)])+\bar{\rho}^1 d^2(\theta(T,Y))\geq 0. \quad (5)$$

The inequality (5) together with hypothesis (i) implies

$$\sum_{i\in K}\bar{\alpha}_i^1(T,Y)\mathcal{E}_i(T,\lambda,\mu,\nu)\geq \sum_{i\in K}\bar{\alpha}_i^1(T,Y)\mathcal{E}_i(Y,\lambda,\mu,\nu),$$

which in view of (2) and (3) yields

$$\sum_{i\in K}\bar{\alpha}_i^1(T,Y)\mu_i(F_i(T)-\lambda G_i(T)+\sum_{j\in I_\circ}\nu_j H_j(T))\geq 0.$$

Since $T\in X_\circ$ and $\nu\geq 0$, we get

$$\sum_{i\in K}\bar{\alpha}_i^1(T,Y)\mu_i(F_i(T)-\lambda G_i(T))\geq 0,$$

Now by Theorem 5.1, we obtain $\psi(T)\geq\lambda$. □

Theorem 6.2 (Strong Duality). *Let T^* be a regular optimal solution of (P), let $\mathcal{F}(T,T^*;DF(T^*))=\sum_{p\in N}\langle D_pF(T^*),\ \chi_{T_p}-\chi_{T_p^*}\rangle$ for any differentiable function $F:\mathcal{A}^n\to R$ and $T\in\mathcal{A}^n$, and assume that the assumptions of weak duality (Theorem 6.1) hold for all feasible solutions of (DII). Then there exist $\lambda^*\in R_+$, $\mu^*\in U$, and $\nu^*\in R_+^m$ such that $(T^*,\lambda^*,\mu^*,\nu^*)$ is an optimal solution of (DII) and the objective values of (P) and (DII) are equal.*

Proof. By Theorem 2.1, there exist $\lambda^*\in R_+$, $\mu^*\in U$, and $\nu^*\in R_+^m$ such that $(T^*,\lambda^*,\mu^*,\nu^*)$ is a feasible solution of (DII). Since $\psi(T^*)=\lambda^*$, it follows from the weak duality (Theorem 6.1) that $(T^*,\lambda^*,\mu^*,\nu^*)$ is an optimal solution of (DII). □

Theorem 6.3 (Strict Converse Duality). *Let T^* be a regular optimal solution of (P) and let $(Y^*,\lambda^*,\mu^*,\nu^*)$ be an optimal solution of (DII) such that*

(i) $[(\mathcal{E}_1(\cdot,\lambda^*,\mu^*,\nu^*),\ldots,\mathcal{E}_k(\cdot,\lambda^*,\mu^*,\nu^*)),(\mathcal{L}_1(\cdot,\nu^*),\ldots,\mathcal{L}_r(\cdot,\nu^*))]$ is $(\mathcal{F},\bar{\alpha},\bar{\rho},\theta)$-V-strictly pseudoquasi type-I at Y^*, and

(ii) $\bar{\rho}^1+\bar{\rho}^2\geq 0$.

Also, suppose that, for any differentiable function $F:\mathcal{A}^n\to R$, $\mathcal{F}(T,T^*;DF(T^*))=\sum_{p\in N}\langle D_pF(T^*),\ \chi_{T_p}-\chi_{T_p^*}\rangle$, $T\in\mathcal{A}^n$. Then $Y^*=T^*$, that is, Y^* is an optimal solution of (P), and $\psi(T^*)=\lambda^*$.

Proof. The proof follows on the similar lines of Theorem 5.4. □

7. Duality model III

This section deals with the following parameter-free dual model for (P) and corresponding weak, strong and strict converse duality theorems.

(DIII) Maximize $\phi(Y,\mu,\nu)=\dfrac{\sum_{i\in K}\mu_iF_i(Y)}{\sum_{i\in K}\mu_iG_i(Y)}$

subject to

$$\mathcal{F}\left(T,Y;\sum_{i\in K}\mu_i[\Gamma(Y,\mu)DF_i(Y)-\Theta(Y,\mu)DG_i(Y)]+\sum_{j\in M}\nu_jDH_j(Y)\right)$$

$$\geq 0,\ \forall\ T\in\mathcal{A}^n, \tag{1}$$

$$\mu_i[\Gamma(Y,\mu)F_i(Y)-\Theta(Y,\mu)G_i(Y)]\geq 0,\ i\in K, \tag{2}$$

$$\nu_jH_j(Y)\geq 0,\ j\in M. \tag{3}$$

$$Y\in\mathcal{A}^n,\ \mu\in U,\ \nu\in R^m_+.$$

We use the functions $\Lambda_i(\cdot,T,\mu)$ and $\mathcal{C}_j(\cdot,\nu)$ introduced in Section 4. Throughout this section, we assume that $\Theta(Y,\mu)\geq 0$ and $\Gamma(Y,\mu)>0$ for all Y and μ such that (Y,μ,ν) is a feasible solution of the considered dual problem.

Theorem 7.1 (Weak Duality). *Let T and (Y,μ,ν) be the feasible solutions of (P) and (DIII), respectively. If*

(i) $[(\Lambda_1(\cdot,T,\mu),\ldots,\Lambda_k(\cdot,T,\mu)),(\mathcal{C}_1(\cdot,\nu),\ldots,\mathcal{C}_m(\cdot,\nu))]$ is $(\mathcal{F},\bar{\alpha},\bar{\rho},\theta)$ - V - pseudo quasi type-I at Y, and (ii) $\bar{\rho}^1+\bar{\rho}^2\geq 0$, then $\psi(T)\geq\phi(Y,\mu,\nu)$.

