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Multiobjective mixed symmetric duality involving cones

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ABSTRACT

A pair of multiobjective mixed symmetric dual programs is formulated over arbitrary cones. Weak, strong, converse and self-duality theorems are proved for these programs under K-preinvexity and K-pseudoinvexity assumptions. This mixed symmetric dual formulation unifies the symmetric dual formulations of Suneja et al. (2002) [14] and Khurana (2005) [15].

1. Introduction

The notion of symmetric duality in which the dual of the dual is the primal, found its way in nonlinear programming in the excellent work of Dorn [1]. However, it was significantly developed and studied by Dantzig et al. [2], Mond [3] and Mond and Weir [4]. Subsequently, symmetric duality was generalized to multiobjective case by Weir and Mond [5] and as well as by Gulati et al. [6]. Bazaraa and Goode [7] generalized the results in [2] to arbitrary cones. Mond and Weir [8] presented two pairs of symmetric dual multiobjective programming problems with non-negative orthant as the cone and established appropriate duality results for efficient solutions. Nanda and Das [9] discussed symmetric duality for fractional programming problems over arbitrary cones assuming the functions to be pseudoinvex. Das and Nanda [10] studied symmetric duality in multiobjective programming with cone constraints. Subsequently, Kim et al. [11] derived symmetric duality results for multiobjective programs under pseudoinvex/strictly pseudoinvex functions and arbitrary cones.

Chandra and Abha [12] and Chandra and Kumar [13] pointed out some logical shortcomings in the dual formulations and the proofs of the duality theorems of Das and Nanda [10], Kim et al. [11] and Nanda and Das [9], respectively, and observed that these results are highly restricted as they are not valid even for convex case. Suneja et al. [14] formulated a pair of Wolfe type multiobjective symmetric dual programs over arbitrary cones, in which the objective function is optimized with respect to an arbitrary closed convex cone by assuming the functions involved to be cone-convex. Later on, Khurana [15] formulated a pair of Mond–Weir type multiobjective symmetric dual programs over arbitrary cones and derived symmetric duality theorems involving cone-pseudoinvex and strongly cone-pseudoinvex functions. Recently, Kim and Kim [16] extended the results of Suneja et al. [14] and Khurana [15] to nondifferentiable multiobjective symmetric dual programs for weak efficiency involving cone-invex and cone-pseudoinvex functions.

Motivated by Bector et al. [17], Ahmad [18], Suneja et al. [14] and Khurana [15], we formulate a pair of multiobjective mixed symmetric dual programs over arbitrary cones and establish weak, strong and converse duality theorems under cone-invexity and cone-pseudoinvexity. At the end, it is also shown that under additional property of skew symmetry, (VP) and (VD) are self-duals; and a self-duality theorem is stated. Our mixed symmetric dual formulation unifies two existing symmetric dual formulations discussed in [14,15].

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2. Notations and preliminaries

Let R^n denote the n-dimensional Euclidean space. The following conventions for vectors $x, y \in R^n$ will be followed throughout this paper: $x \leqq y \Leftrightarrow x_i \leqq y_i, i = 1, 2, \ldots, n$; $x \leq y \Leftrightarrow x \leqq y$ and $x \neq y$; $x < y \Leftrightarrow x_i < y_i, i = 1, 2, \ldots, n$. For $N = \{1, 2, \ldots, n\}$ and $M = \{1, 2, \ldots, m\}$, let $J_1 \subseteq N, K_1 \subseteq M$ and $J_2 = N \setminus J_1$ and $K_2 = M \setminus K_1$. Let $|J_1|$ denote the number of elements in J_1. The other symbols $|J_2|$, $|K_1|$ and $|K_2|$ are defined similarly. Let $x^1 \in R^{|J_1|}, x^2 \in R^{|J_2|}$. Then any $x \in R^n$ can be written as (x^1, x^2). Similarly for $y^1 \in R^{|K_1|}$ and $y^2 \in R^{|K_2|}, y \in R^m$ can be written as (y^1, y^2). It may be noted that if $J_1 = \emptyset$, then $|J_1| = 0, J_2 = N$ and therefore $|J_2| = n$. In this case, $R^{|J_1|}$, $R^{|J_2|}$ and $R^{|J_1|} \times R^{|K_1|}$ will be zero-dimensional, n-dimensional and $|K_1|$-dimensional Euclidean spaces, respectively. The other situations are $J_2 = \emptyset, K_1 = \emptyset$ or $K_2 = \emptyset$. These give particular cases of the problems considered in this paper and are discussed in Section 3.

