

How to determine the law of the solution to a SPDE driven by a Lévy space-time noise

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Abstract

We consider a stochastic partial differential equation (SPDE) on a lattice

$$\partial_t X = (\Delta - m^2)X - \lambda X^p + \eta$$

where η is a space-time Lévy noise. A perturbative (in the sense of formal power series) strong solution is given by a tree expansion, whereas the correlation functions of the solution are given by a perturbative expansion with coefficients that are represented as sums over a certain class of graphs, called Parisi-Wu graphs. The perturbative expansion of the truncated (connected) correlation functions is obtained via a Linked Cluster Theorem as a sums over connected graphs only. The moments of the stationary solution can be calculated as well. In all these solutions the cumulants of the single site distribution of the noise enter as multiplicative constants. To determine them, e.g. by comparison with an empirical correlation function, one can fit these constants (e.g. by the methods of least squares) and thereby one (approximately) determines

laws of the solution and the driving noise.

Key words : SPDEs, Perturbation theory, Parisi-Wu method, Lévy noise.

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1 Introduction and overview

In this work we give a sufficiently general perturbative solution to the Stochastic Partial Differential Equation (SPDE)

$$\begin{cases} \frac{\partial X(t, \mathbf{x})}{\partial t} = \Delta X(t, \mathbf{x}) - m^2 X(t, \mathbf{x}) - \lambda X^p(t, \mathbf{x}) + \eta(t, \mathbf{x}) & , \lambda > 0, m > 0, p \in \mathbb{N} \\ X(0, \mathbf{x}) = f(\mathbf{x}), (t, \mathbf{x}) \in]0, \infty[\times L_\delta \end{cases} \quad (1)$$

where $\eta(t, \mathbf{x})$ is a general space-time noise of Lévy type with moments of arbitrary order such that one can extract information on the distribution of η by comparison with empirical data. To avoid so-called ultra-violet (UV) divergences, we consider this equation on a spacial lattice $L_\delta = \{\delta z, z \in \mathbb{Z}^d\}$ with lattice spacing $\delta > 0$.

Nonlinear SPDEs driven by non Gaussian noise have been discussed both from the physical [14, Section 4.2 and Refs.] and the mathematical side, see e.g. [3, 4]. But so far there seems to be no recipe to find out which kind of noise – given the general structure of the equation, including, say, the numerical values of all coefficients the right hand side – explains a set of empirical data best. But this is a necessary step on the way to determine the solution. It is the aim of this article to give such a recipe by calculating the correlation (moment) functions of the solution X perturbatively. To do so, we generalize the classical Feynman graph approach of Parisi and Wu [17] on the Gaussian stochastic quantization equation to the general Lévy case, using a generalized class of graphs, henceforth called Parisi-Wu graphs. Statistically relevant quantities as expectation, variance and higher order cumulants of the noise enter in the expansion as simple constants – like additional coupling constants – and can therefore be extracted rather easily by

comparison with the data.

The restriction to equation (1) is not essential at all for the combinatorial graph calculus that we develop. The following generalizations are rather obvious: i) the nonlinearity X^p can be replaced by a polynomial, e.g. the Mexican hat potential $2X(X^2 - a)$ in equations of reaction-diffusion type; ii) nonlinearities include derivatives, like $|\nabla X|^2$ in the KPZ-equation [5, 14]; iii) in addition the linear operator content on the right hand side is of higher order as in the non-linear description of nuclear beam epitaxy, where the right hand side of (1) is $-K\Delta^2 X - \lambda\Delta|\nabla X|^2 + \eta$ [5, 14]; iv) the left hand side is of second order $\partial_t^2 X$, as in models of thermalization and symmetry breaking in the early universe [6]. All these changes can be incorporated in our formalism by adaptation of the so-called Feynman rules without changing the combinatorial core of this paper. Hence, it is mostly for notational convenience that here we only consider the SPDE (1).

A perturbative expansion in the sense of formal power series without any control on the convergence of the series, as done in this work, can certainly be criticized from a mathematical point of view. In fact, even the existence of a solution to (1) for the general Lévy case to our best knowledge has not been studied though the restrictions on the nonlinearities in the continuum [4] that exclude polynomial interactions are related to the above mentioned UV-singularities and most likely can be dropped in the lattice regularized case. In this work we happily follow the tradition of perturbative quantum field theory (QFT) that, in spite of this, such expansions deliver (after suitable resummation [20, Chapters 42.5–42.7]) highly precise numerics.

The paper consists of seven sections. In the next section we give a solution of (1) in the sense of formal power series in the parameter λ

$$X(t, \mathbf{x}) = \sum_{j=0}^{\infty} (-\lambda)^j X_j(t, \mathbf{x}). \quad (2)$$

Here X_0 is the solution of the linear equation ($\lambda = 0$) and the X_j , under suitable conditions on the stochastic driving force η , are determined recursively.

In section 3 we represent this solution as a sum over rooted trees with two types of leaves, cf. Appendix A for the graph theoretic notions used in this article. One type of leaf is standing for the noise η and the other for the initial condition, cf. the following first order example for

$p = 3$

$$X(t, \mathbf{x}) = \text{diag}(\otimes) + \text{diag}(\ominus) + \lambda \left[\text{diag}(\bullet) + \text{diag}(\otimes) + 3 \text{diag}(\ominus) + 3 \text{diag}(\bullet) \right] + o(\lambda^2) \quad (3)$$

In section 4 we then develop the graphical calculus for the perturbative evaluation of the correlation functions, generalizing what in physics is called the Parisi-Wu method from Gaussian to Lévy noise. While only the well-known Feynman graphs occur in the Gaussian case, we have to pass on to a more general class of graphs in order to deal also with Lévy type noise. We also prove a Linked Cluster Theorem that shows that the truncated correlation functions (cumulants of the process X) are given by the sum over connected Parisi-Wu graphs, only. So far no detailed assumptions on the stochastic driving force were needed.

In section 5 we show how the rules for the analytic evaluation of graphs simplifies if the noise η is to be taken as a white noise of Lévy type. We also show that a Lévy noise η fulfills the requirement for the existence of the perturbative solution as specified in section 2.

In section 6 we study the limit where t goes to infinity and give the perturbative expansion of equilibrium correlation functions.

In section 7 we briefly study the question how to extract statistical information on the noise from an empirical correlation function, that e.g. could be obtained by X-ray analysis or sampling methods. To illustrate this point we solve a least square minimization problem to first order perturbation theory and give some indications, how to perform higher order calculations.

2 Formal solution of the SPDE

In this section we give a formal solution of stochastic partial differential equation (1).

We first fix some notation. Let $\Gamma = \mathbb{R} \times L_\delta = \{x = (t, \mathbf{x}) ; t \in \mathbb{R}, \mathbf{x} \in L_\delta\}$ and $\Pi_\delta^d \cong [0, \frac{2\pi}{\delta}]^d$ where the opposite edges are being identified. Let $B = L_\delta$ or $B = \Gamma$. By $S(B)$ we denote the Schwartz Space of all rapidly decreasing functions on B endowed with the Schwartz topology, its topological dual is the Space of tempered distribution noted by $S'(B)$. We denote by $\langle \cdot, \cdot \rangle$ the dual pairing between $S(B)$ and $S'(B)$.

Now we introduce the following notation : For $f \in S(\Gamma)$, $\int f(x) dx = \sum_{x \in L_\delta} \delta^d \int_{\mathbb{R}} f(t, \mathbf{x}) dt$, $x = (t, \mathbf{x})$. For $f \in S(L_\delta)$, $\int f(\mathbf{x}) d\mathbf{x} = \sum_{x \in L_\delta} \delta^d f(\mathbf{x})$. Furthermore, for $A \subset \Gamma$ ($A \subset L_\delta$), $\int_A f(x) dx = \int 1_A(x) f(x) dx$ ($\int_A f(\mathbf{x}) d\mathbf{x} = \int 1_A(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}$) where 1_A is the characteristic function of the set A and $f \in S(\Gamma)$ ($f \in S(L_\delta)$).

Let $\mathcal{F} : S(\Gamma) \longrightarrow \mathbb{R} \times \Pi_\delta^d$ be the Fourier transform on $S(\Gamma)$

i.e.

$$\mathcal{F}(f)(E, \mathbf{p}) = \int e^{itE} e^{i\mathbf{p} \cdot \mathbf{x}} f(x) dx, \quad f \in S(\Gamma). \quad (4)$$

The inverse Fourier transform of, \mathcal{F} , is given by

$$f(t, \mathbf{x}) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}} \int_{\Pi_\delta^d} e^{-iEt} e^{-i\mathbf{p} \cdot \mathbf{x}} \mathcal{F}(f)(E, \mathbf{p}) dE d\mathbf{p}. \quad (5)$$

and let $\tilde{\mathcal{F}} : S(L_\delta) \longrightarrow \Pi_\delta^d$ be the Fourier transform on $S(L_\delta)$

i.e :

$$\tilde{\mathcal{F}}(f)(\mathbf{p}) = \int e^{i\mathbf{p} \cdot \mathbf{x}} f(\mathbf{x}) d\mathbf{x}, \quad f \in S(L_\delta). \quad (6)$$

The inverse Fourier transform of, $\tilde{\mathcal{F}}$ is given by :

$$f(\mathbf{x}) = \frac{1}{(2\pi)^d} \int_{\Pi_\delta^d} e^{-i\mathbf{p} \cdot \mathbf{x}} \tilde{\mathcal{F}}(f)(\mathbf{p}) d\mathbf{p}. \quad (7)$$

By definition the action, of the Lattice Laplacian Δ on a test function $f \in S(L_\delta)$ is as follows :

$$\Delta f(\mathbf{x}) = \delta^{-2} [-2d f(\mathbf{x}) + \sum_{|\mathbf{x}-\mathbf{y}|=\delta} f(\mathbf{y})]. \quad (8)$$

Then

$$\tilde{\mathcal{F}}(\Delta f)(\mathbf{p}) = \delta^{-2} \left[-2d + 2 \sum_{j=1}^d \cos(\delta \mathbf{p}_j) \right] \tilde{\mathcal{F}}(f)(\mathbf{p}). \quad (9)$$

Where $e_j = (0, \dots, 1, 0, \dots)$ is the canonical basis of \mathbb{R}^d and $\mathbf{p}_j = \mathbf{p} \cdot e_j$.

