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Continuous Optimization

Symmetric duality for multiobjective fractional variational problems involving cones

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Abstract

In this paper, a pair of multiobjective fractional variational symmetric dual problems over cones is formulated. Weak, strong and converse duality theorems are established under generalized $\mathscr{F}$-convexity assumptions. Moreover, self duality theorem is also discussed.

Keywords: Multiobjective programming; Fractional variational programming; Symmetric duality; Generalized $\mathscr{F}$-convexity; Weakly efficient solutions

1. Introduction

The notion of symmetric duality in nonlinear programming, in which the dual of the dual is the primal, was first introduced by Dorn [13], but significantly developed and studied by Dantzig et al. [11], Mond and Weir [23], and Chandra et al. [7]. Bazaraa and Goode [4] generalized the results of Dantzig et al. [11] to arbitrary cones. Nanda and Das [24] studied symmetric duality in fractional programming involving arbitrary cones assuming the functions to be pseudoinvex. Chandra and Kumar [10] pointed out some logical shortcomings in the proofs of duality theorems of Nanda and Das [24]. Suneja et al. [26] formulated a pair of multiobjective symmetric dual programs over arbitrary cones and proved various duality results for cone–convex functions. Recently, Khurana [19] discussed multiobjective symmetric duality results for Mond–Weir type problems under generalized cone–invex functions.

Mond and Hanson [22] and Bector et al. [5] extended symmetric duality to variational programming, giving continuous analogous of the results of Dantzig et al. [11] and Mond and Weir [23], respectively. Smart and Mond [25] studied symmetric duality for variational problems with invexity, omitting the nonnegativity constraints taken by Mond and Hanson [22]. Gulati et al. [16] presented a pair of multiobjective symmetric dual

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variational problems and discussed duality results under generalized invexity. In [15], Gulati et al. generalized the results of Mond and Hanson [22] and Bector et al. [5] by constraining some of the primal and dual variables to belong to arbitrary sets of integers. Recently, Ahmad and Husain [3] formulated minimax mixed integer multiobjective symmetric dual variational programs over cones and obtained appropriate duality results.

Chandra and Husain [9] studied symmetric duality for fractional variational problems. In [17], Gulati et al. established usual duality results for static and continuous symmetric dual fractional programming problems without nonnegativity constraints. Recently, Kim et al. [20] and Ahmad [1] discussed symmetric duality results for multiobjective fractional variational programs under invexity and pseudoinvexity, respectively.

Generalizing convex functions, Hanson and Mond [18] introduced functions which satisfy certain convexity type properties with sublinear functionals. Egudo and Mond [14] named these functions as $\mathscr{F}$-convex, $\mathscr{F}$-pseudoconvex, and $\mathscr{F}$-quasiconvex functions. Examples of these functions have been given in [14,18] also. Later on, Chandra et al. [8] used these definitions in another form to discuss symmetric duality. Motivated by Hanson and Mond [18], Egudo and Mond [14], and Chandra et al. [8], we propose the continuous version of generalized $\mathscr{F}$-convexity, and use this concept to prove symmetric duality results for multiobjective fractional variational symmetric problems involving arbitrary cones. At the end, self duality theorem is also proved.

2. Notations and preliminaries

Let $I = [a, b]$ be a real interval, and $C_1 \subset R^n, C_2 \subset R^m$, be closed convex cones with nonempty interiors having polars C_1^* and C_2^*. Let for each $i \in K = \{1, 2, \ldots, k\}$, $f^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ and $g^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$, where $x : I \to R^n$ and $y : I \to R^m$, with derivatives $\dot{x}$ and $\dot{y}$, are twice continuously differentiable functions. Superscripts denote vector components; subscripts denote partial derivatives. The symbols $f_x^i, f_{\dot{x}}^i, f_y^i$ and $f_{\dot{y}}^i$ denote gradient vectors of the scalar function $f^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ with respect to $x, \dot{x}, y$ and $\dot{y}$ for $i \in K$. For instance,

$$f_x^i = \left(\frac{\partial f^i}{\partial x^1}, \ldots, \frac{\partial f^i}{\partial x^n}\right)^{\mathrm{T}}, \quad f_{\dot{x}}^i = \left(\frac{\partial f^i}{\partial \dot{x}^1}, \ldots, \frac{\partial f^i}{\partial \dot{x}^n}\right)^{\mathrm{T}}.$$

Similarly, $g_x^i, g_{\dot{x}}^i, g_y^i$ and $g_{\dot{y}}^i$ can be defined.

Let $S(I, R^n)$ denotes the space of piecewise smooth functions x with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by

$$u = Dx \Longleftrightarrow x(t) = \alpha + \int_0^t u(s)\mathrm{d}s,$$

where α is a given boundary value. Therefore, $\frac{\mathrm{d}}{\mathrm{d}t} \equiv D$ except at discontinuities. Denote by $Y(I, R^m)$, the space of piecewise smooth functions $y : I \to R^m$ with the norm as that of space $S(I, R^n)$.

Consider the following multiobjective variational problem:

$$\begin{aligned}
\text{(P) Minimize} \quad & \left(\int_a^b \phi^1(t, x(t), \dot{x}(t))\mathrm{d}t, \ldots, \int_a^b \phi^k(t, x(t), \dot{x}(t))\mathrm{d}t\right) \\
\text{subject to} \quad & x(a) = 0 = x(b), \\
& \dot{x}(a) = 0 = \dot{x}(b), \\
& h(t, x(t), \dot{x}(t)) \leqq 0, \quad t \in I,
\end{aligned}$$

where $\phi : I \times R^n \times R^n \to R^k$ and $h : I \times R^n \times R^n \to R^m$ are differentiable functions.