Let $f : R^{|J_1|} \times R^{|K_1|} \to R^l$ and $g : R^{|J_2|} \times R^{|K_2|} \to R^l$. Then $\nabla_{x^1} f(x, y)$ and $\nabla_{x^2} g(x, y)$ denote the $l \times |J_1|$ and $l \times |J_2|$ matrices of first order partial derivatives. For the scalar function λf with $\lambda \in R^l$, $\nabla_{x^1}(\lambda f)$ and $\nabla_{y^1}(\lambda f)$ denote gradient vectors with respect to x^1 and y^1, respectively; $\nabla_{y^1 y^1}(\lambda f)$ and $\nabla_{x^1 x^1}(\lambda f)$ denote respectively the $|K_1| \times |K_1|$ and $|J_1| \times |J_1|$ matrices of second order partial derivatives. $\nabla_{x^2}(\lambda g)$, $\nabla_{y^2}(\lambda g)$, $\nabla_{y^2 y^2}(\lambda g)$ and $\nabla_{x^2 y^2}(\lambda g)$ are defined similarly. It should be clear from the context that there is no notational distinction between row and column vectors.

Consider the following multiobjective programming problem:

$$\textbf{(P)} \quad K\text{-Minimize } f(x)$$
$$\text{Subject to } -g(x) \in Q, \quad x \in C,$$

where $f : R^n \to R^l, g : R^n \to R^m$ and $C \subset R^n, K$ and Q are closed convex cones with nonempty interiors in R^l and R^m, respectively.

Definition 2.1. A feasible point u is said to be a weakly efficient solution of (P), if there exists no feasible x such that

$$f(u) - f(x) \in \operatorname{int} K.$$

Definition 2.2. The positive polar cone K^* of K is defined by

$$K^* = \{z \in R^l : xz \geqq 0, \text{ for all } x \in K\}.$$

We rewrite the following definitions and remark from [16] in different forms:
Let $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ be open and $S_1 \times S_2 \subseteq R^n \times R^m$. Let $\psi(x, y) : S_1 \times S_2 \to R^l$ be differentiable.

Definition 2.3. $\psi(x, y)$ is said to be K-preinvex in x if for each $v \in S_2$, there exists a function $\eta_1 : S_1 \times S_1 \to R^n$ such that

$$\psi(x, v) - \psi(u, v) - \eta_1(x, u) \nabla_x \psi(u, v) \in K, \quad \text{for all } x, u \in S_1,$$

and ψ is said to be K-preinvex in y if for each $x \in S_1$, there exists a function $\eta_2 : S_2 \times S_2 \to R^m$ such that

$$\psi(u, y) - \psi(u, v) - \eta_2(y, v) \nabla_y \psi(u, v) \in K, \quad \text{for all } y, v \in S_2.$$

Remark 2.1. If ψ is K-preinvex in x with respect to η_1, then $\psi(x, v) - \psi(u, v) - \eta_1(x, u) \nabla_x \psi(u, v) \in K$. Moreover, $\forall \lambda \in K^*$,

$$(\lambda\psi)(x, v) - (\lambda\psi)(u, v) - \eta_1(x, u)\nabla_x(\lambda\psi)(u, v) \geqq 0.$$

Definition 2.4. $\psi(x, y)$ is said to be K-pseudoinvex in x if for each $v \in S_2$, there exists a function $\eta_3 : S_1 \times S_1 \to R^n$ such that

$$-\eta_3(x, u) \nabla_x \psi(u, v) \notin \operatorname{int} K \Rightarrow -(\psi(x, v) - \psi(u, v)) \notin \operatorname{int} K, \quad \text{for all } x, u \in S_1,$$

and ψ is said to be K-pseudoinvex in y if for each $x \in S_1$, there exists a function $\eta_4 : S_2 \times S_2 \to R^m$ such that

$$-\eta_4(y, v) \nabla_y \psi(u, v) \notin \operatorname{int} K \Rightarrow -(\psi(u, y) - \psi(u, v)) \notin \operatorname{int} K, \quad \text{for all } v, y \in S_2.$$

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3. Mixed symmetric dual programs

We formulate the following pair of multiobjective mixed symmetric dual programs and prove duality theorems:

(VP) K-Minimize $\left[f(x^1, y^1) + g(x^2, y^2) - \left[y^1 \nabla_{y^1} \lambda f(x^1, y^1)\right] e\right]$

Subject to

$$-\nabla_{y^1} \lambda f(x^1, y^1) \in C_3^*, \tag{1}$$

$$-\nabla_{y^2} \lambda g(x^2, y^2) \in C_4^*, \tag{2}$$

$$y^2 \nabla_{y^2} \lambda g(x^2, y^2) \geqq 0, \tag{3}$$

$$\lambda \in K^*, \qquad e \in \operatorname{int} K, \qquad \lambda e = 1, \qquad x^1 \in C_1, \qquad x^2 \in C_2. \tag{4}$$

