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SUFFICIENCY AND DUALITY IN MULTIOBJECTIVE PROGRAMMING WITH GENERALIZED (F, ρ)-CONVEXITY

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Abstract. A multiobjective nonlinear programming problem is considered. Sufficiency theorems are derived for efficient and properly efficient solutions under generalized (F, ρ)-convexity assumptions. Weak, strong and strict converse duality theorems are established for a general Mond–Weir type dual relating properly efficient solutions of the primal and dual problems.

1. Introduction and preliminaries

The problem to be considered here is the following multiobjective nonlinear programming problem:

(VP) Minimize $f(x) = [f_1(x), f_2(x), \ldots, f_k(x)]$

subject to $x \in X = \{x \in S : g_i(x) \leqq 0, i \in M\}$,

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where S is a non-empty open convex subset of $\mathbb{R}^n$ and $f : S \to \mathbb{R}^k$ and $g : S \to \mathbb{R}^m$ are differentiable functions.

Mathematical programs involving several conflicting objectives have been the subject of extensive study in the recent literature. By defining a restricted form of efficiency, called proper efficiency. Geoffrion [2] established an equivalence between a convex multiobjetive nonlinear programming problem and a related parametric single objective program. Using parametric equivalence, Weir [9] formulated Wolfe and Mond–Weir type dual problems and established various duality results for properly efficient solution under the convexity and generalized convexity assumptions. The problems of [9] serve as the multiobjective version of the problems of Bector and Klassen [1], Mahajan and Vartak [5] and Mond and Weir [6].

Hanson and Mond [4] proved the Kuhn–Tucker sufficient optimality conditions and Wolfe duality theorems for a scalar nonlinear program under generalized F-convexity. Gulati and Islam [3] derived sufficiency theorems for efficient and properly efficient solutions of (VP) under the Hanson and Mond [4] assumptions. Preda [7] introduced the concept of generalized (F, ρ)-convexity, an extension of F-convexity defined by Hanson and Mond [4] and generalized ρ-convexity defined by Vial [8], and he used the concept to obtain duality results for efficient solutions.

In the present paper, we derive a fairly large number of sufficiency theorems for efficient and properly efficient solutions of (VP) under various generalized (F, ρ)-convexity assumptions. A generalized Mond–Weir type dual is also formulated for (VP) and duality relations are established for properly efficient solutions of the primal and the dual problems.

Throughout this paper, we use the following notations. The index sets $K = \{1, 2, \ldots, k\}$, $L = \{1, 2, \ldots, l\}$ and $M = \{1, 2, \ldots, m\}$. For $\bar{x} \in X$, the index sets $I = \{i \in M : g_i(\bar{x}) = 0\}$ and $J = \{i \in M : g_i(\bar{x}) < 0\} = M - I$. Let g_I denotes the vector of active constraints at $\bar{x}$. For $r \in K$, the set $K_r = K - \{r\}$. Lower case letters are used to denote vectors or vector functions. Subscripts denote components of vectors or vector functions and superscripts indicate the specific vectors. No notational distinction is made between row and column vectors. For a vector valued function $g : \mathbb{R}^n \to \mathbb{R}^m$, the symbol $\nabla g(\bar{x})$ denotes an $m \times n$ Jacobian matrix of g at $\bar{x}$. If x and $y \in \mathbb{R}^n$, then $x \geqq y \Leftrightarrow x_i \geqq y_i$, $i = 1, 2, \ldots, n$; $x \geq y \Leftrightarrow x \geqq y$ and $x \neq y$; $x > y \Leftrightarrow x_i > y_i$, $i = 1, 2, \ldots, n$.

The following definitions are from Geoffrion [2]

Definition 1.1. A point $\bar{x} \in X$ is said to be an efficient solution of (VP) if there exists no $x \in X$ such that $f(x) \leq f(\bar{x})$.

Definition 1.2. An efficient solution $\bar{x}$ of (VP) is said to be properly efficient if there exists a scalar $M > 0$ such that for each $r \in K$ and $x \in X$

satisfying $f_r(x) < f_r(\bar{x})$, we have

$$f_r(\bar{x}) - f_r(x) \leqq M\,[f_j(x) - f_j(\bar{x})]$$

for at least one j satisfying $f_j(\bar{x}) < f_j(x)$.

For readers convenience, we write the following definitions of the generalized (F, ρ) convexity from [7]:

Definition 1.3. A functional $F : S \times S \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is sublinear if for any $x, \bar{x} \in S$,

(i) $\quad F(x, \bar{x};\ a + b) \leqq F(x, \bar{x};\ a) + F(x, \bar{x};\ b) \quad$ for any $a, b \in \mathbb{R}^n$,

and

(ii) $\quad F(x, \bar{x};\ \alpha a) = \alpha F(x, \bar{x};\ a) \quad$ for any $\alpha \in \mathbb{R},\ \alpha \geq 0,$ and $a \in \mathbb{R}^n$.

From (ii) it follows that $F(x, \bar{x};\ 0) = 0$.

Let F be sublinear functional and the numerical function $\phi : S \to \mathbb{R}$ be differentiable at $\bar{x} \in X$ and $\rho \in \mathbb{R}$. Let $d(\cdot, \cdot) : S \times S \longrightarrow \mathbb{R}$.