Let X denotes the set of all feasible solutions of (P), i.e.,

$$X = \{x \in S(I, R^n) | x(a) = 0 = x(b), \dot{x}(a) = 0 = \dot{x}(b), h(t, x(t), \dot{x}(t)) \leqq 0, \quad t \in I\}.$$

Defintion 1. A point $x^0 \in X$ is said to be an efficient solution of (P), if there exists no other $x \in X$ such that

$$\int_a^b \phi^i(t, x^0(t), \dot{x}^0(t))\mathrm{d}t > \int_a^b \phi^i(t, x(t), \dot{x}(t))\mathrm{d}t, \quad \text{for some} \quad i \in K,$$
$$\int_a^b \phi^j(t, x^0(t), \dot{x}^0(t))\mathrm{d}t \geqq \int_a^b \phi^j(t, x(t), \dot{x}(t))\mathrm{d}t, \quad \text{for all} \quad j \in K.$$

Defintion 2 [6]. A point $x^0 \in X$ is said to be a weakly efficient solution of (P), if there exists no other $x \in X$ such that

$$\int_a^b \phi^i(t, x^0(t), \dot{x}^0(t))\mathrm{d}t > \int_a^b \phi^i(t, x(t), \dot{x}(t))\mathrm{d}t, \quad \text{for all} \quad i \in K.$$

Defintion 3. A functional $\mathscr{F} : I \times R^n \times R^n \times R^n \times R^n \times R^n \to R$ is said to be sublinear in its sixth argument, if for any $x, \dot{x}, u, \dot{u} \in S(I, R^n)$,

(i) $\mathscr{F}(t, x, \dot{x}, u, \dot{u}; \alpha_1 + \alpha_2) \leqq \mathscr{F}(t, x, \dot{x}, u, \dot{u}; \alpha_1) + \mathscr{F}(t, x, \dot{x}, u, \dot{u}; \alpha_2)$, for any $\alpha_1, \alpha_2 \in R^n$; and
(ii) $\mathscr{F}(t, x, \dot{x}, u, \dot{u}; \beta\alpha) = \beta\mathscr{F}(t, x, \dot{x}, u, \dot{u}; \alpha)$, for any $\beta \in R, \beta \geqq 0$ and $\alpha \in R^n$.

Let $\psi : I \times R^n \times R^n \times R^m \times R^m \to R$ be a differentiable function.

Defintion 4. The functional $\int_a^b \psi(t, x(t), \dot{x}(t), y(t), \dot{y}(t))\mathrm{d}t$ is said to be $\mathscr{F}$-pseudoconvex in x and $\dot{x}$ for fixed y and $\dot{y}$, if

$$\int_a^b \mathscr{F}(t, x, \dot{x}, u, \dot{u}; \psi_x(t, u, \dot{u}, y, \dot{y}) - D\psi_{\dot{x}}(t, u, \dot{u}, y, \dot{y}))\mathrm{d}t \geqq 0 \Rightarrow \int_a^b \psi(t, x, \dot{x}, y, \dot{y})\mathrm{d}t \geqq \int_a^b \psi(t, u, \dot{u}, y, \dot{y})\mathrm{d}t,$$

for all $x, u : I \to R^n$ and for some arbitrary sublinear functional $\mathscr{F}$.

The functional $\int_a^b \psi(t, x(t), \dot{x}(t), y(t), \dot{y}(t))\mathrm{d}t$ is said to be strictly $\mathscr{F}$-pseudoconvex in x and $\dot{x}$ for fixed y and $\dot{y}$, if

$$\int_a^b \mathscr{F}(t, x, \dot{x}, u, \dot{u}; \psi_x(t, u, \dot{u}, y, \dot{y}) - D\psi_{\dot{x}}(t, u, \dot{u}, y, \dot{y}))\mathrm{d}t \geqq 0 \Rightarrow \int_a^b \psi(t, x, \dot{x}, y, \dot{y})\mathrm{d}t > \int_a^b \psi(t, u, \dot{u}, y, \dot{y})\mathrm{d}t,$$

for all $x, u : I \to R^n$ and for some arbitrary sublinear functional $\mathscr{F}$.

Defintion 5. The functional $\int_a^b \psi(t, x(t), \dot{x}(t), y(t), \dot{y}(t))\mathrm{d}t$ is said to be $\mathscr{F}$-pseudoconcave in y and $\dot{y}$ for fixed x and $\dot{x}$, if

$$\int_a^b \mathscr{F}(t, v, \dot{v}, y, \dot{y}; -\{\psi_y(t, x, \dot{x}, v, \dot{v}) - D\psi_{\dot{y}}(t, x, \dot{x}, v, \dot{v})\})\mathrm{d}t \geqq 0 \Rightarrow \int_a^b \psi(t, x, \dot{x}, v, \dot{v})\mathrm{d}t \leqq \int_a^b \psi(t, x, \dot{x}, y, \dot{y})\mathrm{d}t,$$

for all $y, v : I \to R^m$ and for some arbitrary sublinear functional $\mathscr{F}$.

Similarly, strict $\mathscr{F}$-pseudoconcavity of the functional $\int_a^b \psi(t, x(t), \dot{x}(t), y(t), \dot{y}(t))\mathrm{d}t$ can be defined.

Example. The function $\psi : I \times R \times R \times R \times R \to R$ defined by

$$\psi(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) = (x(t) + y^3(t))t$$

is $\mathscr{F}$-pseudoconvex in x and $\dot{x}$ for fixed y and $\dot{y}$ on $I = [a, b], a < b$, with respect to the sublinear functional

$$\mathscr{F}(t, x(t), \dot{x}(t), y(t), \dot{y}(t); z) = (x(t) - y(t))z.$$

The function ψ is $\mathscr{F}$-pseudoconcave also in y and $\dot{y}$ for fixed x and $\dot{x}$ on $I = [a, b], a < b$, with respect to the sublinear functional

$$\mathscr{F}(t,x(t),\dot{x}(t),y(t),\dot{y}(t);z)=\frac{1}{3}\left(x(t)-\frac{y^3(t)}{x^2(t)}\right)z.$$

Defintion 6. Let $x \in C \subset R^n$. Then C is a cone if and only if $\lambda x \in C$, for all $\lambda \geqq 0$. Moreover, C is called a convex cone if it is convex.