(VD) K-Maximize $\left[f(u^1, v^1) + g(u^2, v^2) - \left[u^1 \nabla_{x^1} \lambda f(u^1, v^1)\right] e\right]$

Subject to

$$\nabla_{x^1} \lambda f(u^1, v^1) \in C_1^*, \tag{5}$$

$$\nabla_{x^2} \lambda g(u^2, v^2) \in C_2^*, \tag{6}$$

$$u^2 \nabla_{x^2} \lambda g(u^2, v^2) \leqq 0, \tag{7}$$

$$\lambda \in K^*, \qquad e \in \operatorname{int} K, \qquad \lambda e = 1, \qquad v^1 \in C_3, \qquad v^2 \in C_4, \tag{8}$$

where $f : R^{|J_1|} \times R^{|K_1|} \to R^l$ and $g : R^{|J_2|} \times R^{|K_2|} \to R^l$ are twice differentiable functions and $e = (1, 1, \ldots, 1)$. C_1, C_2, C_3, and C_4 are closed convex cones with nonempty interiors in $R^{|J_1|}$, $R^{|J_2|}$, $R^{|K_1|}$, and $R^{|K_2|}$, respectively. K is a closed convex pointed cone in R^l such that $\operatorname{int} K \neq \emptyset$, and K^* is its positive polar cone.

Remark 3.1. If we set $J_2 = \emptyset$, $K_2 = \emptyset$, then (VP) and (VD) reduce to the Wolfe type symmetric dual programs of Suneja et al. [14]. Similarly, for $J_1 = \emptyset$ and $K_1 = \emptyset$, we get the Mond–Weir type symmetric dual programs discussed in [15].

Theorem 3.1 (*Weak Duality*). *Let $(x^1, x^2, y^1, y^2, \lambda)$ be feasible for (VP) and $(u^1, u^2, v^1, v^2, \lambda)$ be feasible for (VD). Let*

(i) *$f(., v^1)$ be K-preinvex with respect to η_1 for fixed v^1, and $-f(x^1, .)$ be K-preinvex with respect to η_2 for fixed x^1 with $\eta_1(x^1, u^1) + u^1 \in C_1$ and $\eta_2(v^1, y^1) + y^1 \in C_3$; and*

(ii) *$g(., v^2)$ be K-pseudoinvex with respect to η_3 for fixed v^2, and $-g(x^2, .)$ be K-pseudoinvex with respect to η_4 for fixed x^2 with $\eta_3(x^2, u^2) + u^2 \in C_2$ and $\eta_4(v^2, y^2) + y^2 \in C_4$.*

Then

$$\left[f(x^1, y^1) + g(x^2, y^2) - \left[y^1 \nabla_{y^1} \lambda f(x^1, y^1)\right] e\right] - \left[f(u^1, v^1) + g(u^2, v^2) - \left[u^1 \nabla_{x^1} \lambda f(u^1, v^1)\right] e\right] \notin -\operatorname{int} K.$$

Proof. By the K-preinvexity of $f(., v^1)$ and $-f(x^1, .)$ along with Remark 2.1, we have

$$\lambda f(x^1, v^1) - \lambda f(u^1, v^1) \geqq \eta_1(x^1, u^1) \nabla_{x^1} \lambda f(u^1, v^1),$$

and

$$\lambda f(x^1, y^1) - \lambda f(x^1, v^1) \geqq -\eta_2(v^1, y^1) \nabla_{y^1} \lambda f(x^1, y^1).$$

Adding the above inequalities, we obtain

$$\lambda f(x^1, y^1) - \lambda f(u^1, v^1) \geqq \eta_1(x^1, u^1) \nabla_{x^1} \lambda f(u^1, v^1) - \eta_2(v^1, y^1) \nabla_{y^1} \lambda f(x^1, y^1). \tag{9}$$

Since $\eta_1(x^1, u^1) + u^1 \in C_1$ and $\nabla_{x^1} \lambda f(u^1, v^1) \in C_1^*$, we have

$$\eta_1(x^1, u^1) \nabla_{x^1} \lambda f(u^1, v^1) \geqq -u^1 \nabla_{x^1} \lambda f(u^1, v^1). \tag{10}$$

Similarly, $\eta_2(v^1, y^1) + y^1 \in C_3$ and $-\nabla_{y^1} \lambda f(x^1, y^1) \in C_3^*$ imply

$$-\eta_2(v^1, y^1) \nabla_{y^1} \lambda f(x^1, y^1) \geqq y^1 \nabla_{y^1} \lambda f(x^1, y^1). \tag{11}$$

