

Continuous Optimization

Minimax mixed integer symmetric duality  
for multiobjective variational problems

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**Abstract**

A Mond–Weir type multiobjective variational mixed integer symmetric dual program over arbitrary cones is formulated. Applying the separability and generalized F-convexity on the functions involved, weak, strong and converse duality theorems are established. Self duality theorem is proved. A close relationship between these variational problems and static symmetric dual minimax mixed integer multiobjective programming problems is also presented.

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**1. Introduction**

The concept of symmetric duality was introduced and developed by Dorn [8] and Dantzig et al. [6]. Bazaraa and Goode [2] generalized the results in [6] to arbitrary cones. Mond and Weir [19] presented two pairs of symmetric duals multiobjective programming problems for efficient solutions and established appropriate duality results with the nonnegative orthant as the cone. Nanda and Das [20] presented the symmetric dual fractional programming problem for arbitrary cones assuming the functions to be pseudoinvex. Das and Nanda [7] studied symmetric duality in multiobjective programming with cone constraints. Subsequently, Kim et al. [14] derived symmetric duality results for multiobjective programs under pseudoinvex functions and arbitrary cones.

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Chandra and Abha [3] and Chandra and Kumar [4] pointed out some logical shortcomings in the formulations of the duals and the proofs of the duality theorems of Das and Nanda [7], Kim et al. [14] and Nanda and Das [20] respectively, and observed that these results are highly restricted as they are not valid even for convex case. Recently, Suneja et al. [21] formulated a pair of multiobjective symmetric dual programs over arbitrary cones involving cone-convex functions.

Balas [1] generalized the results of Dantzig et al. [6] by constraining some of the primal and dual variables to belong to arbitrary sets of integers and thus introduced minimax symmetric dual programs. Mishra and Das [17] generalized the results of Balas [1] to arbitrary cones. Later on, Mishra et al. [16] extended these results for convex cone domains involving pseudoconvexity/pseudoconcavity. Kumar et al. [15] formulated a modified pair of minimax symmetric dual programs on the lines of Mond and Weir [19]. Recently, Kim and Song [13] formulated two pairs of multiobjective mixed integer symmetric dual programs for arbitrary cones and established duality results.

Mond and Hanson [18] extended symmetric duality to variational problems, giving continuous analogues of the results of [6]. Kim and Lee [12] presented a pair of multiobjective symmetric dual variational problems and discussed duality results for efficient solutions assuming invexity. In [11], Gulati et al. constructed a different pair of multiobjective symmetric dual variational programs in which duality results are obtained for properly efficient solutions under pseudoconvexity/pseudoconcavity assumptions.

Motivated from the work of Balas [1], Gulati et al. [10] established symmetric duality results for Wolfe and Mond–Weir type single objective minimax mixed integer symmetric variational programs. In [5], Chen extended Wolfe type minimax mixed integer symmetric variational programs in [10] to multiobjective case over arbitrary cones and proved appropriate duality theorems in order to relate these programs.

The purpose of this paper is to study Mond–Weir type minimax mixed integer symmetric dual programs for multiobjective variational programming problems involving arbitrary cones and to establish weak, strong, converse and self duality theorems under F-pseudoconvexity/F-pseudoconcavity assumptions on the functions involved.

## 2. Notations and preliminaries

Let  $I = [a, b]$  be a real interval,  $x : I \rightarrow R^n$  and  $y : I \rightarrow R^m$  are differentiable functions having derivatives  $\dot{x}$  and  $\dot{y}$ , respectively. Let  $U \subset R^{n_1}$  and  $V \subset R^{m_1}$  be two arbitrary sets of integers and  $C_1 \subset R^{n_2}$  and  $C_2 \subset R^{m_2}$  be closed convex cones with nonempty interiors.

Let  $f^i(t, x, \dot{x}, y, \dot{y}) = f^i(t, x^1, \dot{x}^1, x^2, \dot{x}^2, y^1, \dot{y}^1, y^2, \dot{y}^2)$ ,  $i = 1, 2, \dots, k$  be twice continuously differentiable function with respect to  $x^2, \dot{x}^2, y^2, \dot{y}^2$ , where  $x^1 \in U, y^1 \in V, x^2 \in C_1, y^2 \in C_2$ . Superscripts denote vector components and subscripts denote partial derivatives.  $f_{x^2}^i, f_{\dot{x}^2}^i, f_{y^2}^i$  and  $f_{\dot{y}^2}^i$  denote gradient vectors of  $f^i$  with respect to  $x^2, \dot{x}^2, y^2$  and  $\dot{y}^2$ .  $f_{x^2 x^2}^i, f_{\dot{x}^2 \dot{x}^2}^i, f_{y^2 y^2}^i$ , and  $f_{\dot{y}^2 \dot{y}^2}^i$  denote the Hessian matrices of  $f^i$  with respect to  $x^2, \dot{x}^2, y^2$  and  $\dot{y}^2$ . Other Hessian matrices  $f_{x^2 y^2}^i, f_{x^2 \dot{y}^2}^i, f_{\dot{x}^2 y^2}^i, f_{\dot{x}^2 \dot{y}^2}^i, f_{y^2 x^2}^i, f_{y^2 \dot{x}^2}^i, f_{\dot{y}^2 x^2}^i$  and  $f_{\dot{y}^2 \dot{x}^2}^i$  are defined similarly.