Defintion 7. Let $C \subset R^n$ be a cone. Then C^* is said to be a polar of C, if

$$C^* = \left\{p \in R^n | p^{\mathrm{T}}x \leqq 0, \quad \text{for all } x \in C\right\}.$$

In the sequel, we will write $\mathscr{F}(t,x,u;\xi)$ for $\mathscr{F}(t,x,\dot{x},u,\dot{u};\xi)$ and $\mathscr{F}(t,v,y;\eta)$ for $\mathscr{F}(t,v,\dot{v},y,\dot{y};\eta)$.

3. Symmetric duality

We present the following pair of multiobjective fractional variational symmetric dual programs:

$$\begin{aligned}
\text{(SP)}\quad \text{Minimize}\quad & \left[\frac{\int_a^b f^1(t,x,\dot{x},y,\dot{y})\mathrm{d}t}{\int_a^b g^1(t,x,\dot{x},y,\dot{y})\mathrm{d}t},\frac{\int_a^b f^2(t,x,\dot{x},y,\dot{y})\mathrm{d}t}{\int_a^b g^2(t,x,\dot{x},y,\dot{y})\mathrm{d}t},\ldots,\frac{\int_a^b f^k(t,x,\dot{x},y,\dot{y})\mathrm{d}t}{\int_a^b g^k(t,x,\dot{x},y,\dot{y})\mathrm{d}t}\right]\\
\text{subject to}\quad & x(a)=0=x(b),y(a)=0=y(b),\\
& \dot{x}(a)=0=\dot{x}(b),\dot{y}(a)=0=\dot{y}(b),\\
& \sum_{i=1}^k \lambda^i\left\{G^i(x,y)\left(f_y^i-Df_{\dot{y}}^i\right)-F^i(x,y)\left(g_y^i-Dg_{\dot{y}}^i\right)\right\}\in C_2^*,\quad t\in I,\\
& y(t)^{\mathrm{T}}\sum_{i=1}^k \lambda^i\{G^i(x,y)(f_y^i-Df_{\dot{y}}^i)-F^i(x,y)(g_y^i-Dg_{\dot{y}}^i)\}\geqq 0,\quad t\in I,\\
& \lambda>0,x(t)\in C_1,\quad t\in I.
\end{aligned}$$

$$\begin{aligned}
\text{(SD)}\quad \text{Maximize}\quad & \left[\frac{\int_a^b f^1(t,u,\dot{u},v,\dot{v})\mathrm{d}t}{\int_a^b g^1(t,u,\dot{u},v,\dot{v})\mathrm{d}t},\frac{\int_a^b f^2(t,u,\dot{u},v,\dot{v})\mathrm{d}t}{\int_a^b g^2(t,u,\dot{u},v,\dot{v})\mathrm{d}t},\ldots,\frac{\int_a^b f^k(t,u,\dot{u},v,\dot{v})\mathrm{d}t}{\int_a^b g^k(t,u,\dot{u},v,\dot{v})\mathrm{d}t}\right]\\
\text{subject to}\quad & u(a)=0=u(b),v(a)=0=v(b),\\
& \dot{u}(a)=0=\dot{u}(b),\dot{v}(a)=0=\dot{v}(b),\\
& -\sum_{i=1}^k \lambda^i\{G^i(u,v)(f_x^i-Df_{\dot{x}}^i)-F^i(u,v)(g_x^i-Dg_{\dot{x}}^i)\}\in C_1^*,\quad t\in I,\\
& u(t)^{\mathrm{T}}\sum_{i=1}^k \lambda^i\{G^i(u,v)(f_x^i-Df_{\dot{x}}^i)-F^i(u,v)(g_x^i-Dg_{\dot{x}}^i)\}\leqq 0,\quad t\in I,\\
& \lambda>0,v(t)\in C_2,\quad t\in I,
\end{aligned}$$

where, $f^i: I\times C_1\times C_1\times C_2\times C_2\to R_+$, and $g^i: I\times C_1\times C_1\times C_2\times C_2\to R_+\setminus\{0\},\quad i\in K$, are twice continuously differentiable functions, and

$$F^i(x,y)=\int_a^b f^i(t,x,\dot{x},y,\dot{y})\mathrm{d}t,\quad G^i(x,y)=\int_a^b g^i(t,x,\dot{x},y,\dot{y})\mathrm{d}t.$$

Remark. If $C_1=R_+^n, C_2=R_+^m$, and $k=1$, then the programs (SP) and (SD) reduce to those considered by Gulati et al. [17] with the omission of $x\geqq 0$ and $v\geqq 0$ in (SP) and (SD), respectively.

On using an abstract version of Dinkelbach's results [12], we define for each $i\in K$,

$$s^i = \frac{F^i(x,y)}{G^i(x,y)} = \frac{\int_a^b f^i(t,x,\dot{x},y,\dot{y})\mathrm{d}t}{\int_a^b g^i(t,x,\dot{x},y,\dot{y})\mathrm{d}t},$$

$$r^i = \frac{F^i(u,v)}{G^i(u,v)} = \frac{\int_a^b f^i(t,u,\dot{u},v,\dot{v})\mathrm{d}t}{\int_a^b g^i(t,u,\dot{u},v,\dot{v})\mathrm{d}t},$$

and express the problems (SP) and (SD) in the following equivalent forms:

(SP)′ Minimize $s = (s^1, s^2, \ldots, s^k)$

subject to

$$x(a) = 0 = x(b), \quad y(a) = 0 = y(b), \tag{1}$$

$$\dot{x}(a) = 0 = \dot{x}(b), \quad \dot{y}(a) = 0 = \dot{y}(b), \tag{2}$$

$$\int_a^b f^i(t,x,\dot{x},y,\dot{y})\mathrm{d}t - s^i \int_a^b g^i(t,x,\dot{x},y,\dot{y})\mathrm{d}t = 0, \quad i \in K, \tag{3}$$

$$\sum_{i=1}^k \lambda^i\{(f_y^i - Df_{\dot{y}}^i) - s^i(g_y^i - Dg_{\dot{y}}^i)\} \in C_2^*, \quad t \in I, \tag{4}$$

$$y(t)^{\mathrm{T}} \sum_{i=1}^k \lambda^i\{(f_y^i - Df_{\dot{y}}^i) - s^i(g_y^i - Dg_{\dot{y}}^i)\} \geqq 0, \quad t \in I, \tag{5}$$

$$\lambda > 0, x(t) \in C_1, \quad t \in I. \tag{6}$$

(SD)′ Maximize $r = (r^1, r^2, \ldots, r^k)$

subject to

$$u(a) = 0 = u(b), \quad v(a) = 0 = v(b), \tag{7}$$

$$\dot{u}(a) = 0 = \dot{u}(b), \quad \dot{v}(a) = 0 = \dot{v}(b), \tag{8}$$

$$\int_a^b f^i(t,u,\dot{u},v,\dot{v})\mathrm{d}t - r^i \int_a^b g^i(t,u,\dot{u},v,\dot{v})\mathrm{d}t = 0, \quad i \in K, \tag{9}$$

$$-\sum_{i=1}^k \lambda^i\{(f_x^i - Df_{\dot{x}}^i) - r^i(g_x^i - Dg_{\dot{x}}^i)\} \in C_1^*, \quad t \in I, \tag{10}$$

$$u(t)^{\mathrm{T}} \sum_{i=1}^k \lambda^i\{(f_x^i - Df_{\dot{x}}^i) - r^i(g_x^i - Dg_{\dot{x}}^i)\} \leqq 0, \quad t \in I, \tag{11}$$

$$\lambda > 0, v(t) \in C_2, \quad t \in I. \tag{12}$$

Let P and Q denote the sets of feasible solutions of (SP)′ and (SD)′, respectively.

In the subsequent analysis, weak, strong and converse duality theorems are discussed in terms of (SP)′ and (SD)′, but equally apply to (SP) and (SD). In the following theorem, it is assumed that:

$$\mathscr{F}(t,x,u;\xi) + u^{\mathrm{T}}\xi \geqq 0, \quad \text{for all } x,u \in C_1, -\xi \in C_1^*, \quad \text{and} \quad t \in I \tag{13}$$

and

$$\mathscr{F}(t,v,y;\eta) + y^{\mathrm{T}}\eta \geqq 0, \quad \text{for all } v,y \in C_2, -\eta \in C_2^*, \quad \text{and} \quad t \in I. \tag{14}$$

Theorem 1 (Weak duality). *Let* $(x,y,\lambda,s) \in P$ *and* $(u,v,\lambda,r) \in Q$. *If either*

(i) $\int_a^b \sum_{i=1}^k \lambda^i\{f^i(t,\cdot,\cdot,y,\dot{y}) - r^i g^i(t,\cdot,\cdot,y,\dot{y})\}\mathrm{d}t$ *is* $\mathscr{F}$*-pseudoconvex in* x *and* $\dot{x}$ *and* $\int_a^b \sum_{i=1}^k \lambda^i\{f^i(t,x,\dot{x},\cdot,\cdot) - s^i g^i(t,x,\dot{x},\cdot,\cdot)\}\mathrm{d}t$ *is strictly* $\mathscr{F}$*-pseudoconcave in* y *and* $\dot{y}$; *or*

(ii) $\int_a^b \sum_{i=1}^k \lambda^i\{f^i(t,\cdot,\cdot,y,\dot{y}) - r^i g^i(t,\cdot,\cdot,y,\dot{y})\}\mathrm{d}t$ *is strictly* $\mathscr{F}$*-pseudoconvex in* x *and* $\dot{x}$ *and* $\int_a^b \sum_{i=1}^k \lambda^i\{f^i(t,x,\dot{x},\cdot,\cdot) - s^i g^i(t,x,\dot{x},\cdot,\cdot)\}\mathrm{d}t$ *is* $\mathscr{F}$*-pseudoconcave in* y *and* $\dot{y}$,

holds, *then*

$$s \nleq r.$$

Proof. (i). By taking $\xi = \sum_{i=1}^{k}\lambda^i\{(f_x^i - Df_{\dot{x}}^i) - r^i(g_x^i - Dg_{\dot{x}}^i)\}$ and using (13), we get $\mathscr{F}(t,x,u;\sum_{i=1}^{k}\lambda^i\{(f_x^i - Df_{\dot{x}}^i) - r^i(g_x^i - Dg_{\dot{x}}^i)\}) \geqq -u^{\mathrm{T}}\sum_{i=1}^{k}\lambda^i\{(f_x^i - Df_{\dot{x}}^i) - r^i(g_x^i - Dg_{\dot{x}}^i)\} \geqq 0$, by (11), which implies

$$\int_a^b \mathscr{F}(t,x,u;\sum_{i=1}^{k}\lambda^i\{(f_x^i - Df_{\dot{x}}^i) - r^i(g_x^i - Dg_{\dot{x}}^i)\})\mathrm{d}t \geqq 0.$$

This, in view of $\mathscr{F}$-pseudoconvexity of $\int_a^b\sum_{i=1}^{k}\lambda^i\{f^i(t,\cdot,\cdot,y,\dot{y}) - r^ig^i(t,\cdot,\cdot,y,\dot{y})\}\mathrm{d}t$ in x and $\dot{x}$ gives

$$\int_a^b\sum_{i=1}^{k}\lambda^i\{f^i(t,x,\dot{x},v,\dot{v}) - r^ig^i(t,x,\dot{x},v,\dot{v})\}\mathrm{d}t \geqq \int_a^b\sum_{i=1}^{k}\lambda^i\{f^i(t,u,\dot{u},v,\dot{v}) - r^ig^i(t,u,\dot{u},v,\dot{v})\}\mathrm{d}t.$$

This, in view of (9) yields

$$\int_a^b\sum_{i=1}^{k}\lambda^i\{f^i(t,x,\dot{x},v,\dot{v}) - r^ig^i(t,x,\dot{x},v,\dot{v})\}\mathrm{d}t \geqq 0. \tag{15}$$