In view of (10) and (11), it follows from (9) that

$$\left[\lambda f(x^1, y^1) - y^1 \nabla_{y^1} \lambda f(x^1, y^1)\right] - \left[\lambda f(u^1, v^1) - u^1 \nabla_{x^1} \lambda f(u^1, v^1)\right] \geqq 0. \tag{12}$$

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From (6) and $\eta_3(x^2, u^2) + u^2 \in C_2$, we have

$$\begin{aligned}\eta_3(x^2, u^2) \nabla_{x^2} \lambda g(u^2, v^2) &\geqq -u^2 \nabla_{x^2} \lambda g(u^2, v^2)\\ &\geqq 0 \quad (\text{by } (7)).\end{aligned}$$

This gives

$$\eta_3(x^2, u^2) \nabla_{x^2} g(u^2, v^2) \notin -\text{int}\, K.$$

Since $g(., v^2)$ is K-pseudoinvex with respect to η_3 for fixed v^2, we have

$$g(x^2, v^2) - g(u^2, v^2) \notin -\text{int}\, K.$$

Then $\lambda \in K^*$ and $\lambda \neq 0$ imply

$$\lambda g(x^2, v^2) - \lambda g(u^2, v^2) \geqq 0. \tag{13}$$

Similarly, from (2) and $\eta_4(v^2, y^2) + y^2 \in C_4$,

$$\begin{aligned}-\eta_4(v^2, y^2) \nabla_{y^2} \lambda g(x^2, y^2) &\geqq y^2 \nabla_{y^2} \lambda g(x^2, y^2)\\ &\geqq 0 \quad (\text{by } (3)).\end{aligned}$$

This gives

$$-\eta_4(v^2, y^2) \nabla_{y^2} g(x^2, y^2) \notin -\text{int}\, K.$$

As $-g(x^2, .)$ is K-pseudoinvex with respect to η_4 for fixed x^2, we get

$$g(x^2, y^2) - g(x^2, v^2) \notin -\text{int}\, K.$$

Therefore $\lambda \in K^*$ and $\lambda \neq 0$ gives

$$\lambda g(x^2, y^2) - \lambda g(x^2, v^2) \geqq 0. \tag{14}$$

Combining (13) and (14) to get

$$\lambda g(x^2, y^2) - \lambda g(u^2, v^2) \geqq 0. \tag{15}$$

Relations (12) and (15) together yield

$$[\lambda f(x^1, y^1) + \lambda g(x^2, y^2) - y^1 \nabla_{y^1} \lambda f(x^1, y^1)] - [\lambda f(u^1, v^1) + \lambda g(u^2, v^2) - u^1 \nabla_{x^1} \lambda f(u^1, v^1)] \geqq 0. \tag{16}$$

Suppose contrary to the result that

$$\left[f(x^1, y^1) + g(x^2, y^2) - \left[y^1 \nabla_{y^1} \lambda f(x^1, y^1)\right] e\right] - \left[f(u^1, v^1) + g(u^2, v^2) - \left[u^1 \nabla_{x^1} \lambda f(u^1, v^1)\right] e\right] \in -\text{int}\, K,$$

which by $\lambda \in K^*$, $\lambda \neq 0$, and $\lambda e = 1$ yields

$$[\lambda f(x^1, y^1) + \lambda g(x^2, y^2) - y^1 \nabla_{y^1} \lambda f(x^1, y^1)] - [\lambda f(u^1, v^1) + \lambda g(u^2, v^2) - u^1 \nabla_{x^1} \lambda f(u^1, v^1)] < 0,$$

which contradicts (16). Hence the result. □

Theorem 3.2 (*Strong Duality*). *Let $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda})$ be a weakly efficient solution of (VP), and $\lambda = \bar{\lambda}$ fixed in (VD). Let*

(a) *the Hessian matrices $\nabla_{y^1y^1} \bar{\lambda} f(\bar{x}^1, \bar{y}^1)$ and $\nabla_{y^2y^2} \bar{\lambda} g(\bar{x}^2, \bar{y}^2)$ be positive definite; and*
(b) *the set $\left\{\nabla_{y^2} g^1(\bar{x}^2, \bar{y}^2), \nabla_{y^2} g^2(\bar{x}^2, \bar{y}^2), \ldots, \nabla_{y^2} g^l(\bar{x}^2, \bar{y}^2)\right\}$ be linearly independent.*

Then $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda})$ is feasible for (VD) and the corresponding objective values of (VP) and (VD) are equal.

Also, if the assumptions of weak duality (Theorem 3.1) hold for all feasible solutions of (VP) and (VD), then $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda})$ is a weakly efficient solution of (VD).