Denote by  $X$  the space of twice continuously differentiable functions  $x : I \rightarrow R^n$  with norm  $\|x\| = \|x\|_\infty + \|Dx\|_\infty + \|D^2x\|_\infty$ , where the differentiation operator  $D$  is given by

$$u = Dx \iff x(t) = \alpha + \int_a^t u(s) ds,$$

where  $\alpha$  is a given boundary value. Therefore  $D \equiv \frac{d}{dt}$  except at discontinuities. Denote by  $Y$  the space of twice continuously differentiable functions  $y : I \rightarrow R^m$  with the norm as that of the space  $X$ .

Consider the following multiobjective variational problem:

$$\begin{aligned}
 \text{(VP):} \quad & \text{Minimize} \quad \left( \int_a^b \phi^1(t, x(t), \dot{x}(t)) dt, \dots, \int_a^b \phi^k(t, x(t), \dot{x}(t)) dt \right), \\
 & \text{subject to} \\
 & \quad x(a) = 0, \quad x(b) = 0, \\
 & \quad \dot{x}(a) = 0, \quad \dot{x}(b) = 0, \\
 & \quad h(t, x(t), \dot{x}(t)) \leq 0, \quad t \in I,
 \end{aligned}$$

where  $\phi : I \times X \times X \rightarrow R^k$  and  $h : I \times X \times X \rightarrow R^m$  are differentiable functions.

Let  $Q$  denote the set of all feasible solutions of (VP), i.e.,

$$Q = \{x \in X | x(a) = 0, x(b) = 0; \dot{x}(a) = 0, \dot{x}(b) = 0; h(t, x(t), \dot{x}(t)) \leq 0, t \in I\}.$$

**Definition 1** (Geoffrion [9]). A point  $x^0 \in Q$  is said to be an efficient solution of (VP), if for all  $x \in Q$ ,

$$\begin{aligned}
 & \int_a^b \phi^i(t, x^0(t), \dot{x}^0(t)) dt \geq \int_a^b \phi^i(t, x(t), \dot{x}(t)) dt \quad \text{for all } i = 1, 2, \dots, k \\
 & \Rightarrow \int_a^b \phi^i(t, x^0(t), \dot{x}^0(t)) dt = \int_a^b \phi^i(t, x(t), \dot{x}(t)) dt \quad \text{for all } i = 1, 2, \dots, k.
 \end{aligned}$$

**Definition 2.** A point  $x^0 \in Q$  is said to be a weak efficient solution of (VP), if for all  $x \in Q$ ,

$$\begin{aligned}
 & \int_a^b \phi^i(t, x^0(t), \dot{x}^0(t)) dt > \int_a^b \phi^i(t, x(t), \dot{x}(t)) dt \quad \text{for all } i = 1, 2, \dots, k \\
 & \Rightarrow \int_a^b \phi^i(t, x^0(t), \dot{x}^0(t)) dt = \int_a^b \phi^i(t, x(t), \dot{x}(t)) dt \quad \text{for all } i = 1, 2, \dots, k.
 \end{aligned}$$

**Definition 3.** A functional  $F : I \times X \times X \times X \times X \times R^n \rightarrow R$  is sublinear, if for any  $x, \dot{x}, u, \dot{u} \in X$ ,

- (i)  $F(t, x, \dot{x}, u, \dot{u}; \zeta_1 + \zeta_2) \leq F(t, x, \dot{x}, u, \dot{u}; \zeta_1) + F(t, x, \dot{x}, u, \dot{u}; \zeta_2)$ , for any  $\zeta_1, \zeta_2 \in R^n$ ; and
- (ii)  $F(t, x, \dot{x}, u, \dot{u}; \beta\zeta) = \beta F(t, x, \dot{x}, u, \dot{u}; \zeta)$ , for any  $\beta \in R, \beta \geq 0$  and  $\zeta \in R^n$ .

From (ii), it follows that  $F(t, x, \dot{x}, u, \dot{u}; 0) = 0$ .

**Definition 4.** The functional  $\int_a^b f(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) dt$  is said to be F-pseudoconvex in  $x$  and  $\dot{x}$  for fixed  $y$  and  $\dot{y}$ , if

$$\int_a^b F(t, x, \dot{x}, u, \dot{u}; f_x(t, u, \dot{u}, y, \dot{y}) - Df_{\dot{x}}(t, u, \dot{u}, y, \dot{y})) dt \geq 0 \Rightarrow \int_a^b f(t, x, \dot{x}, y, \dot{y}) dt \geq \int_a^b f(t, u, \dot{u}, y, \dot{y}) dt$$

for all  $x, u : I \rightarrow R^n$  and for some arbitrary sublinear functional  $F$ .

**Definition 5.** The functional  $\int_a^b f(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) dt$  is said to be F-pseudoconcave in  $y$  and  $\dot{y}$  for fixed  $x$  and  $\dot{x}$ , if

$$\int_a^b F(t, v, \dot{v}, y, \dot{y}; -\{f_y(t, x, \dot{x}, v, \dot{v}) - Df_{\dot{y}}(t, x, \dot{x}, v, \dot{v})\}) dt \geq 0 \Rightarrow \int_a^b f(t, x, \dot{x}, v, \dot{v}) dt \leq \int_a^b f(t, x, \dot{x}, y, \dot{y}) dt$$

for all  $y, v : I \rightarrow R^m$  and for some arbitrary sublinear functional.

**Definition 6.** A cone  $C_i^*$  is said to be polar of  $C_i$  for  $i = 1, 2$ , if

$$C_i^* = \{w | w^T x \leq 0, \text{ for all } x \in C_i\}.$$

The following concept of separability is required in the sequel which has been extensively used by Balas [1] and Gulati et al. [10].