Taking $\eta = -\sum_{i=1}^{k}\lambda^i\{(f_y^i - Df_{\dot{y}}^i) - s^i(g_y^i - Dg_{\dot{y}}^i)\}$ and using (14), we obtain

$$\mathscr{F}(t,v,y;-\sum_{i=1}^{k}\lambda^i\{(f_y^i - Df_{\dot{y}}^i) - s^i(g_y^i - Dg_{\dot{y}}^i)\}) \geqq y^{\mathrm{T}}\sum_{i=1}^{k}\lambda^i\{(f_y^i - Df_{\dot{y}}^i) - s^i(g_y^i - Dg_{\dot{y}}^i)\} \geqq 0,\ (\text{by (5)}),$$

which shows that

$$\int_a^b \mathscr{F}(t,v,y;-\sum_{i=1}^{k}\lambda^i\{(f_y^i - Df_{\dot{y}}^i) - s^i(g_y^i - Dg_{\dot{y}}^i)\})\mathrm{d}t \geqq 0.$$

The strict $\mathscr{F}$-pseudoconcavity of $\int_a^b\sum_{i=1}^{k}\lambda^i\{f^i(t,x,\dot{x},\cdot,\cdot) - s^ig^i(t,x,\dot{x},\cdot,\cdot)\}\mathrm{d}t$ in y and $\dot{y}$ along with (3) yields

$$\int_a^b\sum_{i=1}^{k}\lambda^i\{f^i(t,x,\dot{x},v,\dot{v}) - s^ig^i(t,x,\dot{x},v,\dot{v})\}\mathrm{d}t < 0. \tag{16}$$

Combining (15) and (16), we get

$$\int_a^b\sum_{i=1}^{k}\lambda^i(s^i - r^i)g^i(t,x,\dot{x},v,\dot{v})\mathrm{d}t > 0. \tag{17}$$

Suppose, if possible, that $s \leqslant r$, i.e., $s^i < r^i$, for some $i \in K$ and $s^j \leqq r^j$, for all $j \neq i$, then from $\lambda > 0$ and $\int_a^b g^i(t,x,\dot{x},v,\dot{v})\mathrm{d}t > 0$, we have

$$\int_a^b\sum_{i=1}^{k}\lambda^i(s^i - r^i)g^i(t,x,\dot{x},v,\dot{v})\mathrm{d}t < 0,$$

which contradicts (17). Hence

$s \nleqslant r$.

(ii). The proof is same as that of part (i). □

Theorem 2 (Strong duality). *Let*

(a1): $(\bar{x},\bar{y},\bar{\lambda},\bar{s})$ *be a weakly efficient solution of* $(\mathrm{SP})'$,

(a2): $\left[\Phi(t)^{\mathrm{T}}\left\{\sum_{i=1}^{k}\bar{\lambda}^i\left(f_{yy}^i - \bar{s}^ig_{yy}^i\right) - D\sum_{i=1}^{k}\bar{\lambda}^i\left(f_{\dot{y}y}^i - \bar{s}^ig_{\dot{y}y}^i\right)\right\} + D\left\{\Phi(t)^{\mathrm{T}}D\sum_{i=1}^{k}\bar{\lambda}^i(f_{\dot{y}\dot{y}}^i - \bar{s}^ig_{\dot{y}\dot{y}}^i)\right\} + D^2\{-\Phi(t)^{\mathrm{T}}\sum_{i=1}^{k}\bar{\lambda}^i(f_{\dot{y}\dot{y}}^i - \bar{s}^ig_{\dot{y}\dot{y}}^i)\}\right]\Phi(t) = 0$ *implies* $\Phi(t) = 0$, *where* $\Phi(t) = (\gamma(t) - \xi(t)\bar{y}(t))$, $t \in I$; *and*

(a3): *the set* $\{((f_y^1 - \bar{s}^1g_y^1) - D(f_{\dot{y}}^1 - \bar{s}^1g_{\dot{y}}^1)),\ldots,((f_y^k - \bar{s}^kg_y^k) - D(f_{\dot{y}}^k - \bar{s}^kg_{\dot{y}}^k))\}$ *be linearly independent.*

Then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{s}) \in Q$ *with* $\lambda = \bar{\lambda}$; *and the objective values of* (SP)′ *and* (SD)′ *are equal. If, in addition, the assumptions of weak duality* (Theorem 1) *are satisfied, then* $(\bar{x}, \bar{y}, \bar{s})$ *is an efficient solution of* (SD)′ *with* $\lambda = \bar{\lambda}$.

Proof. Since $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{s})$ is a weakly efficient solution of (SP)′, by Fritz John optimality conditions [26], there exist $\alpha \in R^k$, $\beta \in R^k$, piecewise smooth $\gamma : I \to C_2$, $\xi : I \to R$ and $\delta \in R^k$ such that

$$\sum_{i=1}^{k} \beta^i \{(f_x^i - \bar{s}^i g_x^i) - D(f_{\dot{x}}^i - \bar{s}^i g_{\dot{x}}^i)\} + (\gamma(t) - \xi(t)\bar{y}(t))^{\mathrm{T}} \sum_{i=1}^{k} \bar{\lambda}^i \{(f_{yx}^i - \bar{s}^i g_{yx}^i) - D(f_{\dot{y}x}^i - \bar{s}^i g_{\dot{y}x}^i)\} - D(\gamma(t)$$
$$- \xi(t)\bar{y}(t))^{\mathrm{T}} \sum_{i=1}^{k} \bar{\lambda}^i \{(f_{y\dot{x}}^i - \bar{s}^i g_{y\dot{x}}^i) - D(f_{\dot{y}\dot{x}}^i - \bar{s}^i g_{\dot{y}\dot{x}}^i) - (f_{\dot{y}x}^i - \bar{s}^i g_{\dot{y}x}^i)\}$$
$$+ D^2\{-(\gamma(t) - \xi(t)\bar{y}(t))^{\mathrm{T}} \sum_{i=1}^{k} \bar{\lambda}^i (f_{\dot{y}\dot{x}}^i - \bar{s}^i g_{\dot{y}\dot{x}}^i)\}(x(t) - \bar{x}(t)) \geqq 0, \quad \forall x(t) \in C_1, \quad t \in I, \tag{18}$$