Proof. Since $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda})$ is a weakly efficient solution of (VP). Hence, by the generalized Fritz John type necessary conditions [14], there exist Lagrange multipliers $\alpha \in K^*$, $\beta \in C_3$, $\nu \in C_4$, $\xi \in R_+$, $\mu \in K$, and $\sigma \in R_+$ such that

$$[\nabla_{x^1} \alpha f(\bar{x}^1, \bar{y}^1) + (\beta - \alpha e \bar{y}^1) \nabla_{x^1y^1} \bar{\lambda} f(\bar{x}^1, \bar{y}^1)](x^1 - \bar{x}^1) \geqq 0, \quad \forall\, x^1 \in C_1, \tag{17}$$

$$[\nabla_{x^2} \alpha g(\bar{x}^2, \bar{y}^2) + (\nu - \xi \bar{y}^2) \nabla_{x^2y^2} \bar{\lambda} g(\bar{x}^2, \bar{y}^2)](x^2 - \bar{x}^2) \geqq 0, \quad \forall\, x^2 \in C_2, \tag{18}$$

$$(\alpha - \alpha e \bar{\lambda}) \nabla_{y^1} f(\bar{x}^1, \bar{y}^1) + (\beta - \alpha e \bar{y}^1) \nabla_{y^1y^1} \bar{\lambda} f(\bar{x}^1, \bar{y}^1) = 0, \tag{19}$$

$$(\alpha - \xi \bar{\lambda}) \nabla_{y^2} g(\bar{x}^2, \bar{y}^2) + (\nu - \xi \bar{y}^2) \nabla_{y^2y^2} \bar{\lambda} g(\bar{x}^2, \bar{y}^2) = 0, \tag{20}$$

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$$(\beta-\alpha e\bar{y}^1)\nabla_{y^1} f^i(\bar{x}^1,\bar{y}^1)+(\nu-\xi\bar{y}^2)\nabla_{y^2} g^i(\bar{x}^2,\bar{y}^2)-\mu^i-\sigma=0,\ i=1,2,\ldots,l, \tag{21}$$

$$\beta\nabla_{y^1}\bar{\lambda}f(\bar{x}^1,\bar{y}^1)=0, \tag{22}$$

$$\nu\nabla_{y^2}\bar{\lambda}g(\bar{x}^2,\bar{y}^2)=0, \tag{23}$$

$$\xi\bar{y}^2\nabla_{y^2}\bar{\lambda}g(\bar{x}^2,\bar{y}^2)=0, \tag{24}$$

$$\mu\bar{\lambda}=0, \tag{25}$$

$$(\alpha,\beta,\nu,\xi,\mu,\sigma)\neq 0. \tag{26}$$

Multiplying (21) by $\left[\alpha^i-\alpha e\bar{\lambda}^i\right]$, $i=1,2,\ldots,l$; summing the resulting expression for all i; using $\bar{\lambda}e=1$ and (25), we obtain

$$\begin{aligned}
&(\beta-\alpha e\bar{y}^1)\sum_{i=1}^{l}\nabla_{y^1} f^i(\bar{x}^1,\bar{y}^1)\left[\alpha^i-\alpha e\bar{\lambda}^i\right]+(\nu-\xi\bar{y}^2)\sum_{i=1}^{l}\nabla_{y^2} g^i(\bar{x}^2,\bar{y}^2)\left[\alpha^i-\alpha e\bar{\lambda}^i\right]\\
&=\sum_{i=1}^{l}\mu^i\left[\alpha^i-\alpha e\bar{\lambda}^i\right]+\sigma\sum_{i=1}^{l}\left[\alpha^i-\alpha e\bar{\lambda}^i\right]\\
&=\left[\sum_{i=1}^{l}\mu^i\alpha^i-\alpha e\sum_{i=1}^{l}\mu^i\bar{\lambda}^i\right]+\sigma\left[\sum_{i=1}^{l}\alpha^i-\alpha e\sum_{i=1}^{l}\bar{\lambda}^i\right]\\
&=\left[\sum_{i=1}^{l}\mu^i\alpha^i-0\right]+\sigma\left[\alpha e-\alpha e\right]\quad\left(\because\sum_{i=1}^{l}\mu^i\bar{\lambda}^i=\mu\bar{\lambda}=0\text{ and }\sum_{i=1}^{l}\bar{\lambda}^i=1\right)\\
&=\sum_{i=1}^{l}\mu^i\alpha^i\\
&=\mu\alpha.
\end{aligned}$$

That is,

$$(\beta-\alpha e\bar{y}^1)\nabla_{y^1} f(\bar{x}^1,\bar{y}^1)\left[\alpha-\alpha e\bar{\lambda}\right]+(\nu-\xi\bar{y}^2)\nabla_{y^2} g(\bar{x}^2,\bar{y}^2)\left[\alpha-\alpha e\bar{\lambda}\right]=\mu\alpha,$$

which along with (23) and (24) gives

$$(\beta-\alpha e\bar{y}^1)\nabla_{y^1} f(\bar{x}^1,\bar{y}^1)\left[\alpha-\alpha e\bar{\lambda}\right]+(\nu-\xi\bar{y}^2)\nabla_{y^2} g(\bar{x}^2,\bar{y}^2)\alpha=\mu\alpha. \tag{27}$$