We define

$$\mu_{\delta, m}^2(\mathbf{p}) = 2\delta^{-2} \left(d - \sum_{j=1}^d \cos(\delta \mathbf{p}_j) \right) + m^2. \quad (10)$$

We introduce two convolutions product " $*$ " and " \star " respectively by

$$f * g(x) = \int f(x-y) g(y) dy, \quad f, g \in S(\Gamma). \quad (11)$$

and

$$f \star g(\mathbf{x}) = \int f(\mathbf{x} - \mathbf{y})g(\mathbf{y}) d\mathbf{y} , f, g \in S(L_\delta). \quad (12)$$

Let $G(t, \mathbf{x})$ be the Green function which satisfies :

$$\begin{cases} \frac{\partial G(t, \mathbf{x})}{\partial t} = \Delta G(t, \mathbf{x}) - m^2 G(t, \mathbf{x}) + \delta(x) & , \\ G(t, \mathbf{x}) = 0, & t < 0 \end{cases} \quad (13)$$

Here $\delta(x)$ is the Dirac distribution defined by $\delta(x) = \delta(t)\delta(\mathbf{x}) = \delta(t)\delta^{-d}\delta_{0, \mathbf{x}}$ where $\delta(t)$ is the Dirac distribution on \mathbb{R} and $\delta_{\mathbf{x}, \mathbf{y}} = \prod_{l=1}^d \delta_{\delta^{-1}x_l, \delta^{-1}y_l}$ with $\delta_{i,j}$ the Kronecker symbol.

Applying to the Fourier transform \mathcal{F} , it is obvious that

$$\mathcal{F}(G)(E, \mathbf{p}) = \frac{1}{-iE + \mu_{\delta, m}^2(\mathbf{p})} . \quad (14)$$

Let $\tilde{G}_t(\mathbf{x})$ be the Green function which satisfies :

$$\begin{cases} \frac{\partial \tilde{G}_t(\mathbf{x})}{\partial t} = \Delta \tilde{G}_t(\mathbf{x}) - m^2 \tilde{G}_t(\mathbf{x}) & , \\ \tilde{G}_0(\mathbf{x}) = \delta(\mathbf{x}). \end{cases} \quad (15)$$

Then by application of the Fourier transform $\tilde{\mathcal{F}}$ we get

$$\tilde{\mathcal{F}}(\tilde{G}_t)(\mathbf{p}) = e^{-t\mu_{\delta, m}^2(\mathbf{p})} . \quad (16)$$

By the use of the residue Theorem, one can see that G and \tilde{G}_t are related by the following equation

$$G(t, \mathbf{x}) = \theta(t)\tilde{G}_t(\mathbf{x}), \quad (17)$$

where $\theta(t) = 1$ if $t > 0$ and $\theta(t) = 0$ else. Let $\Delta_{\Pi_\delta^d}$ be the Laplacian with cyclic boundary condition on $[0, \frac{2\pi}{\delta}] \cong \Pi_\delta^d$. We get

Lemma 2.1. $\forall N \in \mathbb{N}, \exists K_N$ such that :

$$|G(t, \mathbf{x})| \leq K_N \frac{e^{-\frac{tm^2}{2}}}{(1 + |\mathbf{x}|^2)^N}, \quad (t, \mathbf{x}) \in]0, \infty[\times L_\delta.$$

Proof. It suffices to prove

$$\sup_{\mathbf{x}} |G(t, \mathbf{x}) (1 + |\mathbf{x}|^2)^N| < K_N e^{-\frac{tm^2}{2}}. \quad (18)$$

$$\begin{aligned} G(t, \mathbf{x}) (1 + |\mathbf{x}|^2)^N &= \frac{1}{(2\pi)^d} (1 + |\mathbf{x}|^2)^N \theta(t) \int_{\Pi_\delta^d} e^{i\mathbf{p}\cdot\mathbf{x}} e^{-t\mu_{\delta, m}^2(\mathbf{p})} d\mathbf{p} \\ &= \frac{1}{(2\pi)^d} \theta(t) \int_{\Pi_\delta^d} \left[(1 - \Delta_{\Pi_\delta^d})^N e^{i\mathbf{p}\cdot\mathbf{x}} \right] e^{-t\mu_{\delta, m}^2(\mathbf{p})} d\mathbf{p} \\ &= \frac{1}{(2\pi)^d} \theta(t) \int_{\Pi_\delta^d} e^{i\mathbf{p}\cdot\mathbf{x}} \left[(1 - \Delta_{\Pi_\delta^d})^N e^{-t\mu_{\delta, m}^2(\mathbf{p})} \right] d\mathbf{p}. \end{aligned} \quad (19)$$

Then

$$\sup_{x \in L_\delta} |G(t, \mathbf{x}) (1 + |\mathbf{x}|^2)^N| \leq e^{-\frac{tm^2}{2}} \left(\frac{2\pi}{\delta} \right)^d \sup_{t \in (0, \infty), \mathbf{p} \in \Pi_\delta^d} \left| (1 - \Delta_{\Pi_\delta^d})^N e^{-t\mu_{\delta, \frac{m}{\sqrt{2}}}^2(\mathbf{p})} \right|. \quad (20)$$

Now take

$$K_N = \left(\frac{2\pi}{\delta} \right)^d \sup_{t, \mathbf{p}} \left| (1 - \Delta_{\Pi_\delta^d})^N e^{-t\mu_{\delta, \frac{m}{\sqrt{2}}}^2(\mathbf{p})} \right|. \quad (21)$$

■

Definition 2.2. We define a space of all measurable and polynomially bounded functions by :

$$\mathcal{M}_b = \left\{ f : \Gamma \longrightarrow \mathbb{R} \text{ measurable; } \forall k \in \mathbb{N}, \exists N \in \mathbb{N} : \int \frac{|f|^k(x)}{(1 + |x|^2)^N} dx < \infty \right\} \quad (22)$$

Lemma 2.3. \mathcal{M}_b is an Algebra under multiplication.

Proof. For $f, g \in \mathcal{M}_b$, fg and $f + ag$, $a \in \mathbb{R}$, is a measurable function. The vector space structure of \mathcal{M}_b easily follows from the triangular inequality.

Furthermore, for $k \in \mathbb{N}$ fixed, $\exists N \in \mathbb{N}$ such that

$$\int \frac{f^{2k}(x)}{(1 + |x|^2)^N} dx < \infty \quad \text{and} \quad \int \frac{g^{2k}(x)}{(1 + |x|^2)^N} dx < \infty \quad (23)$$

Due to the Cauchy Schwartz inequality,

$$\int \frac{|fg|^k(x)}{(1 + |x|^2)^N} dx \leq \sqrt{\int \frac{f^{2k}(x)}{(1 + |x|^2)^N} dx \int \frac{g^{2k}(x)}{(1 + |x|^2)^N} dx} < \infty. \quad (24)$$

■

Lemma 2.4. *Let $f \in \mathcal{M}_b$ then $G * f \in \mathcal{M}_b$.*

Proof. Let $f \in \mathcal{M}_b$. We have due to Lemma 3.2

$$\begin{aligned}
\int \frac{|G * f|^k(x)}{(1 + |x|^2)^{2kN}} dx &\leq C(N) \int \frac{\int \frac{|f(y_1)|}{(1 + |x - y_1|^2)^N} dy_1 \cdots \int \frac{|f(y_k)|}{(1 + |x - y_k|^2)^N} dy_k}{(1 + |x|^2)^{2kN}} dx \\
&= C(N) \int \frac{\prod_{l=1}^k |f(y_l)}{\prod_{l=1}^k [(1 + |x - y_l|^2)^N (1 + |x|^2)^{2N}]} dx dy_1 \cdots dy_k \\
&\leq 4^k C(N) \left(\int \frac{|f(y)|}{(1 + |y|^2)^N} \right)^k \int \frac{dx}{(1 + |x|^2)^{Nk}}. \tag{25}
\end{aligned}$$

Here the right hand side is finite for N sufficiently large. The last inequality is due to the fact that

$$4(1 + |x - y|^2)(1 + |x|^2)^2 \geq (1 + |x|^2)(1 + |y|^2) \quad \forall x, y \in \Gamma. \tag{26}$$

■

Proposition 2.5. *Suppose that $G * \eta \in \mathcal{M}_b$ a.s.¹. Let $X(t, \mathbf{x}) = \sum_{j=0}^{\infty} (-\lambda)^j X_j(t, \mathbf{x})$ be an expansion of X in the sense of formal power series in the parameter λ . The perturbative solution of the stochastic differential equation (1) is given by*

$$\begin{cases} X_0(t, \mathbf{x}) = G * \eta(t, \mathbf{x}) + \tilde{G}_t * f(\mathbf{x}) \\ X_j(t, \mathbf{x}) = G * \sum_{B_p^j(n_0, n_1, \dots)} \frac{p!}{n_0! n_1! \dots} \prod_{i \geq 0} X_i^{n_i}(t, \mathbf{x}), \quad j \geq 1. \end{cases} \tag{27}$$

with $X_j \in \mathcal{M}_b$ a.s.. Here $B_p^j(n_0, n_1, \dots) = \{n_0, n_1, \dots \geq 0 \mid n_0 + n_1 + \dots = p; \sum_{i \geq 0} i n_i = j - 1\}$.