Definition 1.4. The function ϕ is said to be (F, ρ)-convex at $\bar{x} \in S$, if for all $x \in S$,

$$\phi(x) - \phi(\bar{x}) \geqq F(x, \bar{x}; \nabla\phi(\bar{x})) + \rho d^2(x, \bar{x}).$$

Definition 1.5. The function ϕ is said to be strictly (F, ρ)-convex at $\bar{x} \in S$, if for all $x \in S$, $x \neq \bar{x}$,

$$\phi(x) - \phi(\bar{x}) > F(x, \bar{x}; \nabla\phi(\bar{x})) + \rho d^2(x, \bar{x}).$$

Definition 1.6. The function ϕ is said to be (F, ρ)-quasiconvex at $\bar{x} \in S$, if for all $x \in S$,

$$\phi(x) \leqq \phi(\bar{x}) \Rightarrow F(x, \bar{x}; \nabla\phi(\bar{x})) \leqq -\rho d^2(x, \bar{x}),$$

or equivalently,

$$F(x, \bar{x}; \nabla\phi(\bar{x})) > -\rho d^2(x, \bar{x}) \Rightarrow \phi(x) > \phi(\bar{x}).$$

Definition 1.7. The function ϕ is said to (F, ρ)-pseudoconvex at $\bar{x} \in S$, if for all $x \in S$,

$$F(x, \bar{x}; \nabla\phi(\bar{x})) \geqq -\rho d^2(x, \bar{x}) \Rightarrow \phi(x) \geqq \phi(\bar{x}).$$

Definition 1.8. The function ϕ is said to be strictly (F,ρ)-pseudoconvex at $\bar{x} \in S$, if for all $x \in S$, $x \neq \bar{x}$,

$$F(x,\bar{x};\nabla\phi(\bar{x})) \geqq -\rho d^2(x,\bar{x}) \Rightarrow \phi(x) > \phi(\bar{x}),$$

or equivalently,

$$\phi(x) \leqq \phi(\bar{x}) \Rightarrow F(x,\bar{x};\nabla\phi(\bar{x})) < -\rho d^2(x,\bar{x}).$$

A differentiable numerical function ϕ defined on a set $S \subseteq \mathbb{R}^n$, is said to be (F,ρ)-convex if ϕ is (F,ρ)-convex at every point of S. An m-dimensional vector function $g = (g_1, g_2, \cdots, g_m)$ is said to be (F,ρ)-convex if each g_i, $i = 1, 2, \ldots, m$ is (F,ρ_i)-convex for the same sublinear functional F. Other definitions follow similarly.

Note that, the above definitions are slightly different from those in [7] since we do not assume $d(\cdot,\cdot)$ to be a pseudometric.

2. Sufficiency

Theorem 2.1. *Let f_j, for all $j \in K$ be (F,ρ_j)-convex and let g_I be (F,σ_I)-quasiconvex at $\bar{x} \in X$. If there exist $\bar{u} \in \mathbb{R}^k$ and $\bar{v} \in \mathbb{R}^m$ satisfying*

$$\bar{u}\nabla f(\bar{x}) + \bar{v}\nabla g(\bar{x}) = 0, \tag{2.1}$$

$$\bar{v}g(\bar{x}) = 0, \tag{2.2}$$

$$\bar{u} \geqq 0,\ \bar{v} \geqq 0 \text{ and } (\bar{u}_j, \bar{v}_Q) \geq 0, \quad \text{for all } j \in K, \tag{2.3}$$

where $Q = \{i \in I : g_i$ is strictly (F,σ_i)-convex at $\bar{x}\}$, then $\bar{x}$ is an efficient solution of (VP) provided $\sum_{j\in K} \bar{u}_j\rho_j + \sum_{i\in M} \bar{v}_i\sigma_i \geqq 0$.

Proof. Let there exist $\bar{u} \in \mathbb{R}^k$ and $\bar{v} \in \mathbb{R}^m$ satisfying (2.1) to (2.3). Suppose to the contrary that $\bar{x}$ is not an efficient solution of the problem (VP). Then there exist an $x^0 \in X$ and $r \in K$ such that

$$f_r(x^0) < f_r(\bar{x})$$

and

$$f_j(x^0) \leqq f_j(\bar{x}) \quad \text{for all } j \in K_r.$$

Since for each $j \in K$, f_j is (F,ρ_j)-convex at $\bar{x}$,

$$F(x^0,\bar{x};\nabla f_r(\bar{x})) < -\rho_r d^2(x^0,\bar{x}) \tag{2.4}$$

and

$$F(x^0,\bar{x};\nabla f_j(\bar{x})) \leqq -\rho_j d^2(x^0,\bar{x}) \quad \text{for all } j \in K_r. \tag{2.5}$$

Let $Q' = I - Q = \{i : i \in I, i \notin Q\}$. Since $x^0 \in X$,

$$g_Q(x^0) \leqq 0 = g_Q(\bar{x}).$$

Using strict (F,σ_Q)-convexity of g_Q at $\bar{x}$, we get

$$F(x^0,\bar{x};\nabla g_Q(\bar{x})) < -\sigma_Q d^2(x^0,\bar{x}). \tag{2.6}$$

Similarly, the $(F,\sigma_{Q'})$-quasiconvexity of $g_{Q'}$ at $\bar{x}$ gives

$$F(x^0,\bar{x};\nabla g_{Q'}(\bar{x})) \leqq -\sigma_{Q'} d^2(x^0,\bar{x}). \tag{2.7}$$