Multiplying (19) by $(\beta-\alpha e\bar{y}^1)$, (20) by $(\nu-\xi\bar{y}^2)$, and then adding to get

$$\begin{aligned}
&(\beta-\alpha e\bar{y}^1)\nabla_{y^1} f(\bar{x}^1,\bar{y}^1)(\alpha-\alpha e\bar{\lambda})+(\nu-\xi\bar{y}^2)\nabla_{y^2} g(\bar{x}^2,\bar{y}^2)(\alpha-\xi\bar{\lambda})\\
&\quad+(\beta-\alpha e\bar{y}^1)(\nabla_{y^1y^1}\bar{\lambda}f(\bar{x}^1,\bar{y}^1))(\beta-\alpha e\bar{y}^1)+(\nu-\xi\bar{y}^2)(\nabla_{y^2y^2}\bar{\lambda}g(\bar{x}^2,\bar{y}^2))(\nu-\xi\bar{y}^2)=0.
\end{aligned}$$

Or

$$\begin{aligned}
&(\beta-\alpha e\bar{y}^1)\nabla_{y^1} f(\bar{x}^1,\bar{y}^1)(\alpha-\alpha e\bar{\lambda})+(\nu-\xi\bar{y}^2)\nabla_{y^2} g(\bar{x}^2,\bar{y}^2)\alpha-\xi(\nu-\xi\bar{y}^2)\nabla_{y^2}\bar{\lambda}g(\bar{x}^2,\bar{y}^2)\\
&\quad+(\beta-\alpha e\bar{y}^1)(\nabla_{y^1y^1}\bar{\lambda}f(\bar{x}^1,\bar{y}^1))(\beta-\alpha e\bar{y}^1)+(\nu-\xi\bar{y}^2)(\nabla_{y^2y^2}\bar{\lambda}g(\bar{x}^2,\bar{y}^2))(\nu-\xi\bar{y}^2)=0.
\end{aligned} \tag{28}$$

Using (23), (24) and (27) in (28), we obtain

$$(\beta-\alpha e\bar{y}^1)(\nabla_{y^1y^1}\bar{\lambda}f(\bar{x}^1,\bar{y}^1))(\beta-\alpha e\bar{y}^1)+(\nu-\xi\bar{y}^2)(\nabla_{y^2y^2}\bar{\lambda}g(\bar{x}^2,\bar{y}^2))(\nu-\xi\bar{y}^2)+\mu\alpha=0. \tag{29}$$

Since $\alpha\in K^*$ and $\mu\in K$, $\mu\alpha\geqq 0$, it follows from (29) that

$$(\beta-\alpha e\bar{y}^1)(\nabla_{y^1y^1}\bar{\lambda}f(\bar{x}^1,\bar{y}^1))(\beta-\alpha e\bar{y}^1)+(\nu-\xi\bar{y}^2)(\nabla_{y^2y^2}\bar{\lambda}g(\bar{x}^2,\bar{y}^2))(\nu-\xi\bar{y}^2)\leqq 0, \tag{30}$$

which by hypothesis (a) implies

$$\beta-\alpha e\bar{y}^1=0, \tag{31}$$

and

$$\nu-\xi\bar{y}^2=0. \tag{32}$$

Using (31) and (32) in (21), we get $\mu^i+\sigma=0$, $i=1,2,\ldots,l$. This on multiplying by $\bar{\lambda}^i$, $i=1,2,\ldots,l$, and on using (25) with $\bar{\lambda}e=1$ gives $\sigma=0$, and therefore $\mu=0$.

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From (20) and (32), we have

$$(\alpha - \xi\bar{\lambda}) \nabla_{y^2} g(\bar{x}^2, \bar{y}^2) = 0,$$

which by assumption (b) gives

$$\alpha - \xi\bar{\lambda} = 0. \tag{33}$$

If $\xi = 0$, then (32) and (33) imply $\nu = 0$ and $\alpha = 0$. Whenever $\alpha = 0$, (31) gives $\beta = 0$. Therefore, $(\alpha, \beta, \nu, \xi, \mu, \sigma) = 0$, a contradiction to (26). Hence

$$\xi > 0. \tag{34}$$

From (33), it is clear that $\alpha e = \xi\bar{\lambda}e = \xi$. Since $\xi > 0$, then αe is obviously strictly positive. Thus, from (31) and (32), we get

$$\bar{y}^1 = \frac{\beta}{\alpha e} \in C_3, \text{ and } \bar{y}^2 = \frac{\nu}{\xi} \in C_4. \tag{35}$$