Proof. We have the integrated form of the SPDE

$$X(t, \mathbf{x}) = -\lambda G * (X^p)(t, \mathbf{x}) + G * \eta(t, \mathbf{x}) + \tilde{G}_t * f(\mathbf{x}) \tag{28}$$

It is obvious that the solution of this equation for the zero-th order in λ is given by :

$$X_0(t, \mathbf{x}) = G * \eta(t, \mathbf{x}) + \tilde{G}_t * f(\mathbf{x}), \tag{29}$$

¹For Lévy noise with all moments this will be verified explicitly, cf. Proposition 5.3

where G and \tilde{G} are the Green functions defined earlier. Now we determine the solution of higher order in λ . We have

$$\begin{aligned} \sum_{j=0}^{\infty} (-\lambda)^j X_j(t, \mathbf{x}) &= -\lambda G * \left(\sum_{j=0}^{\infty} (-\lambda)^j X_j \right)^p (t, \mathbf{x}) + G * \eta(t, \mathbf{x}) + \tilde{G}_t \star f(\mathbf{x}) \\ &= -\lambda G * \left(\sum_{\substack{n_0, n_1, \dots \geq 0 \\ n_0 + n_1 + \dots = p}} \frac{p!}{n_0! n_1! \dots} \prod_{i \geq 0} \lambda^{in_i} X_i^{n_i}(t, \mathbf{x}) \right) \\ &\quad + G * \eta(t, \mathbf{x}) + \tilde{G}_t \star f(\mathbf{x}). \end{aligned}$$

By comparison of coefficients we get

$$X_j(t, \mathbf{x}) = G * \sum_{\substack{n_0, n_1, \dots \geq 0 \\ n_0 + n_1 + \dots = p \\ 0 \times n_0 + 1 \times n_1 + \dots = j-1}} \frac{p!}{n_0! n_1! \dots} \prod_{i \geq 0} X_i^{n_i}(t, \mathbf{x}) ; \quad j \geq 1. \quad (30)$$

$X_0 \in \mathcal{M}_b$ a.s. by our assumptions. Using Lemma 2.3 and Lemma 2.4 it follows by induction that $X_j \in \mathcal{M}_b$ a.s. $\forall j \in \mathbb{N}$. ■

3 Tree expansion of the perturbative solution

In this section we first recall some notions of graph theory. We then recall a fundamental result which represents the solution – in the sense of a formal power series – of the stochastic differential equation (1) by a sum over all rooted trees. Some of the graph theoretic standard notions, that are used here, are collected in Appendix A and also in [15, 16].

A tree T is a connected graph without cycle. We also consider graphs with different types of vertices. A rooted tree with root $x \in \Gamma$, denoted by \times , and two types of leaves, the leaves of type one are noted by \otimes , and those of type two by \diamond .

We define a partial order " \leq " on the set of the vertices of T , $V(T)$ by $v \leq w$ if every walk connecting w and x passes through v .

We note $\mathcal{A}(i)$ the set of all rooted trees T with root x and two types of leaves, which has i inner vertices with $p + 1$ legs. See e.g. (3) for the trees from $\mathcal{A}(0)$ and $\mathcal{A}(\infty)$ for $p = 3$.

Definition 3.1. For $T \in \mathcal{A}(i)$, the random variable $\mathcal{B}(T, \eta, x)$, is defined as follows:

1)-Assign $x \in \Gamma$ to the root of the tree T .

- Assign values $y_1, \dots, y_i \in \Gamma$ to the inner vertices.

-Assign values $z_1, \dots, z_k \in \Gamma$, to the leaves of type one and assign a values $\mathbf{z}'_1, \dots, \mathbf{z}'_l \in L_\delta$ to the leaves of type two.

2)- For every edge with two end points, $e = \{v, w\}$, assign a value $G(e) = G(v - w)$, ($v \leq w$) to this edge. G is the Green function defined in (14).

3)- For the j -th leaf multiply with $\eta(t_j, \mathbf{z}_j)$ if this leaf is of type one and multiply with $f(\mathbf{z}_j)$ if this leaf is of type two.

4)- For the $l - th$ inner vertex multiply with the coefficient $\frac{p!}{n_0!(n_0 - n'_0)!n_1! \dots}$, where $n_i, i \in \mathbb{N}$, is the number of rooted subtrees connected to the $l - th$ inner vertex with i inner vertices and n'_0 is the number of the rooted subtrees connected to this vertex with zero inner vertices and a leaf of type one.

5)- Integrate with respect to the Lebesgue measure $dy_1 \cdots dy_p dz_1 \cdots dz_k d\mathbf{z}'_1 \cdots d\mathbf{z}'_l$.

Let $T \in \mathcal{A}(j)$, $j \in \mathbb{N}$, and $C(j) = \{T_{0,1}, \dots, T_{0,n_0} T_{1,1}, \dots; T_{i,1}, \dots, T_{i,n_i} \in \mathcal{A}(i), \sum_{i \geq 0} i n_i = j - 1, n_0 + n_1 + \dots = p\}$. We construct a one to one correspondence between $\mathcal{A}(j)$ and $C(j)$ that is given in the following way:

We first consider the case when $T \in \mathcal{A}(j)$ is given. Let y_1 be the first inner vertex of T and we denote $e_1 = \{x, y_1\}$ the first edge of T . Cut the tree T at y_1 . Let $n_i, i \geq 0$ be the number of the rooted subtrees with root y_1 and i inner vertices denoted by $T_{i,1}, \dots, T_{i,n_i}$. Then $n_0 + n_1 + \dots + n_i + \dots = p$ and $\sum_{i \geq 0} i n_i = j - 1$. Conversely, let $x \in \Gamma$ and $T_{i,1}, \dots, T_{i,n_i} \in \mathcal{A}(i)$, be a collection of rooted trees with root y_1 be given such that $n_0 + n_1 + \dots = p, \sum_{i \geq 0} i n_i = j - 1$. Attach this collection of rooted trees to the edge $e = (x, y_1)$. By definition we obtain a rooted tree $(T, x) \in \mathcal{A}(j)$. For the notions of cutting and attaching we again refer to Appendix A.

The second operation is obviously the inverse of the first one. We have established the following result:

Lemma 3.2. Let $T \in \mathcal{A}(j)$, ($j \in \mathbb{N}$) and $T_{i,1}, \dots, T_{i,n_i} \in \mathcal{A}(i)$, $i \in \mathbb{N}$, the application

$$\mathcal{C} : \mathcal{A}(j) \longrightarrow C(j) \quad (31)$$

is a one to one correspondence and

$$\mathcal{A} : C(j) \longrightarrow \mathcal{A}(j) \quad (32)$$

is its inverse.

Definition 3.3. Let $T \in \mathcal{A}(i)$ and $v \in V(T)$, we define the multiplicity of the vertex v as $M(v) = \frac{p!}{n'_0!(n_0-n'_0)!n_1! \dots}$, where $n_i \geq 0$, is the number of the rooted subtrees connected to v with i inner vertices.

The multiplicity of the rooted tree T is $M(T) = \prod_{v \in V(T)} M(v)$.

For $f \in S(L_\delta)$ we put $F_f(x) = \delta_0(t)f(\mathbf{x})$, $x = (t, \mathbf{x}) \in L_\delta$

Lemma 3.4. Let $T \in \mathcal{A}(j)$ and $v_0 \in V(T)$ the vertex connected to the root x , then the following result hold :

$$\mathcal{B}(T, x, \eta) = M(v_0)G * \left[\prod_{S \in \mathcal{C}(T)} \mathcal{B}(S, \cdot, \eta) \right](x). \quad (33)$$

Where $\mathcal{C}(T)$ is the cut (cf. Appendix A) of the tree T .

Proof.

$$\begin{aligned} \mathcal{B}(T, x, \eta) &= M(T) \int \prod_{e \in E(T)} G(e) \prod_{l_1 \in L_1(T)} \eta(l_1) \prod_{l \in L_2(T)} F_f(l) \bigotimes_{v \in V(T) \setminus \{x\}} dv \\ &= M(v_0) \int G(x - v_0) \prod_{S \in \mathcal{C}(T)} \left[M(S) \int \prod_{e \in E(S)} G(e) \prod_{l_1 \in L_1(S)} \eta(l_1) \right. \\ &\quad \left. \times \prod_{l \in L_2(S)} F_f(l) \bigotimes_{v \in V(S) \setminus \{v_0\}} dv \right] dv_0 = M(v_0)G * \left[\prod_{S \in \mathcal{C}(T)} \mathcal{B}(S, \cdot, \eta) \right](x). \quad (34) \end{aligned}$$

■

Using Proposition 2.5 and Definition 3.1 we get the following theorem:

Theorem 3.5. *The solution of the stochastic differential equation (1) in the sense of formal power series is given by a sum over all rooted trees $T \in \mathcal{A}(j)$ that are evaluated according to the ruled fixed in Definition 3.1 i.e.*

$$X_j(x, \eta) = \sum_{T \in \mathcal{A}(j)} \mathcal{B}(T, x, \eta) \quad ; \quad x \in \Gamma \quad (35)$$

Proof. We prove the assertion by induction. For $j = 0$, we have $X_0(x, \eta) = G * \eta(x) + G * F_f(x)$, which just is the sum of the evaluation of the two trees in $\mathcal{A}(0)$.

Suppose that (35) is true for all $i < j$. By proposition (2.2) we have

$$X_j(x, \eta) = G * \sum_{B_p^j(n_0, n_1, \dots)} \frac{p!}{n_0! n_1! \dots} \prod_{i \geq 0} X_i^{n_i}(x, \eta) \quad (36)$$

Then using the induction hypothesis and Lemmas 3.2 and 3.4 we get

$$\begin{aligned} X_j(x, \eta) &= G * \sum_{B_p^j(n_0, n_1, \dots)} \frac{p!}{n_0! n_1! \dots} \prod_{i \geq 0} \left[\sum_{T_i \in \mathcal{A}(i)} \mathcal{B}(T_i, \cdot, \eta) \right]^{n_i} (x) \\ &= \sum_{B_p^j(n_0, n_1, \dots)} \sum_{\substack{T_{i,1}, \dots, T_{i,n_i} \in \mathcal{A}(i) \\ i=0,1,2,\dots}} \frac{p!}{n_0! n_1! \dots} \left(G * \prod_{i \geq 0} \prod_{l=1}^{n_i} \mathcal{B}(T_{i,l}, \cdot, \eta) \right) (x) \\ &= \sum_{B_p^j(n_0, n_1, \dots)} \sum_{\substack{T_{i,1}, \dots, T_{i,n_i} \in \mathcal{A}(i) \\ i=0,1,2,\dots}} \mathcal{B}(\mathcal{A}(T_{0,1}, \dots, T_{0,n_0}, T_{1,1}, \dots), x, \eta) \\ &= \sum_{T \in \mathcal{A}(j)} \mathcal{B}(T, x, \eta). \end{aligned} \quad (37)$$

■

4 A graphical calculus for the correlation functions

In the following we define a class of graphs that we call Parisi-Wu graphs. We then give a rule which assigns a value to such a graph. We prove a theorem which represents the moments of the solution given in the previous section in terms of a sum over all generalized Parisi-Wu graphs. We finally prove a linked cluster theorem for Parisi-Wu graphs.