Now relations (2.3) to (2.7) and sublinearity of F imply

$$\begin{aligned} F(x^0,\bar{x};\bar{u}\nabla f(\bar{x})+\bar{v}_I\nabla g_I(\bar{x})) &\leqq F(x^0,\bar{x};\bar{u}\nabla f(\bar{x}))+F(x^0,\bar{x};\bar{v}_Q\nabla g_Q(\bar{x})) \\ &\quad + F(x^0,\bar{x};\bar{v}_{Q'}\nabla g_{Q'}(\bar{x})) \\ &< -\Big(\sum_{j\in K}\bar{u}_j\rho_j+\bar{v}_Q\sigma_Q+\bar{v}_{Q'}\sigma_{Q'}\Big)d^2(x^0,\bar{x}) \\ &= -\Big(\sum_{j\in K}\bar{u}_j\rho_j+\sum_{i\in I}\bar{v}_i\sigma_i\Big)d^2(x^0,\bar{x}). \end{aligned}$$

Since $\bar{v} \geqq 0$, $g(\bar{x}) \leqq 0$ and $\bar{v}g(\bar{x}) = 0$ imply $\bar{v}_J = 0$, we obtain

$$F(x^0,\bar{x};\bar{u}\nabla f(\bar{x}) + \bar{v}\nabla g(\bar{x})) < -\Big(\sum_{j\in K}\bar{u}_j\rho_j+\sum_{i\in M}\bar{v}_i\sigma_i\Big)d^2(x^0,\bar{x}) \leqq 0.$$

Therefore,

$$\bar{u}\nabla f(\bar{x})+\bar{v}\nabla g(\bar{x}) \neq 0,$$

a contradiction to (2.1). Hence $\bar{x}$ is an efficient solution of (VP). □

Evidently, the above theorem has a number of important special cases which can readily be identified by the suitable algebraic properties of the (F,ρ)-convex functions. We shall state some of these as corollaries.

Corollary 2.1. *Let $\bar{u}_j f_j$, for all $j \in K$ be (F,ρ_j)-convex and let g_I be (F,σ_I)-quasiconvex at $\bar{x} \in X$. If there exist $\bar{u} \in \mathbb{R}^k$ and $\bar{v} \in \mathbb{R}^m$ satisfying (2.1) to (2.3), then $\bar{x}$ is an efficient solution of (VP) provided $\sum_{j\in K}\rho_j + \sum_{i\in M}\bar{v}_i\sigma_i \geqq 0$.*

Corollary 2.2. *Let $\bar{u}_j f_j$, for all $j \in K$ be (F,ρ_j)-convex and let $\bar{v}_I g_I$ be (F,σ_I)-quasiconvex at $\bar{x} \in X$. If there exist $\bar{u} \in \mathbb{R}^k$ and $\bar{v} \in \mathbb{R}^m$ satisfying (2.1) to (2.3) with*

$$Q = \{i \in I : \bar{v}_i g_i \textit{ is strictly } (F,\sigma_i)\textit{-convex at } \bar{x}\},$$

then $\bar{x}$ is an efficient solution of (VP) provided $\sum_{j\in K}\rho_j + \sum_{i\in M}\sigma_i \geqq 0$.

Theorem 2.2. *Let f_j, for all $j \in K$ be (F, ρ_j)-convex and let g_I be (F, σ_I)-quasiconvex at $\bar{x} \in X$. If there exist $\bar{u} \in \mathbb{R}^k$ and $\bar{v} \in \mathbb{R}^m$ satisfying* (2.1) *to* (2.3) *with*

$$Q = \{i \in I : g_i \text{ is strictly } (F, \sigma_i)\text{-pseudoconvex at } \bar{x}\},$$

then $\bar{x}$ is an efficient solution of (VP) provided $\sum\limits_{j \in K} \bar{u}_j \rho_j + \sum\limits_{i \in M} \bar{v}_i \sigma_i \geqq 0.$

Proof. Suppose to the contrary that $\bar{x}$ is not an efficient solution of (VP). Then as in the proof of Theorem 2.1,

$$F(x^0, \bar{x}; \nabla f_r(\bar{x})) < -\rho_r d^2(x^0, \bar{x}) \tag{2.8}$$

and

$$F(x^0, \bar{x}; \nabla f_j(\bar{x})) \leqq -\rho_j d^2(x^0, \bar{x}) \quad \text{for all } j \in K_r. \tag{2.9}$$

As $x^0 \in X, \quad g_Q(x^0) \leqq 0 = g_Q(\bar{x})$.