Proof. By (3) and $\bar{\alpha}^2_j(T,Y)>0,\ j\in M$, we get $-\sum_{j\in M}\bar{\alpha}^2_j(T,Y)\nu_jH_j(Y)\leq 0$, which along with hypothesis (i) yields

$$\mathcal{F}(T,Y;\sum_{j\in M}\nu_jDH_j(Y))+\bar{\rho}^2d^2(\theta(T,Y))\leq 0. \tag{4}$$

The inequality (4) together with (1), sublinearity of $\mathcal{F}$, and hypothesis (ii) implies

$$\mathcal{F}(T,Y;\sum_{i\in K}\mu_i[\Gamma(Y,\mu)DF_i(Y)-\Theta(Y,\mu)DG_i(Y)])+\bar{\rho}^1d^2(\theta(T,Y))\geq 0,$$

which by virtue of hypothesis (i) gives

$$\sum_{i\in K}\bar{\alpha}^1_i(T,Y)\mu_i(\Gamma(Y,\mu)F_i(T)-\Theta(Y,\mu)G_i(T))$$

$$\geq\sum_{i\in K}\bar{\alpha}^1_i(T,Y)\mu_i(\Gamma(Y,\mu)F_i(Y)-\Theta(Y,\mu)G_i(Y)),$$

or

$$\sum_{i\in K}\bar{\alpha}^1_i(T,Y)\mu_i(\Gamma(Y,\mu)F_i(T)-\Theta(Y,\mu)G_i(T))\geq 0,\ (by\ (2)). \tag{5}$$

From Lemma 2.1, we have

$$\psi(T)\equiv\max_{1\leq i\leq k}\frac{F_i(T)}{G_i(T)}=\max_{1\leq i\leq k}\frac{\bar{\alpha}^1_i(T,Y)F_i(T)}{\bar{\alpha}^1_i(T,Y)G_i(T)}\ (as\ \bar{\alpha}^1_i(T,Y)>0,\ i\in K)$$

$$=\max_{a\in U}\frac{\sum_{i\in K}a_i\bar{\alpha}^1_i(T,Y)F_i(T)}{\sum_{i\in K}a_i\bar{\alpha}^1_i(T,Y)G_i(T)}$$

$$\geq \frac{\sum_{i\in K}\mu_i\bar{\alpha}_i^1(T,Y)F_i(T)}{\sum_{i\in K}\mu_i\bar{\alpha}_i^1(T,Y)G_i(T)} \geq \frac{\Theta(Y,\mu)}{\Gamma(Y,\mu)},\ (by\ (5)).$$

Hence $\psi(T) \geq \dfrac{\Theta(Y,\mu)}{\Gamma(Y,\mu)} = \phi(Y,\mu,\nu)$. □

Theorem 7.2 (Weak Duality)**.** *Let T and (Y,μ,ν) be the feasible solutions of (P) and (DIII), respectively. If*

(i) $[(\Lambda_1(\cdot,T,\mu),\ldots,\Lambda_k(\cdot,T,\mu)),(\mathcal{C}_1(\cdot,\nu),\ldots,\mathcal{C}_m(\cdot,\nu))]$ *is $(\mathcal{F},\tilde{\alpha},\tilde{\rho},\theta)$ - V - prestrict quasistrictlypseudo type-I at Y, and*

(ii) $\tilde{\rho}^1+\tilde{\rho}^2\geq 0$,

then $\psi(T)\geq\phi(Y,\mu,\nu)$.

Proof. The proof follows on the similar lines of Theorem 7.1. □

Theorem 7.3 (Strong Duality)**.** *Let T^* be a regular optimal solution of (P), let $\mathcal{F}(T,T^*;DF(T^*))=\sum_{p\in N}\langle D_pF(T^*),\ \chi_{T_p}-\chi_{T_p^*}\rangle$ for any differentiable function $F:\mathcal{A}^n\to R$ and $T\in\mathcal{A}^n$, and assume that the assumptions of any of the weak duality theorems (Theorem 7.1 or 7.2) hold for all feasible solutions of (DIII). Then there exist $\mu^*\in U$ and $\nu^*\in R_+^m$ such that (T^*,μ^*,ν^*) is an optimal solution of (DIII) and the objective values of (P) and (DIII) are equal.*

Proof. By Theorem 2.2, there exist $\mu^*\in U$ and $\nu^*\in R_+^m$ such that (T^*,μ^*,ν^*) is a feasible solution of (DIII). Since $\psi(T^*)=\phi(Y^*,\mu^*,\nu^*)$, it follows from the weak duality theorem (Theorem 7.1 or 7.2) that (T^*,μ^*,ν^*) is an optimal solution of (DIII). □

Theorem 7.4 (Strict Converse Duality)**.** *Let T^* be a regular optimal solution of (P) and (Y^*,μ^*,ν^*) be an optimal solution of (DIII) such that*

(i) $[(\Lambda_1(\cdot,T^*,\mu^*),\ldots,\Lambda_k(\cdot,T^*,\mu^*)),(\mathcal{C}_1(\cdot,\nu^*),\ldots,\mathcal{C}_m(\cdot,\nu^*))]$ *is $(\mathcal{F},\bar{\alpha},\bar{\rho},\theta)$ - V - strictlypseudo quasi type-I at Y^*, and*

(ii) $\bar{\rho}^1+\bar{\rho}^2\geq 0$.

Also, suppose that, for any differentiable function $F:\mathcal{A}^n\to R$, $\mathcal{F}(T,T^*;DF(T^*))=\sum_{p\in N}\langle D_pF(T^*),\ \chi_{T_p}-\chi_{T_p^*}\rangle,\ T\in\mathcal{A}^n$. Then $Y^*=T^*$, that is, Y^* is an optimal solution of (P), and $\psi(T^*)=\phi(Y^*,\mu^*,\nu^*)$.

Proof. The proof is similar to that of Theorem 5.4. □

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