Definition 4.1. A generalized Parisi-Wu graph of m -th order with n exterior vertices x_1, \dots, x_n of fertility one, is a graph that contains n rooted trees $T_1 \in \mathcal{A}(j_1), \dots, T_n \in \mathcal{A}(j_n)$ as subgraphs, such that any leaf of type one is connected to exactly one vertex of a new type, called inner empty, of arbitrary fertility. We denote such a graph by \mathcal{G} and the set of all generalized Parisi-Wu graphs of m -th order and n roots by $P(m, n)$.

If we denote by $Q = Q(\mathcal{G})$ the set of the empty vertices then

$$V(\mathcal{G}) = \bigcup_{k=1}^n V(T_k) \cup Q, \quad E(\mathcal{G}) = \bigcup_{k=1}^n E(T_k) \cup E(T_k, Q) \quad (38)$$

where $E(T_k, Q)$ is a set of edges $e = \{v, q\}$, $v \in L_1(T_k)$, $q \in Q$ such that $p(v) = 2$, $\forall v \in L_1(T_k)$. By definition, inner vertices are distinguishable and have non distinguishable legs whereas empty vertices are non distinguishable and have non distinguishable legs².

The name generalized Parisi-Wu graph has been chosen to distinguish the graphs used here from Gaussian Parisi-Wu graphs implicitly defined in [17], which, in a topological sense, coincide with the ordinary Feynman graphs.

Proposition 4.2. For $j_1, \dots, j_n \in \mathbb{N}$, let $T_1 \in \mathcal{A}(j_1), \dots, T_n \in \mathcal{A}(j_n)$ n rooted trees with roots $x_1, \dots, x_n \in \Gamma$, then the following result holds in the sense of formal power series

$$\left\langle \prod_{i=1}^n X(x_i, \eta) \right\rangle = \sum_{m=0}^{\infty} \lambda^m \sum_{\substack{j_1, \dots, j_n \geq 0 \\ j_1 + \dots + j_n = m \\ T_1 \in \mathcal{A}(j_1), \dots, T_n \in \mathcal{A}(j_n)}} \left\langle \mathcal{B}_1(T_1, x_1, \eta) \cdots \mathcal{B}_n(T_n, x_n, \eta) \right\rangle \quad (39)$$

Proof.

$$\begin{aligned} \left\langle \prod_{i=1}^n X(x_i, \eta) \right\rangle &= \left\langle \sum_{j=0}^{\infty} \lambda^j X_j(x_1, \eta) \cdots \sum_{j=0}^{\infty} \lambda^j X_j(x_n, \eta) \right\rangle \\ &= \sum_{m=0}^{\infty} \lambda^m \sum_{\substack{j_1, \dots, j_n \geq 0 \\ j_1 + \dots + j_n = m}} \langle X_{j_1}(x_1, \eta) \cdots X_{j_n}(x_n, \eta) \rangle \end{aligned}$$

²The notion of distinguishable and non distinguishable legs are explained in Appendix A.

$$= \sum_{m=0}^{\infty} \lambda^m \sum_{\substack{j_1, \dots, j_n \geq 0 \\ j_1 + \dots + j_n = m \\ T_1 \in \mathcal{A}(j_1), \dots, T_n \in \mathcal{A}(j_n)}} \langle \mathcal{B}_1(T_1, x_1, \eta) \cdots \mathcal{B}_n(T_n, x_n, \eta) \rangle \quad (40)$$

The last equality is an immediate consequence of Theorem 3.5. ■

Let $J \subset \mathbb{N}$ be a finite set. The collection of all partition of J is denoted by $\mathcal{D}(J)$. A partition is a decomposition of J into disjoint, nonempty subsets, i.e

$$I \in \mathcal{D}(J) \iff \exists k \in \mathbb{N}, I = \{I_1, \dots, I_k\}, I_j \cap I_l = \emptyset \forall 1 \leq j < l \leq k, \cup_{l=1}^k I_l = J. \quad (41)$$

Definition 4.3. Let $z_1, \dots, z_k \in \Gamma$, I a partition of the set $\{1, \dots, n\}$, $I \in \mathcal{D}(\{1, \dots, n\})$,

$I = \{I_1, \dots, I_k\}$ the truncated moments functions $\langle \eta(z_1) \cdots \eta(z_n) \rangle^T$ are recursively defined by :

$$\langle \prod_{i=1}^n \eta(z_i) \rangle = \sum_{\substack{I \in \mathcal{D}(\{1, \dots, n\}) \\ I = \{I_1, \dots, I_k\}}} \prod_{l=1}^k \langle I_l \rangle^T \quad (42)$$

where $\langle I_l \rangle^T = \langle \prod_{j \in I_l} \eta(z_j) \rangle^T$.

Definition 4.4. For $\mathcal{G} \in P(m, n)$ and $x_1, \dots, x_n \in \Gamma$ we define the number $\mathcal{P}[\mathcal{G}](x_1, \dots, x_n)$ as follows:

1- Assign values $x_1, \dots, x_n \in \Gamma$ to the roots of the trees T_1, \dots, T_n .

-Assign values $y_1, \dots, y_m \in \Gamma$, to the inner vertices.

-Assign values $z_1, \dots, z_k \in \Gamma$, to the leaves of type one and assign values $\mathbf{z}'_1, \dots, \mathbf{z}'_l \in L_\delta$ to the leaves of type two.

2-For every edge in a tree T_r , $r = 1, \dots, n$ with two end points $e = \{v, w\}$, $v \leq w$, assign value $G(e)$ to this edge.

3-For each inner empty vertices with l legs connected to leaves with arguments q_1, \dots, q_l multiply with

$$\langle \eta(q_1) \cdots \eta(q_l) \rangle^T. \quad (43)$$

4-For the j -th leaf of type two multiply with $f(\mathbf{z}'_j)$.

5-For each vertex $v \in V(\mathcal{G})$ that is an inner vertex of a tree multiply with the corresponding

multiplicity $M(v)$, (see Def.3.3).

6-Integrate with respect to the Lebesgue measure $dy_1 \cdots dy_m dz_1 \cdots dz_k dz'_1 \cdots dz'_l$.

Let $m, n \in \mathbb{N}$, $I \in \mathcal{D}(\bigcup_{j=1}^n L_1(T_j))$, $I = \{I_1, \dots, I_k\}$ and $T_1 \in \mathcal{A}(j_1), \dots, T_n \in \mathcal{A}(j_n)$, such that $j_1 + \dots + j_n = m$. We construct a one to one correspondence between pairs $(T_1, T_2, \dots, T_n; I)$ and the generalized Parisi-Wu graph $\mathcal{G} \in P(m, n)$ in the following way:

Let the Parisi-Wu graph \mathcal{G} is given and we construct $(T_1, \dots, T_n; I)$. By definition \mathcal{G} contains n rooted trees, $T_1 \in \mathcal{A}(j_1), \dots, T_n \in \mathcal{A}(j_n)$ such that $j_1 + \dots + j_n = m$. Suppose that there are k empty vertices in the graph \mathcal{G} . Give an arbitrary number $l = 1, \dots, k$ to each empty vertex. For the l -th empty vertex let I_l be the subset in $\bigcup_{j=1}^n L_1(T_j)$ corresponding to the leaves connected to this vertex. Then $I = \{I_1, \dots, I_k\} \in \mathcal{D}(\bigcup_{i=1}^n L_1(T_i))$.

Conversely, let T_1, \dots, T_n, I be given such that $T_1 \in \mathcal{A}(j_1), \dots, T_n \in \mathcal{A}(j_n)$ and $I \in \mathcal{D}(\bigcup_{j=1}^n L_1(T_j))$, $I = \{I_1, \dots, I_k\}$. Let $Q = \{1, \dots, k\}$ be the set of the empty vertices of \mathcal{G} . Let $V(\mathcal{G}) = \sum_{j=1}^n V(T_j) \cup Q$ be the set of all vertices of \mathcal{G} and $E(T_j, Q)$ be the set of all edges which connect the j -th rooted tree T_j , ($1 \leq j \leq n$) with the empty vertices in Q . Then $E(T_j, Q) = \{(v, q) : v \in L_1(T_j), q \in Q; v \in I_q\}$. Let $E(\mathcal{G}) = \sum_{j=1}^n E(T_j) \cup E(T_j, Q)$ be the set of all edge of \mathcal{G} .

We have established the following result :

Lemma 4.5. *Let $m, n \in \mathbb{N}$ and $M(j) = \{T_1 \in \mathcal{A}(j_1), \dots, T_n \in \mathcal{A}(j_n); I \in \mathcal{D}(\sum_{j=1}^n L(T_j)), I = \{I_1, \dots, I_k\}\}$. Then the mapping N constructed above*

$$N : P(m, n) \longrightarrow M(j) \tag{44}$$

is a one to one correspondence.