The strict (F, σ_Q)-pseudoconvexity of g_Q at $\bar{x}$ gives

$$F(x^0, \bar{x}; \nabla g_Q(\bar{x})) < -\sigma_Q d^2(x^0, \bar{x}). \tag{2.10}$$

Since $(\bar{u}_j, \bar{v}_Q) \geq 0$ for all $j \in K$ and F is sublinear, relations (2.8) to (2.10) imply that

$$\begin{aligned} &F(x^0, \bar{x}; \bar{u}\nabla f(\bar{x})) + F(x^0, \bar{x}; \bar{v}_Q \nabla g_Q(\bar{x})) \\ &< -(\sum_{j \in K} u_j \rho_j + \bar{v}_Q \sigma_Q) d^2(x^0, \bar{x}). \end{aligned} \tag{2.11}$$

Now the $(F, \sigma_{Q'})$-quasiconvexity of $g_{Q'}$ at $\bar{x}$ and sublinearity of F, we get

$$F(x^0, \bar{x}; \bar{v}_{Q'} \nabla g_{Q'}(\bar{x})) \leqq -\bar{v}_{Q'} \sigma_{Q'} d^2(x^0, \bar{x}). \tag{2.12}$$

where $Q' = I - Q = \{i : i \in I, i \notin Q\}$.

Relations (2.11) and (2.12), and sublinearity of F yield

$$F(x^0, \bar{x}; \bar{u}\nabla f(\bar{x}) \ + \ \bar{v}_I \nabla g_I(\bar{x})) < -(\sum_{j \in K} \bar{u}_j \rho_j + \sum_{i \in I} \bar{v}_i \sigma_i) d^2(x^0, \bar{x}).$$

Also, $\bar{v}_J = 0$, where $J = \{i : g_i(\bar{x}) < 0\}$. Therefore

$$F(x^0, \bar{x}; \bar{u}\nabla f(\bar{x}) \ + \ \bar{v} \nabla g(\bar{x})) < -(\sum_{j \in K} \bar{u}_j \rho_j + \sum_{i \in M} \bar{v}_i \sigma_i) d^2(x^0, \bar{x}).$$

Since $\sum_{j \in K} \bar{u}_j \rho_j + \sum_{i \in M} \bar{v}_i \sigma_i \geqq 0$, the above inequality implies

$$F(x^0, \bar{x}; \bar{u}\nabla f(\bar{x}) \ + \ \bar{v} \nabla g(\bar{x})) < 0.$$

Therefore,

$$\bar{u}\nabla f(\bar{x}) + \bar{v} \nabla g(\bar{x}) \neq 0,$$

a contradiction to (2.1). Hence $\bar{x}$ is an efficient solution of (VP). □

Now we state the sufficiency theorems as corollaries without proof for efficient solution of (VP).

Corollary 2.3. *Let $\bar{u}_j f_j$, for all $j \in K$ be (F, ρ_j)-convex and let g_I be (F, σ_I)-quasiconvex at $\bar{x} \in X$. If there exist $\bar{u} \in \mathbb{R}^k$ and $\bar{v} \in \mathbb{R}^m$ satisfying* (2.1) *to* (2.3), *then $\bar{x}$ is an efficient solution of (VP) provided $\sum_{j\in K} \rho_j + \sum_{i\in M} \bar{v}_i\sigma_i \geqq 0$.*

Corollary 2.4. *Let f_j, for all $j \in K$ be (F, ρ_j)-convex and let $\bar{v}_I g_I$ be (F, σ_I)-quasiconvex at $\bar{x} \in X$. If there exist $\bar{u} \in \mathbb{R}^k$ and $\bar{v} \in \mathbb{R}^m$ satisfying* (2.1) *to* (2.3) *with*

$$Q = \{i \in I : \bar{v}_i g_i \text{ is strictly } (F, \sigma_i)\text{-pseudoconvex at } \bar{x}\},$$

then $\bar{x}$ is an efficient solution of (VP) provided $\sum_{j\in K} u_j\rho_j + \sum_{i\in M} \sigma_i \geqq 0$.

In the above theorems we established the **efficiency** of $\bar{x}$ by exhibiting a contradiction. If Q is empty i.e., none of the components of g_I is strictly (F, σ)-convex (or strictly (F, σ)-pseudoconvex) at $\bar{x}$, then in (2.3) the vector $\bar{u} > 0$. We consider this case in the next theorem, which gives a stronger result.

Theorem 2.3. *Let $\bar{u}f$ be (F, ρ)-pseudoconvex and $\bar{v}_I g_I$ be (F, σ)-quasiconvex at $\bar{x} \in X$. If there exist $\bar{u} \in \mathbb{R}^k$ and $\bar{v} \in \mathbb{R}^m$ satisfying*

$$\bar{u}\nabla f(\bar{x}) + \bar{v}\nabla g(\bar{x}) = 0 \tag{2.13}$$

$$\bar{v}g(\bar{x}) = 0 \tag{2.14}$$

$$\bar{u} > 0,\ \bar{v} \geqq 0, \tag{2.15}$$

then $\bar{x}$ is a properly efficient solution of (VP) provided $\rho + \sigma \geqq 0$.