Let  $z^r$ ,  $r = 1, 2, \dots, p$ , be piecewise smooth functions belonging to arbitrary space  $Z^r$ ,  $r = 1, 2, \dots, p$ , equipped with norm  $\|z^r\| = \|z^r\|_\infty + \|Dz^r\|_\infty$ . A function  $L(t, z^1(t), \dot{z}^1(t), z^2(t), \dot{z}^2(t), \dots, z^p(t), \dot{z}^p(t))$  will be called additively separable with respect to  $(z^1(t), \dot{z}^1(t))$  if there exist functions  $M(t, z^1(t), \dot{z}^1(t))$  (independent of  $z^2(t), \dot{z}^2(t), \dots, z^p(t), \dot{z}^p(t)$ ) and  $N(t, z^2(t), \dot{z}^2(t), \dots, z^p(t), \dot{z}^p(t))$  (independent of  $z^1(t), \dot{z}^1(t)$ ) such that

$$L(t, z^1(t), \dot{z}^1(t), z^2(t), \dot{z}^2(t), \dots, z^p(t), \dot{z}^p(t)) = M(t, z^1(t), \dot{z}^1(t)) + N(t, z^2(t), \dot{z}^2(t), \dots, z^p(t), \dot{z}^p(t)).$$

The following form of Fritz John necessary conditions proposed by Suneja et al. [21] is required to prove the strong and converse duality theorems.

**Lemma 1.** Let  $P$  be a convex set with nonempty interior in  $R^n$  and suppose that  $C$  is a closed convex cone in  $R^m$  having a nonempty interior. Let  $R$  and  $S$  be two vector valued functions defined on  $P$ . If  $z_0$  is a weak efficient solution of the following problem:

$$\begin{aligned} &\text{Minimize } R(z) = (R_1(z), R_2(z), \dots, R_k(z)), \\ &\text{subject to } S(z) \in C, \quad z \in P, \end{aligned}$$

then there exists a nonzero vector  $(r_0, r)$  such that

$$[r_0^T R_z(z_0) + r^T S_z(z_0)](z - z_0) \geq 0, \text{ for each } z \in P$$

and  $r^T S(z_0) = 0$ ,  $r_0 \geq 0$ ,  $r \in C^*$ .

### 3. Mond–Weir type mixed integer symmetric duality

In this section, we constrain some of the primal and dual variables to belong to the arbitrary sets of integers, i.e.,  $U$  and  $V$ . Suppose that the first  $n_1$  ( $0 \leq n_1 \leq n$ ) components of  $x$  belong to  $U$  and the first  $m_1$  ( $0 \leq m_1 \leq m$ ) components of  $y$  belong to  $V$ . So we write  $(x, y) = (x^1, x^2, y^1, y^2)$ , where  $x^1 = (x_1, x_2, \dots, x_{n_1}) \in U$ , and  $y^1 = (y_1, y_2, \dots, y_{m_1}) \in V$ ,  $x^2$  and  $y^2$  being the remaining components of  $x$  and  $y$  such that  $x^2 \in C_1$  and  $y^2 \in C_2$ .

Let  $X^1$  and  $Y^1$  be the arbitrary sets of continuously differentiable functions  $x^1 : I \rightarrow R^{n_1}$  and  $y^1 : I \rightarrow R^{m_1}$  ( $0 \leq n_1 \leq n, 0 \leq m_1 \leq m$ ) equipped with the norms which are prescribed for the spaces  $X$  and  $Y$ . Partitioning vector functions  $x$  and  $y$  as above, we introduce the following multiobjective variational minimax mixed integer symmetric dual program:

$$\begin{aligned} \text{Primal (MP)} \quad & \text{Max}_{x^1} \text{Min}_{x^2, y} \int_a^b f(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) dt \\ & = \left[ \int_a^b f^1(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) dt, \dots, \int_a^b f^k(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) dt \right], \end{aligned}$$

subject to

$$\begin{aligned} x(a) = 0 = x(b); \quad y(a) = 0 = y(b), \\ \dot{x}(a) = 0 = \dot{x}(b); \quad \dot{y}(a) = 0 = \dot{y}(b), \\ x^1(t) \in U, \quad y^1(t) \in V, \quad x^2(t) \in C_1, \quad t \in I, \\ [(\lambda^T f)_{y^2}(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) - D(\lambda^T f)_{\dot{y}^2}(t, x(t), \dot{x}(t), y(t), \dot{y}(t))] \in C_2^*, \\ y^2(t)^T [(\lambda^T f)_{y^2}(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) - D(\lambda^T f)_{\dot{y}^2}(t, x(t), \dot{x}(t), y(t), \dot{y}(t))] \geq 0, \\ \lambda > 0. \end{aligned}$$

**Dual (MD)**       $\text{Min Max}_{u,v^2} \int_a^b f(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) dt$

$$= \left[ \int_a^b f^1(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) dt, \dots, \int_a^b f^k(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) dt \right],$$

subject to

$$\begin{aligned} u(a) = 0 = u(b); \quad v(a) = 0 = v(b), \\ \dot{u}(a) = 0 = \dot{u}(b); \quad \dot{v}(a) = 0 = \dot{v}(b), \\ u^1(t) \in U, \quad v^1(t) \in V, \quad v^2(t) \in C_2, \quad t \in I, \\ - [(\lambda^T f)_{x^2}(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) - D(\lambda^T f)_{\dot{x}^2}(t, u(t), \dot{u}(t), v(t), \dot{v}(t))] \in C_1^*, \\ u^2(t)^T [(\lambda^T f)_{x^2}(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) - D(\lambda^T f)_{\dot{x}^2}(t, u(t), \dot{u}(t), v(t), \dot{v}(t))] \leq 0, \\ \lambda > 0. \end{aligned}$$

We shall denote by  $G$  and  $H$ , the sets of feasible solutions of the primal and dual multiobjective variational problems (MP) and (MD), respectively.