and the equalities (19)–(24) and (26)–(28) (with $\bar{q}^i = \bar{s}^i$) in [1] hold. Following the proof of Theorem 2 [1], we obtain

$$\gamma(t) - \xi(t)\bar{y}(t) = 0, \quad t \in I, \tag{19}$$

and

$$\beta = \xi(t)\bar{\lambda}, \quad t \in I. \tag{20}$$

For $\xi(t) = 0$, we reach a contradiction, and therefore $\xi(t) > 0$. So, we have

$$\bar{y}(t) = \frac{\gamma(t)}{\xi(t)} \in C_2, \quad t \in I. \tag{21}$$

Now, (18) along with (19) and (20), and with $\xi(t) > 0$, gives

$$\sum_{i=1}^{k} \bar{\lambda}^i \{(f_x^i - \bar{s}^i g_x^i) - D(f_{\dot{x}}^i - \bar{s}^i g_{\dot{x}}^i)\}(x(t) - \bar{x}(t)) \geqq 0, \quad t \in I. \tag{22}$$

Let $x(t) \in C_1$. Then $x(t) + \bar{x}(t) \in C_1$, $t \in I$ and so (22) shows that for every $x(t) \in C_1$,

$$\sum_{i=1}^{k} \bar{\lambda}^i \{(f_x^i - \bar{s}^i g_x^i) - D(f_{\dot{x}}^i - \bar{s}^i g_{\dot{x}}^i)\} x(t) \geqq 0, \quad t \in I,$$

i.e.,

$$-\sum_{i=1}^{k} \bar{\lambda}^i \{(f_x^i - \bar{s}^i g_x^i) - D(f_{\dot{x}}^i - \bar{s}^i g_{\dot{x}}^i)\} \in C_1^*, \quad t \in I. \tag{23}$$

Also, by letting $x(t) = 0$ and $x(t) = 2\bar{x}(t)$ simultaneously in (22), we obtain

$$\bar{x}(t)^{\mathrm{T}} \sum_{i=1}^{k} \bar{\lambda}^i \{(f_x^i - \bar{s}^i g_x^i) - D(f_{\dot{x}}^i - \bar{s}^i g_{\dot{x}}^i)\} = 0, \quad t \in I. \tag{24}$$

Thus, from (21), (23) and (24), it follows that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{s}) \in Q$ with $\lambda = \bar{\lambda}$, and the two objective functionals are equal (i.e., $\bar{s} = \bar{r}$).

If $(\bar{x}, \bar{y}, \bar{s})$ is not an efficient solution of (SD)′ with $\lambda = \bar{\lambda}$, then there exists (u^*, v^*, r^*) feasible of (SD)′ with $\lambda = \bar{\lambda}$, i.e., $(u^*, v^*, \bar{\lambda}, r^*) \in Q$ such that

$$\bar{s} \leqslant r^*,$$

which contradicts weak duality (Theorem 1). Thus $(\bar{x}, \bar{y}, \bar{s})$ is an efficient solution of (SD)′ with $\lambda = \bar{\lambda}$. □

The proof of the following theorem is analogous to Theorem 2, and hence being omitted.

Theorem 3 (Converse duality). *Let*

(a1): $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{r})$ *be a weakly efficient solution of* (SD)$'$,
(a2): $\Big[\Psi(t)^{\mathrm{T}}\Big\{\sum_{i=1}^{k}\bar{\lambda}^i\big(f^i_{xx}-\bar{r}^i g^i_{xx}\big)-D\sum_{i=1}^{k}\bar{\lambda}^i\big(f^i_{\dot{x}x}-\bar{r}^i g^i_{\dot{x}x}\big)\Big\}+D\Big\{\Psi(t)^{\mathrm{T}}D\sum_{i=1}^{k}\bar{\lambda}^i(f^i_{\dot{x}\dot{x}}-\bar{r}^i g^i_{\dot{x}\dot{x}})\Big\}+D^2\Big\{-\Psi(t)^{\mathrm{T}}\sum_{i=1}^{k}\bar{\lambda}^i (f^i_{\dot{x}\dot{x}}-\bar{r}^i g^i_{\dot{x}\dot{x}})\}\Big]\Psi(t)=0$ *implies* $\Psi(t)=0$, *where* $\Psi(t)=(\gamma(t)-\xi(t)\bar{u}(t))$, $t\in I$; *and*
(a3): *the set* $\{((f^1_x-\bar{r}^1g^1_x)-D(f^1_{\dot{x}}-\bar{r}^1g^1_{\dot{x}})),\ldots,((f^k_x-\bar{r}^kg^k_x)-D(f^k_{\dot{x}}-\bar{r}^kg^k_{\dot{x}}))\}$ *be linearly independent.*

Then $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{r}) \in P$ *with* $\lambda=\bar{\lambda}$; *and the objective values of* (SP)$'$*and* (SD)$'$*are equal. If, in addition, the assumptions of weak duality* (Theorem 1) *are satisfied, then* $(\bar{u}, \bar{v}, \bar{r})$ *is an efficient solution of* (SP)$'$ *with* $\lambda=\bar{\lambda}$.

4. Self duality

A mathematical programming problem is said to be self dual, if its dual can be written in the form of the primal.

The function $f^i(t,u,\dot{u},v,\dot{v}): I\times C\times C\times C\times C\to R_+, i\in K$, is said to be skew symmetric, if

$$f^i(t,u,\dot{u},v,\dot{v})=-f^i(t,v,\dot{v},u,\dot{u}), i\in K,\quad t\in I,$$

for all u and v in the domain of f^i, and the function $g^i(t,u,\dot{u},v,\dot{v}): I\times C\times C\times C\times C\to R_+\setminus\{0\}, i\in K$, is said to be symmetric, if

$$g^i(t,u,\dot{u},v,\dot{v})=g^i(t,v,\dot{v},u,\dot{u}),\quad i\in K,\quad t\in I,$$

for all u and v in the domain of g^i.