Proof. Let $J = \{i : g_i(\bar{x}) < 0\}$. Therefore $I \cup J = \{1, 2 \cdots, m\}$.
Also $\bar{v} \geqq 0$, $g(\bar{x}) \leqq 0$ and $\bar{v}g(\bar{x}) = 0 \Rightarrow \bar{v}_J = 0$. Now let $x \in X$. Then

$$\bar{v}_I g_I(x) \leqq 0 = \bar{v}_I g_I(\bar{x}).$$

The (F, σ)-quasiconvexity of $v_I g_I$ at $\bar{x}$ gives

$$F(x, \bar{x}; \bar{v}_I \nabla g_I(\bar{x})) \leqq -\sigma d^2(x, \bar{x}) \text{ for all } x \in X,$$

or

$$F(x, \bar{x}; \bar{v}\nabla g(\bar{x})) \leqq -\sigma d^2(x, \bar{x}) \text{ for all } x \in X \tag{2.16}$$

By the sublinearity of F,

$$\begin{aligned} F(x, \bar{x}; \bar{u}\nabla f(\bar{x})) + F(x, \bar{x}; \bar{v}\nabla g(\bar{x})) &\geqq F(x, \bar{x}; \bar{u}\nabla f(\bar{x})) + \bar{v}\nabla g(\bar{x})) \\ &= 0 \qquad \text{(using (2.13))}. \end{aligned}$$

That is,

$$F(x,\bar{x};\bar{u}\nabla f(\bar{x})) \geqq -F(x,\bar{x};\bar{v}\nabla g(\bar{x}))$$
$$\geqq \sigma d^2(x,\bar{x}) \qquad (\text{using}(2.16)). \tag{2.17}$$

Since $\rho+\sigma \geqq 0$ and from (2.17), we have

$$F(x,\bar{x};\bar{u}\nabla f(\bar{x})) \geqq -\rho d^2(x,\bar{x}).$$

Now the (F,ρ)-pseudoconvexity of $\bar{u}f$ at $\bar{x}$ gives

$$\bar{u}f(x) \geqq \bar{u}f(\bar{x}) \quad \text{for all } x \in X.$$

Hence by Theorem 1 in Geoffrion [2], $\bar{x}$ is a properly efficient solution of (VP). □

Theorem 2.4. *Let there exist $\bar{x} \in X$, $\bar{u} \in \mathbb{R}^k$ and $\bar{v} \in \mathbb{R}^m$ satisfying* (2.13) *to* (2.15). *If*

(*i*) *$\bar{u}f + \bar{v}_I g_I$ is (F,ρ)-pseudoconvex at $\bar{x}$ with $\rho \geqq 0$, or*
(*ii*) *f_j, for all $j \in K$ is (F,ρ_j)-convex and g_I is(F,σ)-quasiconvex at $\bar{x}$ with $\sum_{j\in K} \bar{u}_j\rho_j + \sigma \geqq 0$,*

then $\bar{x}$ is a properly efficient solution of (VP).

Proof. Let the assumption (i) hold. Since $\bar{v}_J = 0$ and F is a sublinear functional, for each $x \in X$, equation (2.13) gives

$$F(x,\bar{x};\bar{u}\nabla f(\bar{x}) + \bar{v}_I\nabla g_I(\bar{x})) = 0. \tag{2.18}$$

Since $\rho \geqq 0$, we have

$$F(x,\bar{x};\bar{u}\nabla f(\bar{x}) + \bar{v}_I\nabla g_I(\bar{x})) + \rho d^2(x,\bar{x}) \geqq 0.$$

By (F,ρ)-pseudoconvexity of $\bar{u}f + \bar{v}_I g_I$ at $\bar{x}$,

$$\bar{u}f(x) + \bar{v}_I g_I(x) \geqq \bar{u}f(\bar{x}) + \bar{v}_I g_I(\bar{x})$$

or

$$\bar{u}f(x) \geqq \bar{u}f(\bar{x}) - \bar{v}_I g_I(x).$$

Also $x \in X$ and $\bar{v}_I \geqq 0$ imply $\bar{v}_I g_I(x) \leqq 0$. Therefore

$$\bar{u}f(x) \geqq \bar{u}f(\bar{x}) \quad \text{for all } x \in X. \tag{2.19}$$

We now prove (2.19) under the assumption (ii). For $x \in X$,

$$g_I(x) \leqq 0 = g_I(\bar{x}).$$

The (F,σ)-quasiconvexity of g_I at $\bar{x}$ gives

$$F(x,\bar{x};\nabla g_I(\bar{x})) \leqq -\sigma_I d^2(x,\bar{x}) \quad \text{for all } x \in X.$$

Since $\bar{v}_I \geqq 0$, $\bar{v}_J = 0$ and sublinearity of F, we get

$$F(x,\bar{x};\bar{v}_I\nabla g_I(\bar{x})) \leqq -\sigma_I d^2(x,\bar{x}),$$

or

$$F(x,\bar{x};\bar{v}\nabla g(\bar{x})) \leqq -\sigma d^2(x,\bar{x}). \tag{2.20}$$

Relations (2.13), (2.20), $\sum_{j\in K} u_j\rho_j + \sigma \geqq 0$ and sublinearity of F imply

$$F(x,\bar{x};\bar{u}\nabla f(\bar{x})) \geqq -\sum_{j\in K} u_j\rho_j d^2(x,\bar{x}).$$

Since f_j, for all $j \in K$ is (F,ρ_j)-convex and $\bar{u} > 0$. Therefore

$$\bar{u}f(x) - \bar{u}f(\bar{x}) \geqq F(x,\bar{x};\bar{u}\nabla f(\bar{x})) + \sum_{j\in K} u_j\rho_j d^2(x,\bar{x}) \geqq 0,$$

or

$$\bar{u}f(x) \geqq \bar{u}f(\bar{x}) \quad \text{for all } x \in X.$$

Hence by Theorem 1 in Geoffrion [2], $\bar{x}$ is a properly efficient solution of (VP). □

Below we state a theorem without proof which includes all possible combinations of f and g following Mond and Weir [6]. Let $I_\alpha \subseteq M, \alpha = 0,1,2,\ldots,p$ with $I_\alpha \cap I_\beta = \phi$, $\alpha \neq \beta$ and $\bigcup_{\alpha=0}^{p} I_\alpha = M$.