Theorem 4.6. *Let $\{X_t, t \geq 0\}$, be the solution of the stochastic differential equation (1), and T_1, \dots, T_n , n rooted trees with roots x_1, \dots, x_n . Then the moments of $X(x)$ are given by a sum over all generalized Parisi-Wu graphs $\mathcal{G} \in P(m, n)$ of m -th order that are evaluated according*

to the ruled fixed in definition (4.4). i.e :

$$\left\langle \prod_{i=1}^n X(x_i, \eta) \right\rangle = \sum_{m=0}^{\infty} \lambda^m \sum_{\mathcal{G} \in P(m, n)} \mathcal{P}[\mathcal{G}](x_1, \dots, x_n) \quad (45)$$

Proof. Let

$$L_1(\mathcal{G}) = \bigcup_{i=1}^n L_1(T_{j_i}), L_2(\mathcal{G}) = \bigcup_{i=1}^n L_2(T_{j_i}), V(\mathcal{G}) = \bigcup_{i=1}^n V(T_{j_i}) \setminus (\{x_1, \dots, x_n\}), E(\mathcal{G}) = \bigcup_{i=1}^n E(T_{j_i})$$

The multiplicity of the graph \mathcal{G} is $M(\mathcal{G}) = \prod_{v \in V(\mathcal{G})} M(v)$.

we have :

$$\begin{aligned} \left\langle \prod_{i=1}^n X(x_i, \eta) \right\rangle &= \sum_{m=0}^{\infty} \lambda^m \left\langle \sum_{\substack{j_1, \dots, j_n \geq 0 \\ j_1 + \dots + j_n = m \\ T_1 \in \mathcal{A}(j_1), \dots, T_n \in \mathcal{A}(j_n)}} \mathcal{B}_1(T_1, x_1, \eta) \cdots \mathcal{B}_n(T_n, x_n, \eta) \right\rangle \\ &= \sum_{m=0}^{\infty} \lambda^m \sum_{\substack{j_1, \dots, j_n \geq 0 \\ j_1 + \dots + j_n = m \\ T_1 \in \mathcal{A}(j_1), \dots, T_n \in \mathcal{A}(j_n)}} M(\mathcal{G}) \int \prod_{e \in E(\mathcal{G})} G(e) \\ &\quad \times \prod_{l \in L_2(\mathcal{G})} F_f(l) \left\langle \prod_{l_1 \in L_1(\mathcal{G})} \eta(l_1) \right\rangle \otimes_{v \in V(\mathcal{G}) \setminus Q(\mathcal{G})} dv \\ &= \sum_{m=0}^{\infty} \lambda^m \sum_{\substack{j_1, \dots, j_n \geq 0 \\ j_1 + \dots + j_n = m \\ T_1 \in \mathcal{A}(j_1), \dots, T_n \in \mathcal{A}(j_n)}} M(\mathcal{G}) \int \prod_{e \in E(\mathcal{G})} G(e) \\ &\quad \times \prod_{l \in L_2(\mathcal{G})} F_f(l) \sum_{\substack{I \in \mathcal{D}(L_1(\mathcal{G})) \\ I = \{I_1, \dots, I_k\}}} \prod_{l=1}^k \left\langle \prod_{l_1 \in I_l} \eta(l_1) \right\rangle^T \otimes_{v \in V(\mathcal{G}) \setminus Q(\mathcal{G})} dv \\ &= \sum_{m=0}^{\infty} \lambda^m \sum_{\substack{j_1, \dots, j_n \geq 0 \\ j_1 + \dots + j_n = m \\ T_1 \in \mathcal{A}(j_1), \dots, T_n \in \mathcal{A}(j_n) \\ I \in \mathcal{D}(L_1(\mathcal{G})) \\ I = \{I_1, \dots, I_k\}}} M(\mathcal{G}) \int \prod_{e \in E(\mathcal{G})} G(e) \\ &\quad \times \prod_{l \in L(\mathcal{G})} F_f(l) \prod_{l=1}^k \left\langle \prod_{l_1 \in I_l} \eta(l_1) \right\rangle^T \otimes_{v \in V(\mathcal{G}) \setminus (Q\mathcal{G})} dv. \quad (46) \end{aligned}$$

Now we apply Lemma 4.5 and Definition 4.4 to conclude. ■

We denote the collections of the connected generalized Parisi-Wu graph of m -th order with n roots by $P_c(m, n)$.

We construct a one to one correspondence between $P(m, n)$ and $\mathcal{N}(m, n) = \{I \in \mathcal{D}(\{1\dots n\}), I = \{I_1, \dots, I_k\}; G_1 \in P_c(m, p_1), \dots, G_k \in P_c(m_k, p_k), m = m_1 + \dots + m_k, n = p_1 + \dots + p_k\}$ in the following way:

We first consider the case when a generalized Parisi-Wu graph $\mathcal{G} \in P(m, n)$ is given. By definition \mathcal{G} contains n rooted trees $T_1 \in \mathcal{A}(j_1), \dots, T_n \in \mathcal{A}(j_n)$, with roots x_1, \dots, x_n and an arbitrary number of empty vertices. For $1 \leq q < j \leq n$ we say that the graph connects q and j , in notation $q \sim_R j$, if x_q and x_j are connected in the generalized Parisi-Wu graph. Obviously \sim_R is an equivalence relation on $\{1, \dots, n\}$. Let $I = \{I_1, \dots, I_k\}$ be the equivalence classes of \sim_R then $I \in \mathcal{D}(\{1\dots n\})$. For $1 \leq l \leq k$ let \mathcal{G}_l be the connected generalized Parisi-Wu graph with m_l inner vertices and $p_l = \#I_l$ roots, then $m = m_1 + \dots + m_k$ and $n = p_1 + \dots + p_k$. Conversely let $I \in \mathcal{D}(\{1\dots n\}), I = \{I_1, \dots, I_k\}, \mathcal{G}_1 \in P_c(m, p_1), \dots, \mathcal{G}_k \in P_c(m_k, p_k), m = m_1 + \dots + m_k$.

Obviously the connected generalized Parisi-Wu graphs $\mathcal{G}_1, \dots, \mathcal{G}_k$ determine a generalized Parisi-Wu graph \mathcal{G} with $m = m_1 + \dots + m_k$ inner vertices and $n = p_1 + \dots + p_k$ roots by taking the disjoint union.

This second operation is the inverse of the first one. We have established the following result:

Lemma 4.7. *The map*

$$M : P(m, n) \longrightarrow \mathcal{N}(m, n) \tag{47}$$

is a one to one correspondence.

The following is known as linked cluster theorem and gives the truncated moments of $X(x)$ as a sum over the connected Parisi-Wu graphs, only.

Theorem 4.8. *Let $P_c(m, n)$ be the set of connected generalized Parisi-Wu graphs, then the truncated moments $\langle X(x_1, \eta) \cdots X(x_n, \eta) \rangle^T$ are given by*

$$\langle X(x_1, \eta) \cdots X(x_n, \eta) \rangle^T = \sum_{m=0}^{\infty} \lambda^m \sum_{\mathcal{G} \in P_c(m, n)} \mathcal{P}[\mathcal{G}](x_1, \dots, x_n). \tag{48}$$

Proof. By Theorem (4.6) we have

$$\langle X(x_1, \eta) \cdots X(x_n, \eta) \rangle = \sum_{m=0}^{\infty} \lambda^m \sum_{\mathcal{G} \in P(m, n)} \mathcal{P}[\mathcal{G}](x_1, \dots, x_n). \quad (49)$$

In the other hand by definition of the truncated moments

$$\langle X(x_1, \eta) \cdots X(x_n, \eta) \rangle = \sum_{\substack{I \in \mathcal{D}(\{1, \dots, n\}) \\ I = \{I_1, \dots, I_k\}}} \prod_{l=1}^k \langle \prod_{j \in I_l} X(x_j, \eta) \rangle^T \quad (50)$$

To prove the theorem it suffices to prove that in equation (50) we can replace the truncated moments by the right hand side of (48), i.e.

$$\langle X(x_1, \eta) \cdots X(x_n, \eta) \rangle = \sum_{\substack{I \in \mathcal{D}(\{1, \dots, n\}) \\ I = \{I_1, \dots, I_k\}}} \prod_{l=1}^k \left(\sum_{m_l=0}^{\infty} \lambda^{m_l} \sum_{\mathcal{G}_l \in P_c(m_l, \#I_l)} \mathcal{P}[\mathcal{G}_l](x_j; j \in I_l) \right). \quad (51)$$

Thus we have to prove that the right hand side of (51) is equal to the right hand side (r.h.s) of (49). We have for the r.h.s of (51)

$$\begin{aligned} & \sum_{\substack{I \in \mathcal{D}(\{1, \dots, n\}) \\ I = \{I_1, \dots, I_k\}}} \sum_{\substack{m_1=0 \\ \mathcal{G}_1 \in PW_c(m_1, p_1)}}^{\infty} \lambda^{m_1} \cdots \sum_{\substack{m_k=0 \\ \mathcal{G}_k \in P_c(m_k, p_k)}}^{\infty} \lambda^{m_k} \mathcal{P}[\mathcal{G}_1](x_j; j \in I_1) \cdots \mathcal{P}[\mathcal{G}_k](x_j; j \in I_k) \\ = & \sum_{\substack{I \in \mathcal{D}(\{1, \dots, n\}) \\ I = \{I_1, \dots, I_k\}}} \sum_{\substack{m_1, \dots, m_k=0 \\ \mathcal{G}_1 \in P_c(m_1, p_1) \\ \vdots \\ \mathcal{G}_k \in P_c(m_k, p_k)}}^{\infty} \lambda^{m_1 + \dots + m_k} \mathcal{P}[\mathcal{G}_1](x_j; j \in I_1) \cdots \mathcal{P}[\mathcal{G}_k](x_j; j \in I_k) \\ = & \sum_{m_1, \dots, m_k=0}^{\infty} \lambda^{m_1 + \dots + m_k} \sum_{\substack{I \in \mathcal{D}(\{1, \dots, n\}) \\ I = \{I_1, \dots, I_k\} \\ \mathcal{G}_1 \in P_c(m_1, p_1) \\ \vdots \\ \mathcal{G}_k \in P_c(m_k, p_k)}} \mathcal{P}[\mathcal{G}_1](x_j; j \in I_1) \cdots \mathcal{P}[\mathcal{G}_k](x_j; j \in I_k) \\ = & \sum_{m=0}^{\infty} \lambda^m \sum_{\substack{I \in \mathcal{D}(\{1, \dots, n\}) \\ I = \{I_1, \dots, I_k\} \\ \mathcal{G}_1 \in P_c(m_1, p_1) \\ \vdots \\ \mathcal{G}_k \in P_c(m_k, p_k) \\ m_1, \dots, m_k=0, 1, \dots \\ m_1 + \dots + m_k = m}} \mathcal{P}[\mathcal{G}_1](x_j; j \in I_1) \cdots \mathcal{P}[\mathcal{G}_k](x_j; j \in I_k) \end{aligned} \quad (52)$$

Now application of Lemma (4.7) concludes the argument. ■

5 Lévy noise

In this section we define first a white noise measure with Lévy characteristic ψ , then we recall a theorem which give the truncated moments of the noise which permit us to derive a simplification of the Feynman rules (Definition 4.4).