Consequently, it follows that

$$f^i_x(t,u,\dot{u},v,\dot{v})=-f^i_y(t,v,\dot{v},u,\dot{u}); f^i_{\dot{x}}(t,u,\dot{u},v,\dot{v})=-f^i_{\dot{y}}(t,v,\dot{v},u,\dot{u}),\quad i\in K,\quad t\in I,$$

and

$$g^i_x(t,u,\dot{u},v,\dot{v})=g^i_y(t,v,\dot{v},u,\dot{u}); g^i_{\dot{x}}(t,u,\dot{u},v,\dot{v})=g^i_{\dot{y}}(t,v,\dot{v},u,\dot{u}),\quad i\in K,\quad t\in I.$$

Now we assume $C_1=C_2=C; C_1^*=C_2^*=C^*$; f^i and $g^i, i\in K$, to be skew symmetric and symmetric, respectively, to show that (SP) and (SD) are self duals.

The problem (SD) may be recast as a minimization problem as:

$$\text{Minimize}\quad -\left[\frac{\int_a^b f^1(t,u,\dot{u},v,\dot{v})\mathrm{d}t}{\int_a^b g^1(t,u,\dot{u},v,\dot{v})\mathrm{d}t},\frac{\int_a^b f^2(t,u,\dot{u},v,\dot{v})\mathrm{d}t}{\int_a^b g^2(t,u,\dot{u},v,\dot{v})\mathrm{d}t},\ldots,\frac{\int_a^b f^k(t,u,\dot{u},v,\dot{v})\mathrm{d}t}{\int_a^b g^k(t,u,\dot{u},v,\dot{v})\mathrm{d}t}\right]$$

$$\text{subject to}\quad u(a)=0=u(b), v(a)=0=v(b),$$

$$\dot{u}(a)=0=\dot{u}(b), \dot{v}(a)=0=\dot{v}(b),$$

$$-\sum_{i=1}^{k}\lambda^i\{G^i(u,v)(f^i_x-Df^i_{\dot{x}})-F^i(u,v)(g^i_x-Dg^i_{\dot{x}})\}\in C^*,\quad t\in I,$$

$$u(t)^{\mathrm{T}}\sum_{i=1}^{k}\lambda^i\{G^i(u,v)(f^i_x-Df^i_{\dot{x}})-F^i(u,v)(g^i_x-Dg^i_{\dot{x}})\}\leqq 0,\quad t\in I,$$

$$\lambda>0,\quad v(t)\in C,\quad t\in I.$$

On using the skew symmetry and symmetry of f^i and g^i, respectively, for each $i\in K$, the above problem is transformed to

$$(\mathrm{SD})^* \quad \text{Minimize} \quad \left[\frac{\int_a^b f^1(t,v,\dot v,u,\dot u)\mathrm{d}t}{\int_a^b g^1(t,v,\dot v,u,\dot u)\mathrm{d}t}, \frac{\int_a^b f^2(t,v,\dot v,u,\dot u)\mathrm{d}t}{\int_a^b g^2(t,v,\dot v,u,\dot u)\mathrm{d}t}, \ldots, \frac{\int_a^b f^k(t,v,\dot v,u,\dot u)\mathrm{d}t}{\int_a^b g^k(t,v,\dot v,u,\dot u)\mathrm{d}t}\right]$$

$$\text{subject to} \quad u(a) = 0 = u(b), v(a) = 0 = v(b),$$

$$\dot u(a) = 0 = \dot u(b), \dot v(a) = 0 = \dot v(b),$$

$$\sum_{i=1}^{k} \lambda^i \{G^i(v,u)(f_y^i - Df_{\dot y}^i) - F^i(v,u)(g_y^i - Dg_{\dot y}^i)\} \in C^*, \quad t \in I,$$

$$u(t)^{\mathrm{T}} \sum_{i=1}^{k} \lambda^i \{G^i(v,u)(f_y^i - Df_{\dot y}^i) - F^i(v,u)(g_y^i - Dg_{\dot y}^i)\} \geqq 0, \quad t \in I,$$

$$\lambda > 0, \quad v(t) \in C, \quad t \in I,$$

which is formally identical to (SP), i.e., the objectives, the constraint functions and the initial conditions of (SP) and (SD)* are identical. Thus, (SP) is a self dual. It can easily be shown that the feasibility of (x, y, λ) for (SP) implies the feasibility of (y, x, λ) for (SD) and conversely.

Theorem 4 (Self duality). *Let $f^i(t,x,\dot x,y,\dot y)$ be skew symmetric and $g^i(t,x,\dot x,y,\dot y)$ be symmetric for each $i \in K$; let $C_1 = C_2 = C$ and $C_1^* = C_2^* = C^*$. Then* (SP) *is a self dual. Also, if* (SP) *and* (SD) *are dual variational problems, and (x^0, y^0, λ^0) is a joint weakly efficient solution, then so is (y^0, x^0, λ^0) and*

$$\frac{\int_a^b f^i(t,x^0,\dot x^0,y^0,\dot y^0)\mathrm{d}t}{\int_a^b g^i(t,x^0,\dot x^0,y^0,\dot y^0)\mathrm{d}t} = 0, \quad i \in K.$$

Proof. Since (x^0, y^0, λ^0) is a joint weakly efficient solution of (SP) and (SD), the objective functional values are equal to

$$\frac{\int_a^b f^i(t,x^0,\dot x^0,y^0,\dot y^0)\mathrm{d}t}{\int_a^b g^i(t,x^0,\dot x^0,y^0,\dot y^0)\mathrm{d}t}, \quad i \in K.$$

As (SP) is a self dual, it follows that (x^0, y^0, λ^0) is feasible for (SP) iff (y^0, x^0, λ^0) is feasible for (SD). Therefore, weak efficiency of (x^0, y^0, λ^0) for (SP) implies the weak efficiency of (y^0, x^0, λ^0) for (SD) and conversely. Also, the two objective values are equal to

$$\frac{\int_a^b f^i(t,y^0,\dot y^0,x^0,\dot x^0)\mathrm{d}t}{\int_a^b g^i(t,y^0,\dot y^0,x^0,\dot x^0)\mathrm{d}t}, \quad i \in K.$$