Theorem 2.5. *Let there exist $\bar{x} \in X$, $\bar{u} \in \mathbb{R}^k$ and $\bar{v} \in \mathbb{R}^m$ satisfying* (2.13) *to* (2.15). *If $\bar{u}f + \sum_{i\in I_0} \bar{v}_i g_i$ is (F,ρ)-pseudoconvex and $\sum_{i\in I_\alpha} \bar{v}_i g_i$, $\alpha = 1,2,\ldots,p$ is (F,σ_α)-quasiconvex with $\rho + \sum_{i\in I_\alpha} \sigma_\alpha \geqq 0$, then $\bar{x}$ is a properly efficient solution of (VP).*

3. Generalized Mond–Weir type duality

We shall use the following result to establish duality results for properly efficient solution of (VP).

Theorem 3.1 ([2]). *Let $\bar{x}$ be a properly efficient solution of (VP) and let g satisfies the Kuhn–Tucker constraint qualification at $\bar{x} \in X$. Then there exist $\bar{u} \in \mathbb{R}^k$ and $\bar{v} \in \mathbb{R}^m$ such that*

$$\bar{u}\nabla f(\bar{x}) + \bar{v}\nabla g(\bar{x}) = 0,$$
$$\bar{v}g(\bar{x}) = 0,$$
$$\bar{u} > 0,\ \bar{u}e = 1,\ \bar{v} \geqq 0,$$

where e is a k-tuple of 1's.

We now prove weak, strong and converse duality theorems between the primal problem (VP) and its following general Mond–Weir [6] type dual problem:

(MD) Maximize $f(y) + \sum_{i \in I_0} v_i g_i(y) e$

subject to

$$u\nabla f(y) + v\nabla g(y) = 0, \tag{3.1}$$

$$\sum_{i \in I_\alpha} v_i g_i(y) \geqq 0, \quad \alpha = 1, 2 \cdots, p, \tag{3.2}$$

$$u > 0, \; ue = 1, \; v \geqq 0, \quad y \in S. \tag{3.3}$$

Let Z be the set of all feasible solutions of the dual problem (MD).

Theorem 3.2 (Weak Duality). *Let $x \in X$ and $(y, u, v) \in Z$. Let $uf + \sum_{i \in I_0} v_i g_i$ be (F, ρ)-pseudoconvex and $\sum_{i \in I_\alpha} v_i g_i$, $\alpha = 1, 2, \ldots, p$, be (F, σ_α)-quasiconvex at y over X with $\rho + \sum_{\alpha=1}^{p} \sigma_\alpha \geqq 0$. Then*

$$uf(x) \geqq uf(y) + \sum_{i \in I_0} v_i g_i(y),$$

and therefore

$$f(x) \nleq f(y) + \sum_{i \in I_0} v_i g_i(y) e.$$

Proof. Since $g(x) \leqq 0$ and $v \geqq 0$,

$$\sum_{i \in I_\alpha} v_i g_i(x) \leqq 0 \leqq \sum_{i \in I_\alpha} v_i g_i(y), \; \alpha = 1, 2, \ldots, p.$$

(F, σ)-quasiconvexity of $\sum_{i \in I_\alpha} v_i g_i$ at y implies

$$F(x, y; \sum_{i \in I_\alpha} v_i \nabla g_i(y)) \leqq -\sigma_\alpha d^2(x, y), \; \alpha = 1, 2, \ldots, p. \tag{3.4}$$

By (3.1) and sublinearity of F,

$$\begin{aligned} 0 &= F(x, y; u\nabla f(y) + v\nabla g(y)) \\ &\leqq F(x, y; u\nabla f(y) + \sum_{i \in I_0} v_i \nabla g_i(y)) + \sum_{\alpha=1}^{p} F(x, y; \sum_{i \in I_\alpha} v_i \nabla g_i(y)) \\ &\leqq F(x, y; u\nabla f(y) + \sum_{i \in I_0} v_i \nabla g_i(y)) - \sum_{\alpha=1}^{p} \sigma_\alpha d^2(x, y) \quad \text{(using (3.4))}. \end{aligned}$$

Since $\rho + \sum_{\alpha=1}^{p} \sigma_\alpha \geqq 0$, we have

$$F(x, y; u\nabla f(y) + \sum_{i \in I_0} v_i \nabla g_i(y)) \geqq -\rho d^2(x, y).$$

The (F,ρ)-pseudoconvexity of $uf+\sum_{i\in I_0} v_i g_i$ at y over X, implies

$$uf(x)+\sum_{i\in I_0} v_i g_i(x) \geqq uf(y)+\sum_{i\in I_0} v_i g_i(y).$$

Also $x\in X$ and $v\geqq 0$. The above inequality yields

$$uf(x)\geqq uf(y)+\sum_{i\in I_0} v_i g_i(y),$$

and therefore

$$f(x)\nleq f(y)+\sum_{i\in I_0} v_i g_i(y)e.$$

□

The assumption that $\sum_{i\in I_\alpha} v_i g_i$, $\alpha=1,2,\ldots,p$ is (F,σ_α)-quasiconvex is very important, as we see in the previous Theorem 3.2. Of course to get the desired result without this condition, other conditions should be enforced, which leads to the following theorem.