From Eqs. (17), (31), (33) and (34),

$$[\nabla_{x^1} \bar{\lambda} f(\bar{x}^1, \bar{y}^1)](x^1 - \bar{x}^1) \geqq 0, \quad \forall\, x^1 \in C_1. \tag{36}$$

Let $x^1 \in C_1$. Then $x^1 + \bar{x}^1 \in C_1$, as C_1 is a closed convex cone, and so (36) shows that $\forall\, x^1 \in C_1$

$$x^1 \nabla_{x^1} \bar{\lambda} f(\bar{x}^1, \bar{y}^1) \geqq 0,$$

which implies

$$\nabla_{x^1} \bar{\lambda} f(\bar{x}^1, \bar{y}^1) \in C_1^*. \tag{37}$$

Moreover, Eqs. (18) and (32)–(34) give

$$[\nabla_{x^2} \bar{\lambda} g(\bar{x}^2, \bar{y}^2)](x^2 - \bar{x}^2) \geqq 0, \quad \forall\, x^2 \in C_2. \tag{38}$$

Let $x^2 \in C_2$. Then $x^2 + \bar{x}^2 \in C_2$, as C_2 is a closed convex cone, and so (38) implies that $\forall x^2 \in C_2$

$$x^2 \nabla_{x^2} \bar{\lambda} g(\bar{x}^2, \bar{y}^2) \geqq 0,$$

which yields

$$\nabla_{x^2} \bar{\lambda} g(\bar{x}^2, \bar{y}^2) \in C_2^*. \tag{39}$$

Also, letting $x^2 = 0$ and $x^2 = 2\bar{x}^2$ simultaneously in (38) to get

$$\bar{x}^2 \nabla_{x^2} \bar{\lambda} g(\bar{x}^2, \bar{y}^2) = 0. \tag{40}$$

The Eqs. (35), (37), (39) and (40) give the feasibility of $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda})$ for (VD).
Similarly, by putting $x^1 = 0$ and $x^1 = 2\bar{x}^1$ simultaneously in (36), we get

$$\bar{x}^1 \nabla_{x^1} \bar{\lambda} f(\bar{x}^1, \bar{y}^1) = 0.$$

Moreover, (22), (31) and $\alpha e > 0$ yield

$$\bar{y}^1 \nabla_{y^1} \bar{\lambda} f(\bar{x}^1, \bar{y}^1) = 0.$$

Therefore $\bar{x}^1 \nabla_{x^1} \bar{\lambda} f(\bar{x}^1, \bar{y}^1) = \bar{y}^1 \nabla_{y^1} \bar{\lambda} f(\bar{x}^1, \bar{y}^1) = 0$, which shows that two objective values are equal.

We shall now show the weak efficiency of $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda})$ for (VD), otherwise there exists a feasible solution $(u^1, u^2, v^1, v^2, \bar{\lambda})$ of (VD) such that

$$\left[f(\bar{x}^1, \bar{y}^1) + g(\bar{x}^2, \bar{y}^2) - \left[\bar{x}^1 \nabla_{x^1} \bar{\lambda} f(\bar{x}^1, \bar{y}^1)\right] e\right] - \left[f(u^1, v^1) + g(u^2, v^2) - \left[u^1 \nabla_{x^1} \bar{\lambda} f(u^1, v^1)\right] e\right] \in -\text{int}\, K.$$

Since $\bar{x}^1 \nabla_{x^1} \bar{\lambda} f(\bar{x}^1, \bar{y}^1) = \bar{y}^1 \nabla_{y^1} \bar{\lambda} f(\bar{x}^1, \bar{y}^1) = 0$, we have

$$\left[f(\bar{x}^1, \bar{y}^1) + g(\bar{x}^2, \bar{y}^2) - \left[\bar{y}^1 \nabla_{y^1} \bar{\lambda} f(\bar{x}^1, \bar{y}^1)\right] e\right] - \left[f(u^1, v^1) + g(u^2, v^2) - \left[u^1 \nabla_{x^1} \bar{\lambda} f(u^1, v^1)\right] e\right] \in -\text{int}\, K,$$

which contradicts weak duality (Theorem 3.1). Hence the result. □

A converse duality theorem may merely be stated, as its proof would run analogous to that of Theorem 3.2.