**Theorem 1** (Weak duality). *Let  $(x, y, \lambda) \in G$  and  $(u, v, \lambda) \in H$ . Assume that*

- (i)  $f^i(t, x, \dot{x}, y, \dot{y})$ ,  $i = 1, 2, \dots, k$  is twice differentiable with respect to  $(x^2, \dot{x}^2)$  and  $(y^2, \dot{y}^2)$ , respectively,
- (ii)  $f^i(t, x, \dot{x}, y, \dot{y})$ ,  $i = 1, 2, \dots, k$  is additively separable with respect to  $(x^1, \dot{x}^1)$  or  $(y^1, \dot{y}^1)$ ,
- (iii)  $\int_a^b \lambda^T f(t, x, \dot{x}, y, \dot{y}) dt$  is  $F$ -pseudoconvex in  $(x^2, \dot{x}^2)$  for each  $(x^1, \dot{x}^1, y, \dot{y})$  on  $I$ ,
- (iv)  $\int_a^b \lambda^T f(t, x, \dot{x}, y, \dot{y}) dt$  is  $F$ -pseudoconcave in  $(y^2, \dot{y}^2)$  for each  $(x, \dot{x}, y^1, \dot{y}^1)$  on  $I$ ,
- (v)  $F(t, x^2, \dot{x}^2, u^2, \dot{u}^2; \xi^2) + (u^2)^T \xi^2 \geq 0$ , for all  $x^2, \dot{x}^2, u^2, \dot{u}^2 \in C_1$  and  $\xi^2 \in C_1^*$  and
- (vi)  $F(t, v^2, \dot{v}^2, y^2, \dot{y}^2; \eta^2) + (y^2)^T \eta^2 \geq 0$ , for all  $v^2, \dot{v}^2, y^2, \dot{y}^2 \in C_2$  and  $\eta^2 \in C_2^*$ .

Then

$$\int_a^b f(t, x, \dot{x}, y, \dot{y}) dt \not\leq \int_a^b f(t, u, \dot{u}, v, \dot{v}) dt.$$

**Proof.** Let

$$g = \text{Max}_{x^1} \text{Min}_{x^2, y} \left[ \int_a^b f(t, x, \dot{x}, y, \dot{y}) dt \mid (x, y, \lambda) \in G \right]$$

and

$$h = \text{Min}_{v^1} \text{Max}_{u, v^2} \left[ \int_a^b f(t, u, \dot{u}, v, \dot{v}) dt \mid (u, v, \lambda) \in H \right].$$

Since  $f^i(t, x, \dot{x}, y, \dot{y})$ ,  $i = 1, 2, \dots, k$  is additively separable with respect to  $(x^1, \dot{x}^1)$  or  $(y^1, \dot{y}^1)$  (say with respect to  $(x^1, \dot{x}^1)$ ), it follows that:

$$f^i(t, x, \dot{x}, y, \dot{y}) = f_1^i(t, x^1, \dot{x}^1) + f_2^i(t, x^2, \dot{x}^2, y, \dot{y}), \quad i = 1, 2, \dots, k.$$

Therefore  $g$  can be written as

$$g = \text{Max}_{x^1} \text{Min}_{x^2, y} \left[ \int_a^b f_1(t, x^1, \dot{x}^1) dt + \int_a^b f_2(t, x^2, \dot{x}^2, y, \dot{y}) dt \mid (x, y, \lambda) \in G \right]$$

or

$$g = \text{Max}_{x^1} \text{Min}_{y^1} \left[ \int_a^b f_1(t, x^1, \dot{x}^1) dt + \phi(y^1) \mid x^1 \in U, y^1 \in V \right],$$

where

$$\int_a^b f_1(t, x^1, \dot{x}^1) dt = \left[ \int_a^b f_1^1(t, x^1, \dot{x}^1) dt, \int_a^b f_1^2(t, x^1, \dot{x}^1) dt, \dots, \int_a^b f_1^k(t, x^1, \dot{x}^1) dt \right]$$

and

$$\phi(y^1) = \text{Min}_{x^2, y^2} \int_a^b f_2(t, x^2, \dot{x}^2, y, \dot{y}) dt = \left[ \int_a^b f_2^1(t, x^2, \dot{x}^2, y, \dot{y}) dt, \dots, \int_a^b f_2^k(t, x^2, \dot{x}^2, y, \dot{y}) dt \right]$$

subject to

$$x^2(a) = 0 = x^2(b); \quad y^2(a) = 0 = y^2(b), \quad (1)$$

$$\dot{x}^2(a) = 0 = \dot{x}^2(b); \quad \dot{y}^2(a) = 0 = \dot{y}^2(b), \quad (2)$$

$$x^1 \in U, \quad y^1 \in V, \quad x^2 \in C_1, \quad (3)$$

$$[(\lambda^T f_2)_{y^2}(t, x^2, \dot{x}^2, y, \dot{y}) - D(\lambda^T f_2)_{y^2}(t, x^2, \dot{x}^2, y, \dot{y})] \in C_2^*, \quad (4)$$

$$(y^2)^T [(\lambda^T f_2)_{y^2}(t, x^2, \dot{x}^2, y, \dot{y}) - D(\lambda^T f_2)_{y^2}(t, x^2, \dot{x}^2, y, \dot{y})] \geq 0, \quad (5)$$