Let ν be an infinitely divisible probability distribution. By Levy-Khinchine theorem (see e.g. [7]) we know that the Fourier transform (or characteristic function) of ν , denoted by C_ν , satisfies the following formula

$$C_\nu(t) = \int_{\mathbb{R}} e^{ist} d\nu(s) = e^{\psi(t)}, \quad t \in \mathbb{R}$$

where $\psi : \mathbb{R} \rightarrow \mathbb{C}$ is a continuous function, called the Lévy characteristic of ν , which is uniquely represented as follows

$$\psi(t) = i\bar{a}t - \frac{\sigma^2 t^2}{2} + \int_{\mathbb{R} \setminus \{0\}} \left(e^{ist} - 1 - \frac{ist}{1+s^2} \right) dM(s), \quad \forall t \in \mathbb{R}. \quad (53)$$

where $\bar{a} \in \mathbb{R}$, $\sigma^2 \geq 0$ and M is a Lévy measure on $\mathbb{R} \setminus \{0\}$, i.e. $\int_{\mathbb{R} \setminus \{0\}} \min(1, s^2) dM(s) < \infty$. Under the additional assumption that all moments of M exist, i.e. $\int_{\mathbb{R} \setminus \{0\}} s^{2n} dM(s) < \infty$, $n \in \mathbb{N}_0$, one can reparametrize this representation setting $z = \int_{\mathbb{R} \setminus \{0\}} dM(s)$, $r = M/z$, $a = \bar{a} - \int_{\mathbb{R} \setminus \{0\}} \frac{s}{1+s^2} dM(s)$ and one obtains

$$\psi(t) = iat - \frac{\sigma^2 t^2}{2} + z \int_{\mathbb{R} \setminus \{0\}} (e^{ist} - 1) dr(s), \quad \forall t \in \mathbb{R}. \quad (54)$$

The first term in (54) is called deterministic, the second one Gaussian term and the third one Poisson term.

We note by \mathcal{B} the σ -algebra generated by the cylinder sets of $S'(\Gamma)$. Then $S'(\Gamma)$ is a measurable space.

We define a characteristic functional on $S(\Gamma)$, as a functional $C : S(\Gamma) \rightarrow \mathbb{C}$ such that :

1. C is continuous on $S(\Gamma)$;

2. C is positive-definite;

3. $C(0) = 1$.

By the well-known Bochner-Minlos theorem (see [12]) there exists a one to one correspondence between characteristic functional C and probability measures μ on $(S'(\Gamma), \mathcal{B})$ given by the following relation

$$C(f) = \int_{S'(\Gamma)} e^{i\langle f, \xi \rangle} d\mu(\xi), \quad f \in S(\Gamma). \quad (55)$$

We set $\Gamma_{\pm} = \{x \in \Gamma : x = (t, \mathbf{x}), \pm t > 0\}$.

Theorem 5.1. *Let ψ be a Lévy characteristic given by the representation (53) then there exist an unique probability measure P_{ψ} on $(S'(\Gamma), \mathcal{B})$ such that*

$$C_{\eta}(f) = \int_{S'(\Gamma)} e^{i\eta(f, \xi)} dP_{\psi}(\xi) = \exp \left(\int_{\Gamma_+} \psi(f(x)) dx \right), \quad x = (t, \mathbf{x}). \quad (56)$$

Where $\eta(f, \xi) = \langle f, \xi \rangle$, $f \in S(\Gamma)$.

Proof. The right-hand side of (56) is a characteristic functional on $S(\Gamma)$ (see eg. [12]). The result thus holds by using the Bochner-Minlos Theorem. ■

Definition 5.2. We call P_{ψ} in Theorem 5.1 a generalized white noise measure with Lévy characteristic ψ and $(S'(\Gamma), \mathcal{B}, P_{\psi})$ the generalized white noise probability space associated with ψ .

The associated coordinate process

$$\eta : S(\Gamma) \times (S'(\Gamma), \mathcal{B}, P_{\psi}) \longrightarrow \mathbb{C}, \quad \eta(f)(\omega) = \omega(f) \quad \forall f \in S(\Gamma), \quad \omega \in S'(\Gamma)$$

is called a Lévy noise.

The following result on Lévy noise is essential for our perturbative approach

Proposition 5.3. *There exists a version of $G * \eta$ such that $G * \eta \in \mathcal{M}_b P_{\psi}$ a.s. .*

Proof. The proof requires several steps.

i) By (54), the Lévy noise can be decomposed into independent parts $\eta = \eta^d + \eta^g + \eta^p$, where η^d , η^g and η^p are respectively the deterministic, Gaussian and Poisson noise given by their characteristic function in theorem (5.1). It is sufficient to deal with these parts separately. The statement holds trivially for the deterministic part.

Let us next focus on the Gaussian part. We get the following two results that together show $G * \eta^g \in \mathcal{M}_b$ a.s. .

ii) $G * \eta^g(t, \mathbf{x})$, for $\mathbf{x} \in L_\delta$ fixed, has a continuous extension in t . In particular, $G * \eta^g$ is a measurable function on Γ (a.s.).

By the Linked Cluster theorem we have

$$\langle |G * \eta^g(t, \mathbf{x}) - G * \eta^g(s, \mathbf{x})|^4 \rangle = 3(\langle |G * \eta^g(t, \mathbf{x}) - G * \eta^g(s, \mathbf{x})|^2 \rangle)^2. \quad (57)$$

If we assume that $t > s$, then it is easy to see, using $\langle \eta^g(t, \mathbf{x}) \eta^g(s, \mathbf{y}) \rangle = \sigma^2 \theta(t) \theta(s) \delta(t-s) \delta(\mathbf{x}-\mathbf{y})$, that

$$\begin{aligned} & \langle |G * \eta^g(t, \mathbf{x}) - G * \eta^g(s, \mathbf{x})|^2 \rangle \\ &= \langle (G * \eta^g(t, \mathbf{x}))^2 + (G * \eta^g(s, \mathbf{x}))^2 - 2(G * \eta^g(t, \mathbf{x}))(G * \eta^g(s, \mathbf{x})) \rangle \\ &= \left\langle \left(\int_0^\infty \int \theta(t-t') \tilde{G}_{t-t'}(\mathbf{x}-\mathbf{x}') \eta^g(t', \mathbf{x}') d\mathbf{x}' dt' \right)^2 \right. \\ & \quad + \left(\int_0^\infty \int \theta(s-t') \tilde{G}_{s-t'}(\mathbf{x}-\mathbf{x}') \eta^g(t', \mathbf{x}') d\mathbf{x}' dt' \right)^2 \\ & \quad - 2 \int_0^\infty \int_0^\infty \int \int \theta(t-t') \theta(s-s') \tilde{G}_{t-t'}(\mathbf{x}-\mathbf{x}') \tilde{G}_{s-s'}(\mathbf{x}-\mathbf{x}'') \\ & \quad \left. \times \eta^g(t', \mathbf{x}') \eta^g(s', \mathbf{x}'') d\mathbf{x} d\mathbf{x}'' dt' ds' \right\rangle \\ &= \sigma^2 \int_0^\infty \int \left(\theta(t-t') \tilde{G}_{t-t'}^2(\mathbf{x}-\mathbf{x}') + \theta(s-t') \tilde{G}_{s-t'}^2(\mathbf{x}-\mathbf{x}') \right. \\ & \quad \left. - 2\theta(s-t') \tilde{G}_{t-t'}(\mathbf{x}-\mathbf{x}') \tilde{G}_{s-t'}(\mathbf{x}-\mathbf{x}') \right) d\mathbf{x}' dt' \\ &= \sigma^2 \int_0^\infty \int \theta(s-t') \left(\tilde{G}_{t-t'}(\mathbf{x}') - \tilde{G}_{s-t'}(\mathbf{x}') \right)^2 d\mathbf{x}' dt' + \int_s^t \int \tilde{G}_{t-t'}^2(\mathbf{x}') d\mathbf{x}' dt' \quad (58) \end{aligned}$$

Moreover by the expression of \tilde{G}_t given in (16) we get :

$$\left| (1 + |\mathbf{x}|^2)^d \left(\tilde{G}_t(\mathbf{x}) - \tilde{G}_s(\mathbf{x}) \right) \right|^2 \leq \frac{c^2(t-s)^2}{(2\pi)^{2d}} \quad \text{and} \quad \left| (1 + |\mathbf{x}|^2)^d G_{t-t'}^2(\mathbf{x}) \right| \leq c \frac{\delta^{-2d}}{(2\pi)^{2d}}. \quad (59)$$

Here $c = \sup_{t>0} \int_{\Pi_\delta^d} (1 - \Delta)^d e^{-t\mu_{\delta,m}^2(\mathbf{p})} d\mathbf{p} < \infty$. Inserting these two estimates in (58), we get for $K > 0$ sufficiently large

$$\langle |G * \eta^g(t, \mathbf{x}) - G * \eta^g(s, \mathbf{x})|^4 \rangle \leq K |t - s|^2. \quad (60)$$

Application of Kolmogorov's extension theorem now proves the assertion.

ii) $G * \eta^g(t, \mathbf{x})$ is polynomially bounded (a.s.).

We prove the stronger statement that the expectation of $\int \frac{(G * \eta^g(x))^k}{(1+|x|^2)^N} dx$ is finite. We have for $k \in \mathbb{N}$ even

$$\begin{aligned} \left\langle \int \frac{(G * \eta^g(x))^k}{(1+|x|^2)^N} dx \right\rangle &= A \int \frac{\left(\langle |G * \eta^g(\mathbf{x})|^2 \rangle \right)^{\frac{k}{2}}}{(1+|x|^2)^N} dx \\ &\leq A \sigma^2 \left(\int_{\Gamma} G^2(x) dx \right)^{\frac{k}{2}} \int \frac{dx}{(1+|x|^2)^N} < \infty. \end{aligned} \quad (61)$$

The last inequality hold for an arbitrary $N > d/2$. Here A is the number of pairings of k objects.

It is left over to deal with the poisson part. First we recall the path properties of η^p following [2].

iii) η^p is a signed measure with locally discrete support (a.s.).