Theorem 3.3 (*Converse Duality*). *Let $(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{\lambda})$ be a weakly efficient solution of (VD) and $\lambda = \bar{\lambda}$ fixed in (VP). Let*

(a′) *the Hessian matrices $\nabla_{x^1x^1} \bar{\lambda} f(\bar{u}^1, \bar{v}^1)$ and $\nabla_{x^2x^2} \bar{\lambda} g(\bar{u}^2, \bar{v}^2)$ be negative definite; and*

(b′) *the set* $\left\{\nabla_{x^2} g^1(\bar{u}^2, \bar{v}^2), \nabla_{x^2} g^2(\bar{u}^2, \bar{v}^2), \ldots, \nabla_{x^2} g^l(\bar{u}^2, \bar{v}^2)\right\}$ *be linearly independent.*

Then $(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{\lambda})$ *is feasible for (VP) and the corresponding objective values of (VP) and (VD) are equal.*

Also, if the assumptions of weak duality (Theorem 3.1) hold for all feasible solutions of (VP) and (VD), then $(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{\lambda})$ *is a weakly efficient solution of (VP).*

4. Self-duality

A primal (dual) problem having equivalent dual (primal) formulation is said to be self-dual, that is, if the dual can be recast in the form of the primal. In general, (VP) and (VD) are not self-duals without some added restrictions on f, g, C_1, C_2, C_3 and C_4. If we assume $C_1 = C_2 = C_3 = C_4 = C$; $f : R^{|J_1|} \times R^{|J_1|} \to R^l$ and $g : R^{|J_1|} \times R^{|J_1|} \to R^l$ to be skew symmetric, that is

$$f^i(u^1, v^1) = -f^i(v^1, u^1) \quad \text{and} \quad g^i(u^2, v^2) = -g^i(v^2, u^2), \quad i = 1, 2, \ldots, l,$$

then we shall show that (VP) and (VD) are self-duals. By recasting the dual problem (VD) as a minimization problem, we have

$$\begin{aligned}
&(\mathrm{VD}_0) \quad \text{Minimize } -\left[f(u^1, v^1) + g(u^2, v^2) - \left[u^1 \nabla_{x^1} \lambda f(u^1, v^1)\right] e\right] \\
&\text{Subject to} \\
&\nabla_{x^1} \lambda f(u^1, v^1) \in C^*, \\
&\nabla_{x^2} \lambda g(u^2, v^2) \in C^*, \\
&u^2 \nabla_{x^2} \lambda g(u^2, v^2) \leqq 0, \\
&\lambda \in K^*, e \in \operatorname{int} K, \lambda e = 1, v^1 \in C, v^2 \in C.
\end{aligned}$$

As f and g are skew symmetric, $\nabla_{x^1} \lambda f(u^1, v^1) = -\nabla_{y^1} \lambda f(v^1, u^1)$ and $\nabla_{x^2} \lambda g(u^2, v^2) = -\nabla_{y^2} \lambda g(v^2, u^2)$, above problem becomes:

$$\begin{aligned}
&(\mathrm{VD}_0) \quad \text{Minimize } \left[f(v^1, u^1) + g(v^2, u^2) - \left[u^1 \nabla_{y^1} \lambda f(v^1, u^1)\right] e\right] \\
&\text{Subject to} \\
&-\nabla_{y^1} \lambda f(v^1, u^1) \in C^*, \\
&-\nabla_{y^2} \lambda g(v^2, u^2) \in C^*, \\
&u^2 \nabla_{y^2} \lambda g(v^2, u^2) \geqq 0, \\
&\lambda \in K^*, \qquad e \in \operatorname{int} K, \qquad \lambda e = 1, \qquad v^1 \in C, \qquad v^2 \in C.
\end{aligned}$$

This shows that (VD_0) is formally identical to (VP), that is, the objective and the constraint functions are identical. Thus, the problem (VP) becomes self-dual in the spirit of Dorn [1].

It is obvious that the feasibility of $(x^1, x^2, y^1, y^2, \lambda)$ for (VP) implies the feasibility of $(y^1, y^2, x^1, x^2, \lambda)$ for (VD) and vice versa.

We now state the following self-duality theorem. Its proof follows on the lines of Weir and Mond [5].

Theorem 4.1 (*Self-Duality*). *Let f and g be skew symmetric and let $C_1 = C_2 = C_3 = C_4 = C$. Then (VP) is a self-dual. Also, if (VP) and (VD) are dual problems and* $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda})$ *is a joint weakly efficient solution, then so is* $(\bar{y}^1, \bar{y}^2, \bar{x}^1, \bar{x}^2, \bar{\lambda})$ *and the common objective function value is zero.*

5. Conclusion

A pair of multiobjective mixed symmetric dual programs has been formulated by considering the optimization with respect to an arbitrary cone under the assumptions of cone-invexity and cone-pseudoinvexity. It may be noted that the symmetric duality between (VP) and (VD) can be utilized to establish mixed symmetric duality in integer and other related programming problems.

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