Theorem 3.3 (Weak Duality). *Let $x\in X$ and $(y,u,v)\in Z$. Let $uf+vg$ be (F,ρ)-pseudoconvex at y over X with $\rho\geqq 0$. Then*

$$uf(x)\geqq uf(y)+\sum_{i\in I_0} v_i g_i(y),$$

and therefore

$$f(x)\nleq f(y)+\sum_{i\in I_0} v_i g_i(y)e.$$

Proof. By using $F(x,y;0)=0$ in Definition 1.3 and the equality constraint (3.1) about gradients in (MD), we get

$$F(x,y;u\nabla f(y)+v\nabla g(y))=0. \tag{3.5}$$

Since $\rho\geqq 0$, we have

$$F(x,y;u\nabla f(y)+v\nabla g(y))\geqq -\rho d^2(x,y).$$

By the (F,ρ)-pseudoconvexity of $uf+vg$ at y over X,

$$uf(x)+vg(x)\geqq uf(y)+vg(y).$$

Using equations (3.2), (3.3) and feasibility of x for (VP), we get

$$uf(x)\geqq uf(y)+\sum_{i\in I_0} v_i g_i(y),$$

and therefore

$$f(x)\nleq f(y)+\sum_{i\in I_0} v_i g_i(y)e.$$

□

Theorem 3.4. *Let a weak duality hold between (VP) and (MD). If $\bar{x} \in X$ and $(\bar{y}, \bar{u}, \bar{v}) \in Z$ such that*

$$\bar{u}f(\bar{x}) = \bar{u}[f(\bar{y}) + \sum_{i \in I_0} \bar{v}_i g_i(\bar{y})e]. \tag{3.6}$$

Then $\bar{x}$ is a properly efficient solution of the problem (VP).

Proof. Let x be any feasible solution for (VP). From the weak duality theorem and equation (3.6),

$$\begin{aligned} \bar{u}f(x) &\geqq \bar{u}[f(\bar{y}) + \sum_{i \in I_0} \bar{v}_i g_i(\bar{y})e] \\ &= \bar{u}f(\bar{x}). \end{aligned}$$

Hence by Theorem 1 in Geoffrion [2], $\bar{x}$ is a properly efficient solution for (VP). □

Theorem 3.5. *Let a weak duality hold between (VP) and (MD). If $\bar{x} \in X$ and $(\bar{y}, \bar{u}, \bar{v}) \in Z$ such that*

$$f(\bar{x}) = [f(\bar{y}) + \sum_{i \in I_0} \bar{v}_i g_i(\bar{y})e]. \tag{3.7}$$

Then $\bar{x}$ and $(\bar{y}, \bar{u}, \bar{v})$ are properly efficient solutions for problems (VP) and (MD) respectively.

Proof. Proper efficiency of $\bar{x}$ follows from Theorem 3.4. We first prove that $(\bar{y}, \bar{u}, \bar{v})$ is an efficient solution of (MD). Suppose to the contrary that $(\bar{y}, \bar{u}, \bar{v})$ is not efficient for (MD), then there exists $(y^*, u^*, v^*) \in Z$ such that

$$f(y^*) + \sum_{i \in I_0} v_i^* g_i(y^*)e \geqq f(\bar{y}) + \sum_{i \in I_0} \bar{v}_i g_i(\bar{y})e.$$

Using (3.7), we obtain

$$f(y^*) + \sum_{i \in I_0} v_i^* g_i(y^*)e \geqq f(\bar{x}),$$

a contradiction to the weak duality theorems. Hence $(\bar{y}, \bar{u}, \bar{v})$ is an efficient solution of (MD). Assume now it is not a properly efficient solution of (MD). Then there exist (y^*, u^*, v^*) and a $j \in K$ such that

$$f_j(y^*) + \sum_{i \in I_0} v_i^* g_i(y^*) > f_j(\bar{y}) + \sum_{i \in I_0} \bar{v}_i g_i(\bar{y})$$

and

$$f_j(y^*) + \sum_{i\in I_0} v_i^* g_i(y^*) - f_j(\bar{y}) - \sum_{i\in I_0} \bar{v}_i g_i(\bar{y})$$
$$> M\left[f_r(\bar{y}) + \sum_{i\in I_0} \bar{v}_i g_i(\bar{y}) - f_r(y^*) - \sum_{i\in I_0} v_i^* g_i(y^*)\right]$$

for all $M > 0$ and for all $r \in K_j$ satisfying

$$f_r(\bar{y}) + \sum_{i\in I_0} \bar{v}_i g_i(\bar{y}) > f_r(y^*) + \sum_{i\in I_0} v_i^* g_i(y^*).$$