$$\lambda > 0. \quad (6)$$

Similarly  $h$  can be written as

$$h = \text{Min}_{v^1} \text{Max}_{u^1} \left[ \int_a^b f_1(t, u^1, \dot{u}^1) dt + \psi(v^1) \mid u^1 \in U, v^1 \in V \right],$$

where

$$\psi(v^1) = \text{Max}_{u^2, v^2} \int_a^b f_2(t, u^2, \dot{u}^2, v, \dot{v}) dt = \left[ \int_a^b f_2^1(t, u^2, \dot{u}^2, v, \dot{v}) dt, \dots, \int_a^b f_2^k(t, u^2, \dot{u}^2, v, \dot{v}) dt \right],$$

subject to

$$u^2(a) = 0 = u^2(b); \quad v^2(a) = 0 = v^2(b), \quad (7)$$

$$\dot{u}^2(a) = 0 = \dot{u}^2(b); \quad \dot{v}^2(a) = 0 = \dot{v}^2(b), \quad (8)$$

$$u^1 \in U, \quad v^1 \in V, \quad v^2 \in C_2, \quad (9)$$

$$- [(\lambda^T f_2)_{x^2}(t, u^2, \dot{u}^2, v, \dot{v}) - D(\lambda^T f_2)_{x^2}(t, u^2, \dot{u}^2, v, \dot{v})] \in C_1^*, \quad (10)$$

$$(u^2)^T [(\lambda^T f_2)_{x^2}(t, u^2, \dot{u}^2, v, \dot{v}) - D(\lambda^T f_2)_{x^2}(t, u^2, \dot{u}^2, v, \dot{v})] \leq 0, \quad (11)$$

$$\lambda > 0. \quad (12)$$

Let  $(x, y, \lambda) \in G$  and  $(u, v, \lambda) \in H$ . In order to prove the theorem, it is sufficient to prove that  $\phi(y^1) \not\leq \psi(v^1)$ , for a given  $y^1$ .

On taking  $\xi^2 = (\lambda^T f_2)_{x^2} - D(\lambda^T f_2)_{x^2}$ , we have

$$F[t, x^2, \dot{x}^2, u^2, \dot{u}^2; (\lambda^T f_2)_{x^2} - D(\lambda^T f_2)_{x^2}] \geq - (u^2)^T [(\lambda^T f_2)_{x^2} - D(\lambda^T f_2)_{x^2}] \geq 0 \text{ (by hypothesis (v) and (11)),}$$

which implies

$$\int_a^b F[t, x^2, \dot{x}^2, u^2, \dot{u}^2; (\lambda^T f_2)_{x^2} - D(\lambda^T f_2)_{x^2}] dt \geq 0.$$

This in view of F-pseudoconvexity of  $\int_a^b \lambda^T f_2 dt$  in  $(x^2, \dot{x}^2)$  yields

$$\int_a^b \lambda^T f_2(t, x^2, \dot{x}^2, v, \dot{v}) dt \geq \int_a^b \lambda^T f_2(t, u^2, \dot{u}^2, v, \dot{v}) dt. \tag{13}$$

By taking  $\eta^2 = -\{(\lambda^T f_2)_{y^2} - D(\lambda^T f_2)_{y^2}\}$ , we get

$$F[t, v^2, \dot{v}^2, y^2, \dot{y}^2; -\{(\lambda^T f_2)_{y^2} - D(\lambda^T f_2)_{y^2}\}] \geq (y^2)^T [(\lambda^T f_2)_{y^2} - D(\lambda^T f_2)_{y^2}] \geq 0 \text{ (by hypothesis (vi) and (5)),}$$

$$\Rightarrow \int_a^b F[t, v^2, \dot{v}^2, y^2, \dot{y}^2; -\{(\lambda^T f_2)_{y^2} - D(\lambda^T f_2)_{y^2}\}] dt \geq 0,$$

which by F-pseudoconcavity of  $\int_a^b \lambda^T f_2 dt$  in  $(y^2, \dot{y}^2)$  gives

$$\int_a^b \lambda^T f_2(t, x^2, \dot{x}^2, v, \dot{v}) dt \leq \int_a^b \lambda^T f_2(t, x^2, \dot{x}^2, y, \dot{y}) dt. \tag{14}$$

Adding the inequalities (13) and (14), we obtain

$$\int_a^b \lambda^T f_2(t, x^2, \dot{x}^2, y, \dot{y}) dt \geq \int_a^b \lambda^T f_2(t, u^2, \dot{u}^2, v, \dot{v}) dt.$$

Hence

$$\int_a^b f_2(t, x^2, \dot{x}^2, y, \dot{y}) dt \not\leq \int_a^b f_2(t, u^2, \dot{u}^2, v, \dot{v}) dt. \quad \square$$

**Theorem 2 (Strong duality).** Let  $(\bar{x}(t), \bar{y}(t), \bar{\lambda})$  be an efficient solution of (MP), and fixed  $\lambda = \bar{\lambda}$  in (MD). Assume that

- (i)  $f^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ ,  $i = 1, 2, \dots, k$  is twice differentiable at  $x^2(t)$ ,  $\dot{x}^2(t)$ ,  $y^2(t)$  and  $\dot{y}^2(t)$ ,
- (ii)  $f^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ ,  $i = 1, 2, \dots, k$  is additively separable with respect to  $(x^1(t), \dot{x}^1(t))$  or  $(y^1(t), \dot{y}^1(t))$ ,
- (iii) the system  $[(r_1(t) - \bar{y}^2 r_2)^T \{(\bar{\lambda}^T f_2)_{y^2, y^2} - D(\bar{\lambda}^T f_2)_{y^2, y^2}\} - D\{(r_1(t) - \bar{y}^2 r_2)^T \{(\bar{\lambda}^T f_2)_{y^2, y^2} - D(\bar{\lambda}^T f_2)_{y^2, y^2}\}\}] (r_1(t) - \bar{y}^2 r_2) = 0 \Rightarrow (r_1(t) - \bar{y}^2 r_2) = 0$ , for every  $r_1(t) \in C_2$ ,  $t \in I$ , and
- (iv) the set  $\{(f_{2y^2}^1 - Df_{2y^2}^1), \dots, (f_{2y^2}^k - Df_{2y^2}^k)\}$  is linearly independent.