Let $\Lambda_n \subseteq \Gamma_+$ be a monotone sequence of compact sets s.t $\Lambda_n \uparrow \Gamma_+$ as $n \rightarrow \infty$ and $\Lambda_0 = \emptyset$. For $n \in \mathbb{N}$ let $D_n = \Lambda_n \setminus \Lambda_{n-1}$ and we denote the (Lebesgue) volume of D_n by $|D_n|$. Let $\tilde{\eta}_n^p = \sum_{j=1}^{N_n^z} S_j^n \delta_{X_j^n}$ be a random field, where δ_x is the Dirac measure of mass one in x and N_n^z is a Poisson random variable with intensity $z |D_n|$ i.e

$$P(N_n^z = l) = e^{-z|D_n|} \frac{(z|D_n|)^l}{l!}; \quad l \in \mathbb{N}_0.$$

For $n \in \mathbb{N}$, $\{X_j^n\}_{j \in \mathbb{N}}$ is a family of i.i.d. Γ -valued random variables distributed uniformly on D_n . $\{S_j^n\}_{j,n \in \mathbb{N}}$ is a family of real valued random variables with law given by r . All these random variables are independent of each other. The characteristic functional of $\tilde{\eta}_n^p$ is given by

$$\left\langle e^{i\tilde{\eta}_n^p(f)} \right\rangle = \left\langle e^{i \sum_{j=1}^{N_n^z} S_j^n f(X_j^n)} \right\rangle$$

$$\begin{aligned}
 &= e^{-z|D_n|} \sum_{l=0}^{\infty} \frac{(zD_n)^l}{l!} \left(\int_{D_n} \int_{\mathbb{R} \setminus \{0\}} e^{isf(x)} dr(s) \frac{dx}{|D_n|} \right)^l \\
 &= \exp \left\{ z \int_{D_n} \int_{\Gamma \setminus \{0\}} (e^{isf(x)} - 1) dr(s) dx \right\} = \mathcal{C}_{\eta^p}(f), \quad \forall f \in S(\Gamma), \text{supp } f \subseteq D_n.
 \end{aligned} \tag{62}$$

Hence η^p , when restricted to D_n , coincides in law with η_n^p . By the uniqueness statement of the Bochner-Minlos Theorem and the fact that by construction $\tilde{\eta}_n^p$ takes values in the locally discrete signed measures (a.s.), it follows (cf. [2] for the details) that also η^p with probability one is a locally discrete signed measure.

iv) For $f \in L^1(\Gamma, dx)$ bounded and non-negative, η^p is f -finite (a.s.), i.e. $\int_{\Gamma} f d|\eta^p| < \infty$. Here $|\eta^p| = \eta_+^p + \eta_-^p$ is the modulus of the signed measure η , cf. [13].

This assertion can be seen from the following calculation

$$\begin{aligned}
 \left\langle \int_{\Gamma} f d|\eta^p| \right\rangle &= \sum_{n=0}^{\infty} \left\langle \int_{D_n} f d|\tilde{\eta}_n^p| \right\rangle \\
 &= \sum_{n=0}^{\infty} \left\langle \sum_{j=1}^{N_n^z} |S_j^n| f(X_j^n) \right\rangle \\
 &= \sum_{n=0}^{\infty} e^{-z|D_n|} \sum_{l=1}^{\infty} \frac{(z|D_n|)^l}{(l-1)!} \int_{D_n} f(x) \frac{dx}{|D_n|} \int_{\mathbb{R} \setminus \{0\}} |s| dr(s) \\
 &= z \sum_{n=0}^{\infty} e^{-z|D_n|} e^{z|D_n|} \int_{D_n} f(x) dx \int_{\mathbb{R} \setminus \{0\}} |s| dr(s) \\
 &= z \int_{\mathbb{R} \setminus \{0\}} |s| dr(s) \int_{\Gamma} f(x) dx < \infty.
 \end{aligned} \tag{63}$$

v) $G * \eta^p \in L^1(\Gamma, g_{\epsilon} dx)$ a.s., where $g_{\epsilon} = (1 + |x|^2)^{-(d/2+\epsilon)}$. In particular, $G * \eta^p$ is measurable.

Let $D_n^l = \Lambda_n \setminus \Lambda_l$ for $n > l$, we denote the restriction of the noise η^p to an open set $A \subseteq \Gamma$ by $\eta_{|_A}^p$. Clearly, $G * \eta_{|\Lambda_n}^p \in L^1(\Gamma, g_{\epsilon} dx)$ (a.s.) since G is in $L^1(\Gamma, dx)$ and $\text{supp} \eta_{|\Lambda_n}^p$ is finite (a.s.). The following estimate shows that $G * \eta_{|\Lambda_n}^p$ forms a Cauchy sequence in $L^1(\Gamma, g_{\epsilon} dx)$.

With $\|\cdot\|_{\epsilon,1}$ the L^1 -norm on that space, we get

$$\sup_{n>l} \|G * \eta_{|\Lambda_n}^p - G * \eta_{|\Lambda_l}^p\|_{\epsilon,1} = \sup_{n>l} \|G * \eta_{|D_n^l}^p\|_{\epsilon,1}$$

$$\begin{aligned}
&\leq \sup_{n>l} \int_{\Gamma} |G| * g_{\epsilon} d | \eta_{D_n^l}^p | \\
&= \int_{\Gamma \setminus \Lambda_l} |G| * g_{\epsilon} d | \eta^p | \longrightarrow 0 \text{ as } l \longrightarrow \infty \quad (64)
\end{aligned}$$

since η^p is $|G| * g_{\epsilon}$ -finite (see iv) and note that $L^1(\Gamma, dx)$ is closed under convolutions). Also,

$$\lim_{n \rightarrow \infty} \langle G * \eta_{\Lambda_n}^p, f \rangle = \lim_{n \rightarrow \infty} \langle \eta_{\Lambda_n}^p, G * f \rangle = \langle \eta^p, G * f \rangle = \langle G * \eta^p, f \rangle \quad \forall f \in \mathcal{S}, \quad (65)$$

and by the fact that convergence in $L^1(\Gamma, g_{\epsilon} dx)$ implies convergence in \mathcal{S}' , we get that $G * \eta^p$ coincide with the limit of $G * \eta_{\Lambda_n}^p$ in the Banach space $L^1(\Gamma, g_{\epsilon} dx)$.

vi) $G * \eta^p(t, \mathbf{x})$ is polynomially bounded.

$$\begin{aligned}
\int \frac{\langle (G * \eta^p(x))^n \rangle}{(1 + |x|^2)^N} dx &= \int \sum_{\substack{I \in \mathcal{D}(\{1, \dots, n\}) \\ I = \{I_1, \dots, I_k\}}} \prod_{l=1}^k \frac{\langle (G * \eta^p(x))^{\# I_l} \rangle^T}{(1 + |x|^2)^N} dx \\
&= \sum_{\substack{I \in \mathcal{D}(\{1, \dots, n\}) \\ I = \{I_1, \dots, I_k\}}} \int \prod_{l=1}^k \frac{\langle (G * \eta^p(x))^{\# I_l} \rangle^T}{(1 + |x|^2)^N} dx \\
&\leq C \int \frac{dx}{(1 + |x|^2)^N} < \infty. \quad (66)
\end{aligned}$$

The last inequality hold for an arbitrary $N > d/2$. Here

$$C = b_n \max_{n_l: \sum_{l=1}^k n_l = n} \sup_{(t, \mathbf{x}) \in \Gamma} \prod_{l=1}^k |\langle (G * \eta^p(t, \mathbf{x}))^{n_l} \rangle^T|$$

with b_n the n -th Bell number, i.e. the number of partitions of n objects, and the supremum of the truncated expectation values is finite by virtue of Theorem 5.4 below and Lemma 2.1 ■

Theorem 5.4. *The truncated moment functions of the Lévy noise η are given by the following formula*

$$\langle \eta(x_1) \cdots \eta(x_n) \rangle^T = c_n \int_{\Gamma_+} \delta(x - x_1) \cdots \delta(x - x_n) dx. \quad (67)$$

where

$$\begin{aligned}
c_n &= (-i)^n \frac{d^n \psi(t)}{dt^n} \Big|_{t=0} \\
&= \delta_{n,1} a + \delta_{n,2} \sigma^2 + z \int_{\mathbb{R} \setminus \{0\}} s^n dr(s) \quad (68)
\end{aligned}$$



Figure 1: Construction of a generalized Feynman graph where every leave of type one together with the two edges connected to it is replaced by one edge

$\delta_{n,n'}$ being the Kronecker symbol.

Proof. After application of the linked cluster theorem to (56), the formula follows by a straight forward calculation, cf. [1] for details. ■

The particular form of (67) makes it trivial to carry out the integrals over the leaves of type one of the Parisi-Wu graph. This leads to the following simplification of Feynman rules:

Theorem 5.5. *Suppose that the noise η is of Lévy type. Note that in a Parisi-Wu graph, each leave of type one has exactly two legs. For a given Parisi-Wu graph $\mathcal{G} \in P(m,n)$, one can thus get an equivalent graph \mathcal{G}' where every leave of type one together with the two edges connected to it is replaced by one edge (cf. Figure (1)).*

Let $P'_c(m,n)$ be the collection of all connected graphs obtained in this way from $P_c(m,n)$. The Feynman rules $\mathcal{V}[\mathcal{G}']$ for the graph \mathcal{G}' then simplify in the following way:

- 1- Associate the values x_j , $j = 1, \dots, n$, to the roots, an integration variable y to each inner vertex and an integration variable z to each leave of type 2.
- 2- For each inner vertex v of a tree T_j , multiply with a multiplicity factor $M(v)$ and for each inner empty vertex with l legs multiply with a factor c_l .
- 3- For each leave of type two multiply with a factor $F_f(z)$.
- 4- For each edge multiply with a propagator $G(e)$.
- 5- Then integrate over all inner vertices and all leaves of type two.