This means that

$$\lambda_j = f_j(y^*) + \sum_{i\in I_0} v_i^* g_i(y^*) - f_j(\bar{y}) - \sum_{i\in I_0} \bar{v}_i g_i(\bar{y})$$

can be made arbitrary large whereas

$$\lambda_r = f_r(\bar{y}) + \sum_{i\in I_0} \bar{v}_i g_i(\bar{y}) - f_r(y^*) - \sum_{i\in I_0} v_i^* g_i(y^*)$$

is finite for all $r \in K_j$. Therefore

$$u_j^*\lambda_j > \sum_{r\in K_j} u_r^*\lambda_r,$$

or

$$u^*\left[f(y^*) + \sum_{i\in I_0} v_i^* g_i(y^*)e\right] > u^*\left[f(\bar{y}) + \sum_{i\in I_0} \bar{v}_i g_i(\bar{y})e\right],$$

or using (3.7)

$$u^* f(y^*) + \sum_{i\in I_0} v_i^* g_i(y^*) > u^* f(\bar{x}).$$

Again a contradiction to the weak duality theorems. Hence $(\bar{y}, \bar{u}, \bar{v})$ is a properly efficient solution of (MD). □

Theorem 3.6 (Strong Duality). *Let $\bar{x} \in X$ be a properly efficient solution of the primal problem (VP) and let g satisfy the Kuhn–Tucker constraint qualification at $\bar{x}$. For for each $x \in X$ and $(y, u, v) \in Z$, let there exist a sublinear functional F such that $uf + \sum_{i\in I_0} v_i g_i$ is (F, ρ)-pseudoconvex and $\sum_{i\in I_\alpha} v_i g_i$, $\alpha = 1, 2, \dots, p$ is (F, σ_α)-quasiconvex at y over x with $\rho + \sum_{\alpha=1}^{p} \sigma_\alpha \geqq 0$. Then there exists $(\bar{u}, \bar{v})$ such that $(\bar{x}, \bar{u}, \bar{v})$ is a properly efficient solution of the dual problem (MD), and the corresponding objective values of (VP) and (MD) are equal.*

Proof. Since $\bar{x}$ is a properly efficient solution of (VP) and g satisfy the Kuhn–Tucker constraint qualification, by Theorem 3.1, there exist $\bar{u} \in \mathbb{R}^k$ and $\bar{v} \in \mathbb{R}^m$ such that

$$\begin{aligned} &\bar{u}\nabla f(\bar{x}) + \bar{v}\nabla g(\bar{x}) = 0 \\ &\bar{v}g(\bar{x}) = 0 \\ &\bar{u} > 0,\ \bar{v} \geqq 0,\ \bar{u}e = 1. \end{aligned}$$

Since $\bar{v}g(\bar{x}) = 0$, $g(\bar{x}) \leqq 0$ and $\bar{v} \geqq 0$, it follows that

$$\bar{v}_i g_i(\bar{x}) = 0 \quad \text{for all } i \in M,$$

and

$$\sum_{i\in I_\alpha} \bar{v}_i g_i(\bar{x}) = 0, \quad \alpha = 0, 1, 2, \dots, p.$$

Therefore $(\bar{x}, \bar{u}, \bar{v})$ is a feasible solution of (MD) and the objective values of (VP) and (MD) are equal. Hence by Theorem 3.5, $(\bar{x}, \bar{u}, \bar{v})$ is a properly efficient solution of the dual problem (MD). □

Theorem 3.7 (Strict Converse Duality). *Let $\bar{x} \in X$ and $(\bar{y}, \bar{u}, \bar{v}) \in Z$ be properly efficient solutions for problems (VP) and (MD) respectively such that*

$$\bar{u}f(\bar{x}) = \bar{u}f(\bar{y}) + \sum_{i\in I_0} \bar{v}_i g_i(\bar{y}). \tag{3.8}$$

If $\bar{u}f + \sum_{i\in I_0} \bar{v}_i g_i$ is strictly (F, ρ)-pseudoconvex and $\sum_{i\in I_\alpha} \bar{v}_i g_i$, $\alpha = 1, 2, \cdots, p$ is (F, σ_α)-quasiconvex at $\bar{y}$ with $\rho + \sum_{\alpha=1}^{p} \sigma_\alpha \geqq 0$, then $\bar{y} = \bar{x}$, that is, $\bar{y}$ is a properly efficient solution for (VP).

Proof. Suppose that $\bar{x} \neq \bar{y}$. Since $\bar{u}f + \sum_{i\in I_0} v_i g_i$ is strictly (F, ρ)-pseudoconvex and $\sum_{i\in I_\alpha} \bar{v}_i g_i$, $\alpha = 1, 2, \dots, p$ is (F, σ)-quasiconvex with $\rho + \sum_{\alpha=1}^{p} \sigma_\alpha \geqq 0$, following the proof of Theorem 3.2, we get

$$\bar{u}f(\bar{x}) > \bar{u}f(\bar{y}) + \sum_{i\in I_0} \bar{v}_i g_i(\bar{y}),$$

a contradiction to (3.8). Hence $\bar{x} = \bar{y}$. □

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