Then  $(\bar{x}(t), \bar{y}(t), \bar{\lambda})$  is a feasible solution of (MD) and the objective values of (MP) and (MD) are equal. Furthermore, if the assumptions of Theorem 1 are satisfied, then  $(\bar{x}(t), \bar{y}(t), \bar{\lambda})$  is an efficient solution of (MD).

**Proof.** Since  $(\bar{x}(t), \bar{y}(t), \bar{\lambda})$  is an efficient solution of (MP), hence it is weak efficient, then by Lemma 1, there exist  $r_0 \in R^k$ ,  $r_1(t) \in C_2$ ,  $r_2 \in R$  and  $\delta \in R^k$  such that

$$\begin{aligned} r_0^T [f_{2x^2} - Df_{2x^2}](x^2(t) - \bar{x}^2(t)) + [(r_1(t) - \bar{y}^2 r_2)^T ((\bar{\lambda}^T f_2)_{y^2 x^2} \\ - D(\bar{\lambda}^T f_2)_{y^2 x^2}) - D\{(r_1(t) - \bar{y}^2 r_2)^T ((\bar{\lambda}^T f_2)_{y^2 x^2} - D(\bar{\lambda}^T f_2)_{y^2 x^2})\}](x^2(t) - \bar{x}^2(t)) \geq 0, \\ \text{for all } x^2(t) \in C_1 \quad \text{and } t \in I, \end{aligned} \quad (15)$$

$$\begin{aligned} (r_0 - \bar{\lambda} r_2)^T [f_{2y^2} - Df_{2y^2}] + [(r_1(t) - \bar{y}^2 r_2)^T ((\bar{\lambda}^T f_2)_{y^2 y^2} - D(\bar{\lambda}^T f_2)_{y^2 y^2}) \\ - D\{(r_1(t) - \bar{y}^2 r_2)^T ((\bar{\lambda}^T f_2)_{y^2 y^2} - D(\bar{\lambda}^T f_2)_{y^2 y^2})\}] = 0, \quad \forall t \in I, \end{aligned} \quad (16)$$

$$(r_1(t) - \bar{y}^2 r_2)^T (f_{2y^2} - Df_{2y^2}) - \delta = 0, \quad \forall t \in I, \quad (17)$$

$$r_1(t)^T ((\bar{\lambda}^T f_2)_{y^2} - D(\bar{\lambda}^T f_2)_{y^2}) = 0, \quad \forall t \in I, \quad (18)$$

$$r_2 \bar{y}^2 ((\bar{\lambda}^T f_2)_{y^2} - D(\bar{\lambda}^T f_2)_{y^2}) = 0, \quad \forall t \in I \quad (19)$$

$$\delta^T \bar{\lambda} = 0, \quad (20)$$

$$r_0 \geq 0, \quad r_2 \geq 0, \quad \delta \geq 0, \quad t \in I, \quad (21)$$

$$(r_0, r_1(t), r_2, \delta) \neq 0. \quad (22)$$

As  $\bar{\lambda} > 0$ , it follows from (20) that  $\delta = 0$ . Therefore (17) reduces to

$$(r_1(t) - \bar{y}^2 r_2)^T (f_{2y^2} - Df_{2y^2}) = 0 \quad \forall t \in I. \quad (23)$$

Multiplying (16) by  $(r_1(t) - \bar{y}^2 r_2)$  and using (23), we get

$$\begin{aligned} [(r_1(t) - \bar{y}^2 r_2)^T ((\bar{\lambda}^T f_2)_{y^2 y^2} - D(\bar{\lambda}^T f_2)_{y^2 y^2}) - D\{(r_1(t) - \bar{y}^2 r_2)^T ((\bar{\lambda}^T f_2)_{y^2 y^2} - D(\bar{\lambda}^T f_2)_{y^2 y^2})\}](r_1(t) - \bar{y}^2 r_2) \\ = 0, \end{aligned}$$

which by hypothesis (iii) gives

$$r_1(t) = \bar{y}^2 r_2. \quad (24)$$

From (16) and (24)

$$(r_0 - \bar{\lambda} r_2)^T (f_{2y^2} - Df_{2y^2}) = 0.$$

Since  $\{(f_{2y^2}^1 - Df_{2y^2}^1), \dots, (f_{2y^2}^k - Df_{2y^2}^k)\}$  is linearly independent, then the above equation implies

$$r_0 = \bar{\lambda} r_2. \quad (25)$$

If  $r_2 = 0$ , then from (24) and (25), we obtain  $r_1(t) = r_0 = 0$ . Hence  $(r_0, r_1(t), r_2, \delta) = 0$ , contradicting (22). Thus  $r_2 > 0$ , Eq. (24) yields

$$\bar{y}^2 = \frac{r_1(t)}{r_2} \in C_2.$$

Now, (15) along with (24) and (25) gives

$$\bar{\lambda} r_2 [f_{2x^2} - Df_{2x^2}](x^2(t) - \bar{x}^2(t)) \geq 0.$$

Since  $r_2 > 0$ , the above inequality implies

$$[(\bar{\lambda}^T f_2)_{x^2} - D(\bar{\lambda}^T f_2)_{x^2}](x^2(t) - \bar{x}^2(t)) \geq 0. \quad (26)$$