Thus, we have

$$\frac{\int_a^b f^i(t,x^0,\dot x^0,y^0,\dot y^0)\mathrm{d}t}{\int_a^b g^i(t,x^0,\dot x^0,y^0,\dot y^0)\mathrm{d}t} = \frac{\int_a^b f^i(t,y^0,\dot y^0,x^0,\dot x^0)\mathrm{d}t}{\int_a^b g^i(t,y^0,\dot y^0,x^0,\dot x^0)\mathrm{d}t}, \quad i \in K$$

$$= -\frac{\int_a^b f^i(t,x^0,\dot x^0,y^0,\dot y^0)\mathrm{d}t}{\int_a^b g^i(t,x^0,\dot x^0,y^0,\dot y^0)\mathrm{d}t}, \quad i \in K,$$

(by the skew symmetry of f^i and by the symmetry of g^i). Hence

$$\frac{\int_a^b f^i(t,x^0,\dot x^0,y^0,\dot y^0)\mathrm{d}t}{\int_a^b g^i(t,x^0,\dot x^0,y^0,\dot y^0)\mathrm{d}t} = 0, \quad i \in K. \qquad \square$$

5. Conclusion

We have presented multiobjective fractional variational symmetric dual programs over cones, and obtained symmetric duality results by assuming the functions involved to be $\mathscr{F}$-pseudoconvex/$\mathscr{F}$-pseudoconcave and strictly $\mathscr{F}$-pseudoconvex/strictly$\mathscr{F}$-pseudoconcave. Our results extend the results appeared in [1,20], and some other references cited therein. It is possible to extend these results to a more general class of functions, viz., $(\mathscr{F}, \alpha, \rho, d)$-convex functions [21], and generalized $(\mathscr{F}, \alpha, \rho, d)$-convex functions [2].

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References

[1] I. Ahmad, Symmetric duality for multiobjective fractional variational problems with generalized invexity, Information Sciences 176 (2006) 2192–2207.
[2] I. Ahmad, Z. Husain, Duality in nondifferentiable minimax fractional programming with generalized convexity, Applied Mathematics and Computation 176 (2006) 545–551.
[3] I. Ahmad, Z. Husain, Minimax mixed integer symmetric duality for multiobjective variational problems, European Journal of Operational Research 177 (2007) 71–82.
[4] M.S. Bazaraa, J.J. Goode, On symmetric duality in nonlinear programming, Operations Research 21 (1973) 1–9.
[5] C.R. Bector, S. Chandra, I. Husain, Generalized concavity and duality in continuous programming, Utilitas Mathematica 25 (1984) 171–190.
[6] J.M. Borwein, Optimization with Respect to Partial Orderings, D.Phil. Thesis, Oxford University 1974.
[7] S. Chandra, B.D. Craven, B. Mond, Symmetric dual fractional programming, Zeitschrift für Operations Research 29 (1985) 59–64.
[8] S. Chandra, A. Goyal, I. Husain, On symmetric duality in mathematical programming with $\mathscr{F}$-convexity, Optimization 43 (1998) 1–18.
[9] S. Chandra, I. Husain, Symmetric dual continuous fractional programming, Journal of Information and Optimization Sciences 10 (1989) 241–255.
[10] S. Chandra, V. Kumar, A note on pseudoinvexity and symmetric duality, European Journal of Operational Research 105 (1998) 626–629.
[11] G.B. Dantzig, E. Eisenberg, R.W. Cottle, Symmetric dual nonlinear programs, Pacific Journal of Mathematics 15 (1965) 809–812.
[12] W. Dinkelbach, On nonlinear fractional programming, Management Science 13 (1967) 492–498.
[13] W.S. Dorn, A symmetric dual theorem for quadratic programs, Journal of Operations Research Society of Japan 2 (1960) 93–97.
[14] R.R. Egudo, B. Mond, Duality with generalized convexity, Journal of the Australian Mathematical Society (Series B) 28 (1986) 10–21.
[15] T.R. Gulati, I. Ahmad, I. Husain, Symmetric duality for minimax variational problems, Mathematical Methods of Operations Research 48 (1998) 81–95.
[16] T.R. Gulati, I. Husain, A. Ahmed, Symmetric duality for multiobjective variational problems, Journal of Mathematical Analysis and Applications 210 (1997) 22–38.
[17] T.R. Gulati, I. Husain, A. Ahmed, Symmetric duality with invexity in static and continuous fractional programming, Optimization 40 (1997) 41–56.
[18] M.A. Hanson, B. Mond, Further generalizations of convexity in mathematical programming, Journal of Information and Optimization Sciences 3 (1982) 25–32.
[19] S. Khurana, Symmetric duality in multiobjective programming involving generalized cone–invex functions, European Journal of Operational Research 165 (2005) 592–597.
[20] D.S. Kim, W.J. Lee, S. Schaible, Symmetric duality for invex multiobjective fractional variational problems, Journal of Mathematical Analysis and Applications 289 (2004) 505–521.
[21] Z.A. Liang, H.X. Huang, P.M. Pardalos, Optimality conditions and duality for a class of nonlinear fractional programming problems, Journal of Optimization Theory and Applications 110 (2001) 611–619.
[22] B. Mond, M.A. Hanson, Symmetric duality for variational problems, Journal of Mathematical Analysis and Applications 23 (1968) 161–172.
[23] B. Mond, T. Weir, Generalized concavity and duality, in: S. Schaible, W.T. Ziemba (Eds.), Generalized Concavity in Optimization and Economics, Academic Press, New York, 1981, pp. 263–279.
[24] S. Nanda, L.N. Das, Pseudoinvexity and duality in nonlinear programming, European Journal of Operational Research 88 (1996) 572–577.
[25] I. Smart, B. Mond, Symmetric duality with invexity in variational problems, Journal of Mathematical Analysis and Applications 152 (1990) 536–545.
[26] S.K. Suneja, S. Aggarwal, S. Davar, Multiobjective symmetric duality involving cones, European Journal of Operational Research 141 (2002) 471–479.