Let  $x^2(t) \in C_1$ . Then  $x^2(t) + \bar{x}^2(t) \in C_1$  and so (26) shows that for every  $x^2(t) \in C_1$

$$[(\bar{\lambda}^T f_2)_{x^2} - D(\bar{\lambda}^T f_2)_{x^2}]x^2(t) \geq 0 \text{ i.e., } -[(\bar{\lambda}^T f_2)_{x^2} - D(\bar{\lambda}^T f_2)_{x^2}] \in C_1^*.$$



Also, by letting  $x^2(t) = 0$  and  $x^2(t) = 2\bar{x}^2(t)$  simultaneously in the inequality (26), we get

$$\bar{x}^2(t)^T [(\bar{\lambda}^T f_2)_{x^2} - D(\bar{\lambda}^T f_2)_{x^2}] = 0.$$

Thus  $(\bar{x}(t), \bar{y}(t), \bar{\lambda})$  is a feasible solution of (MD) and the objective functional values are equal. By Theorem 1,  $(\bar{x}(t), \bar{y}(t), \bar{\lambda})$  is an efficient solution of (MD).  $\square$

A converse duality theorem may be stated as its proof would run analogously to that of Theorem 2.

**Theorem 3** (Converse duality). *Let  $(\bar{u}(t), \bar{v}(t), \bar{\lambda})$  be an efficient solution of (MD), and fixed  $\lambda = \bar{\lambda}$  in (MP). Assume that*

- (i)  $f^i(t, u(t), \dot{u}(t), v(t), \dot{v}(t))$ ,  $i = 1, 2, \dots, k$  is twice differentiable at  $u^2(t)$ ,  $\dot{u}^2(t)$ ,  $v^2(t)$  and  $\dot{v}^2(t)$ ,
- (ii)  $f^i(t, u(t), \dot{u}(t), v(t), \dot{v}(t))$ ,  $i = 1, 2, \dots, k$  is additively separable with respect to  $(u^1(t), \dot{u}^1(t))$  or  $(v^1(t), \dot{v}^1(t))$ ,
- (iii) the system  $[(r_1(t) - \bar{u}^2 r_2)^T \{(\bar{\lambda}^T f_2)_{x^2, x^2} - D(\bar{\lambda}^T f_2)_{x^2, x^2}\} - D\{(r_1(t) - \bar{u}^2 r_2)^T \{(\bar{\lambda}^T f_2)_{x^2, x^2} - D(\bar{\lambda}^T f_2)_{x^2, x^2}\}\}] (r_1(t) - \bar{u}^2 r_2) = 0 \Rightarrow (r_1(t) - \bar{u}^2 r_2) = 0$ , for every  $r_1(t) \in C_1$ ,  $t \in I$ , and
- (iv) the set  $\{(f_{2x^2}^1 - Df_{2x^2}^1), \dots, (f_{2x^2}^k - Df_{2x^2}^k)\}$  is linearly independent.

Then  $(\bar{u}(t), \bar{v}(t), \bar{\lambda})$  is a feasible solution of (MP) and the objective values of (MP) and (MD) are equal. Furthermore, if the assumptions of Theorem 1 are satisfied, then  $(\bar{u}(t), \bar{v}(t), \bar{\lambda})$  is an efficient solution of (MP).

#### 4. Self duality

A mathematical programming problem is said to be self dual if it is formally identical with its dual, that is, the dual can be recast in the form of the primal. If we assume

- (i)  $C_2 = C_1$ , and
- (ii)  $f^i(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) : I \times R^n \times R^n \times R^n \times R^n \rightarrow R$ ,  $i = 1, 2, \dots, k$  to be skew symmetric, that is,

$$f^i(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) = -f^i(t, v(t), \dot{v}(t), u(t), \dot{u}(t)), \quad i = 1, 2, \dots, k$$

then we shall show that the programs (MP) and (MD) are self duals. By recasting the dual problem (MD) as maxmin problem, we have

$$\begin{aligned} \text{Dual (MD)} \quad & \text{Max}_{v^1} \text{Min}_{u, v^2} - \int_a^b f(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) dt \\ & = \left[ - \int_a^b f^1(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) dt, \dots, - \int_a^b f^k(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) dt \right], \end{aligned}$$

subject to

$$\begin{aligned} u(a) = 0 = u(b), \quad v(a) = 0 = v(b), \\ \dot{u}(a) = 0 = \dot{u}(b), \quad \dot{v}(a) = 0 = \dot{v}(b), \\ u^1(t) \in U, \quad v^1(t) \in V, \quad v^2(t) \in C_2, \quad t \in I, \\ - [(\lambda^T f)_{x^2}(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) - D(\lambda^T f)_{x^2}(t, u(t), \dot{u}(t), v(t), \dot{v}(t))] \in C_1^*, \\ - u^2(t)^T [(\lambda^T f)_{x^2}(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) - D(\lambda^T f)_{x^2}(t, u(t), \dot{u}(t), v(t), \dot{v}(t))] \geq 0, \\ \lambda > 0. \end{aligned}$$

Since  $f^i(t, u(t), \dot{u}(t), v(t), \dot{v}(t))$ ,  $i = 1, 2, \dots, k$  is skew symmetric, we have

$$f_{x^2}^i(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) = -f_{y^2}^i(t, v(t), \dot{v}(t), u(t), \dot{u}(t)), \quad i = 1, 2, \dots, k$$

and

$$f_{\dot{x}^2}^i(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) = -f_{\dot{y}^2}^i(t, v(t), \dot{v}(t), u(t), \dot{u}(t)), \quad i = 1, 2, \dots, k.$$

The above problem becomes

$$\begin{aligned} \text{Dual (MD)} \quad & \text{Max}_{v^1} \text{Min}_{u, v^2} \int_a^b f(t, v(t), \dot{v}(t), u(t), \dot{u}(t)) dt \\ & = \left[ \int_a^b f^1(t, v(t), \dot{v}(t), u(t), \dot{u}(t)) dt, \dots, \int_a^b f^k(t, v(t), \dot{v}(t), u(t), \dot{u}(t)) dt \right], \end{aligned}$$

subject to

$$u(a) = 0 = u(b), \quad v(a) = 0 = v(b),$$

$$\dot{u}(a) = 0 = \dot{u}(b), \quad \dot{v}(a) = 0 = \dot{v}(b),$$

$$u^1(t) \in U, \quad v^1(t) \in V, \quad v^2(t) \in C_1, \quad t \in I,$$

$$[(\lambda^T f)_{y^2}(t, v(t), \dot{v}(t), u(t), \dot{u}(t)) - D(\lambda^T f)_{y^2}(t, v(t), \dot{v}(t), u(t), \dot{u}(t))] \in C_1^*,$$

$$u^2(t)^T [(\lambda^T f)_{y^2}(t, v(t), \dot{v}(t), u(t), \dot{u}(t)) - D(\lambda^T f)_{y^2}(t, v(t), \dot{v}(t), u(t), \dot{u}(t))] \geq 0,$$

$$\lambda > 0.$$

We observe that (MD) is formally identical to (MP); that is, the objective and the constraint functions of (MP) and (MD) are identical. Therefore (MP) is a self dual. It can easily be seen that the feasibility of  $(x(t), y(t), \lambda)$  for (MP) implies the feasibility of  $(y(t), x(t), \lambda)$  for (MD), and conversely.

**Theorem 4 (Self duality).** *Assume that  $f^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ ,  $i = 1, 2, \dots, k$  is skew symmetric. Then (MP) is a self dual. Also, if (MP) and (MD) are dual variational problems and  $(\bar{x}(t), \bar{y}(t), \bar{\lambda})$  is a joint efficient solution, then so is  $(\bar{y}(t), \bar{x}(t), \bar{\lambda})$  and the common objective functional value is 0.*

**Proof.** Since  $(\bar{x}(t), \bar{y}(t), \bar{\lambda})$  is a joint efficient solution of (MP) and (MD), the objective functional values are equal to

$$\int_a^b f(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{y}(t), \dot{\bar{y}}(t)) dt.$$

From self duality,  $(\bar{x}(t), \bar{y}(t), \bar{\lambda})$  is feasible for (MP) if and only if  $(\bar{y}(t), \bar{x}(t), \bar{\lambda})$  is feasible for (MD). Therefore efficiency of  $(\bar{x}(t), \bar{y}(t), \bar{\lambda})$  for (MP) implies efficiency of  $(\bar{y}(t), \bar{x}(t), \bar{\lambda})$  for (MD) and vice versa. Hence the objective functional values are equal to

$$\int_a^b f(t, \bar{y}(t), \dot{\bar{y}}(t), \bar{x}(t), \dot{\bar{x}}(t)) dt.$$

Therefore

$$\int_a^b f(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{y}(t), \dot{\bar{y}}(t)) dt = \int_a^b f(t, \bar{y}(t), \dot{\bar{y}}(t), \bar{x}(t), \dot{\bar{x}}(t)) dt = - \int_a^b f(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{y}(t), \dot{\bar{y}}(t)) dt.$$

Thus we have

$$\int_a^b f(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{y}(t), \dot{\bar{y}}(t)) dt = 0. \quad \square$$

## 5. Static symmetric dual multiobjective programs

If the time dependency in (MP) and (MD) is relaxed, then they reduce to the following static minimax mixed integer symmetric dual problems:

$$\begin{aligned}
 \text{(MP1):} \quad & \text{Max}_{x^1} \text{Min}_{x^2, y} f(x, y), \\
 & \text{subject to} \\
 & x^1 \in U, \quad y^1 \in V, \quad x^2 \in C_1, \\
 & (\lambda^T f)_{y^2}(x, y) \in C_2^*, \\
 & (y^2)^T (\lambda^T f)_{y^2}(x, y) \geq 0, \\
 & \lambda > 0.
 \end{aligned}$$

$$\begin{aligned}
 \text{(MD1):} \quad & \text{Min}_{v^1} \text{Max}_{u, v^2} f(u, v) \\
 & \text{subject to} \\
 & u^1 \in U, \quad v^1 \in V, \quad v^2 \in C_2, \\
 & -(\lambda^T f)_{x^2}(u, v) \in C_1^*, \\
 & (u^2)^T (\lambda^T f)_{x^2}(u, v) \leq 0, \\
 & \lambda > 0.
 \end{aligned}$$

The above problems (MP1) and (MD1) are the Mond–Weir type mixed integer symmetric dual programs considered in [13], with the omission of  $\lambda^t e = 1$ , as this is not required for the duality theorems to hold.

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