In this way, one obtains in the sense of formal power series

$$\langle X(x_1) \dots X(x_n) \rangle^T = \sum_{m=0}^{\infty} (-\lambda)^m \sum_{\mathcal{G}' \in P'_c(m,n)} \mathcal{V}[\mathcal{G}'](x_1, \dots, x_n) \quad (69)$$

6 Equilibrium correlation function

In this short section we determine the limit of the correlation functions when all time arguments go to infinity simultaneously.

Definition 6.1. Let $P'_{1,c}(m, n)$ be the collection of graphs in $P'_c(m, n)$ that do not possess a leaf of type 2. For $\mathcal{G} \in P'_{1,c}(n, m)$ the value $\mathcal{V}_{\infty}[\mathcal{G}](\mathbf{x}_1, \dots, \mathbf{x}_n)$ is obtained the following Feynman rules:

- 1- Assign values $(0, \mathbf{x}_1), \dots, (0, \mathbf{x}_n) \in L_{\delta}$ to the roots of the trees T_1, \dots, T_n .
 - Assign values $y_1, \dots, y_m \in \Gamma$, to the inner vertices of the tree.
 - Assign values $z_1, \dots, z_k \in \Gamma$ to the inner empty vertices.
- 2- For every vertex, v in a tree, multiply with the multiplicity coefficient $M(v)$ and for each inner empty vertex with l legs by c_l .
- 3-For every edge with two end points $e = \{v, w\}$, ($v \leq w$), assign a value $G(e)$.
- 4-Integrate over Γ_- with respect to the Lebesgue measure $dy_1 \dots dy_m dz_1 \dots dz_k$.

Lemma 6.2. Let $\mathcal{G} \in P'_c(m, n)$ and let $I(\mathcal{G}) = \prod_{e \in E(\mathcal{G})} G(e)$, then $\forall N \in \mathbb{N}$, $\exists K = K(N, m)$ such that

$$|I(\mathcal{G})| \leq \frac{K}{(1 + \max_{v, w \in V(\mathcal{G})} |v - w|^2)^N} . \quad (70)$$

Proof. Let $v', v'' \in V(\mathcal{G})$ such that $|v' - v''|^2 = \max_{\substack{v, w \in V(\mathcal{G}) \setminus \{v_0\} \\ v \neq w}} |v - w|^2$. As \mathcal{G} is connected, there exist a walk W from v' to v'' and let $q \leq m + 1$ be the number of steps from v' to v'' . Let e_1, e_2, \dots, e_q be the edges of the walk W . Then

$$I(\mathcal{G}) = \prod_{e \in W} G(e) \times \prod_{e \in E(\mathcal{G}) \setminus W} G(e) \quad (71)$$

In W , there must be at least one of the q steps that is $\geq \frac{|v'-v''|}{q}$. By the use of Lemma 2.1 we find

$$|I(\mathcal{G})| \leq \frac{K(N, m)}{(1 + |v' - v''|^2)^N}. \quad (72)$$

■

Theorem 6.3. *Let $\mathcal{G} \in W_c(m, n)$, then the perturbation series for the truncated moments converges in the sense of formal power series, when t goes to infinity to :*

$$\langle X_\infty(\mathbf{x}_1) \cdots X_\infty(\mathbf{x}_n) \rangle^T = \lim_{t \rightarrow \infty} \langle X(x_1) \cdots X(x_n) \rangle^T \quad (73)$$

where

$$\langle X_\infty(\mathbf{x}_1) \cdots X_\infty(\mathbf{x}_n) \rangle^T = \sum_{m=0}^{\infty} \lambda^m \sum_{\mathcal{G} \in P'_{1,c}(m, n)} \mathcal{V}_\infty[\mathcal{G}](\mathbf{x}_1, \dots, \mathbf{x}_n). \quad (74)$$

Proof. Let $P'_{2,c}(m, n) = P'_c(m, n) \setminus P'_{1,c}(m, n)$. We have

$$\langle X(x_1) \cdots X(x_n) \rangle^T = \sum_{m=0}^{\infty} \lambda^m \sum_{\mathcal{G} \in P'_{1,c}(m, n)} \mathcal{V}[\mathcal{G}](x_1, \dots, x_n) + \sum_{m=0}^{\infty} \lambda^m \sum_{\mathcal{G} \in P'_{2,c}(m, n)} \mathcal{V}[\mathcal{G}](x_1, \dots, x_n) \quad (75)$$

As we take the limit $t \rightarrow \infty$ in the sense of formal power series, it suffices to show

$$\lim_{t \rightarrow \infty} \mathcal{V}[\mathcal{G}]((t, \mathbf{x}_1), \dots, (t, \mathbf{x}_n)) = \mathcal{V}_\infty[\mathcal{G}](\mathbf{x}_1, \dots, \mathbf{x}_n), \quad \forall \mathcal{G} \in P'_{1,c}(m, n) \quad (76)$$

and $\lim_{t \rightarrow \infty} \mathcal{V}[\mathcal{G}]((t, \mathbf{x}_1), \dots, (t, \mathbf{x}_n)) = 0, \quad \forall \mathcal{G} \in P'_{2,c}(m, n)$

Let first $\mathcal{G} \in P'_{1,c}(m, n)$, we have

$$\mathcal{V}[\mathcal{G}]((t, \mathbf{x}_1), \dots, (t, \mathbf{x}_n)) = M(\mathcal{G}) \int_{\Gamma_{(0,t)}^m} \prod_{e \in E(\mathcal{G})} G(e) \bigotimes_{v \in V(\mathcal{G}) \setminus \{(t, \mathbf{x}_1), \dots, (t, \mathbf{x}_n)\}} dv \quad (77)$$

where $\Gamma_{(a,b)} = \{x = (t, \mathbf{x}) \in \Gamma, a < t \leq b\}$. Now we transform $s \rightarrow s - t$ for $v = (s, \mathbf{v}) \in V(\mathcal{G})$ and let t go to infinity.

Let $\Lambda_e = \{e = \{v, w\}, v = (t_v, \mathbf{v}), w = (t_w, \mathbf{w}), (v \leq w)\}$, we get

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathcal{V}[\mathcal{G}]((t, \mathbf{x}_1), \dots, (t, \mathbf{x}_n)) &= M(\mathcal{G}) \lim_{t \rightarrow \infty} \int_{\Gamma_{(-t,0)}^m} \prod_{\Lambda_e} G(e) \bigotimes_{v \in V(\mathcal{G}) \setminus \{(x_1, \dots, x_n)\}} dv \Big|_{x_l = (0, \mathbf{x}_l)} \\ &= \mathcal{V}_\infty[\mathcal{G}](\mathbf{x}_1, \dots, \mathbf{x}_n) \end{aligned} \quad (78)$$

where the convergence in the last step is due to Lemma 6.2.

Let now $\mathcal{G} \in P'_{2,c}(m, n)$. Then, again by Lemma 6.2, $I(\mathcal{G})$ is rapidly decreasing in the difference of the time argument t of the external vertices and the time argument 0 of the leave of type two. Consequently, the integral over $I(\mathcal{G})$ gives no contribution in the limit $t \rightarrow \infty$. ■

7 Determination of the law of the solution

The main subject of this section is to find statistical information on the law of the solution of equation (1) from empirical data. One approach to achieve this is the least square method, i.e. to solve the minimization problem for the two point function of the stationary distribution

$$\begin{cases} Q(c_1, c_2, \dots) = \int |F_{th}(\mathbf{x}, c_1, c_2, \dots) - F_{em}(\mathbf{x})|^2 d\mathbf{x}, \\ \frac{\partial Q(c_1, c_2, \dots)}{\partial c_j} = 0, \quad j \in \mathbb{N}, \end{cases} \quad (79)$$

where F_{em} is the empirical³ correlation function. Note that by Lemma (6.2) the function $F_{th}(\mathbf{x}) = \langle X_\infty(\mathbf{0})X_\infty(\mathbf{x}) \rangle^T$ to any order of perturbation theory is rapidly decreasing in \mathbf{x} and thus there should be no problem with the convergence of the integral in (79) – if the modeling is not completely wrong, F_{em} should also be of fast decay.

In the remainder of the section, we give the solution to (79) to the first order in perturbation theory for $p = 3$. To simplify the calculation we note first

Lemma 7.1. *Let the measure r be symmetric, i.e. $r(A) = r(-A) \forall A \in \mathcal{B}$. Then one can omit all such generalized Parisi-Wu graphs from the perturbation series that have an empty vertex with an odd number of legs.*

We assume this symmetry of r in the following. On inspection of the first order solution, we obtain by the use of Theorem (5.5):

$$F_{th}(\mathbf{x}, c_2, c_4) = c_2 P_1(\mathbf{x}) + \lambda c_4 P_2(\mathbf{x}) + o(\lambda^2) \quad (80)$$

³In material sciences, the empirical correlation function $F_{em}(t, \mathbf{x}, c_1, c_2, \dots)$ can be measured by X-ray spectroscopy or sampling methods, cf. e.g. [5, 14].

$$F_{th}(\mathbf{x}, c_2, c_4) = \text{---}\times\text{---}\bigcirc\text{---}\times \quad + \quad \text{---}\times\text{---}\bigcirc\text{---}\bigcirc\text{---}\times + o(\lambda^2)$$

Figure 2: Expression of the correlation function in the first order expansion.

where $P_j(\mathbf{x}) = \mathcal{V}_\infty[\mathcal{G}_j](\mathbf{0}, \mathbf{x})/c_{2j}$, $j = 1, 2$ and \mathcal{G}_j the first/second graph in the above first order expansion (cf. Figure 2). Note that P_j does not depend on c_2, c_4 anymore.

Apparently, Q only depends on c_2 and c_4 . We have to solve the equations $\frac{\partial Q(c_2, c_4)}{\partial c_i} = 0, i = 2, 4$

Moreover,

$$Q(c_2, c_4) = \alpha c_2^2 + \beta c_4^2 + 2\gamma c_2 c_4 - 2c_2 a - 2c_4 b + c \quad (81)$$

where $\alpha = \int P_1^2 d\mathbf{x}$, $\beta = \lambda^2 \int P_2^2 d\mathbf{x}$, $\gamma = \lambda \int P_1 P_2 d\mathbf{x}$, $a = \int P_1 F_{em} d\mathbf{x}$, $b = \lambda \int P_2 F_{em} d\mathbf{x}$ and $c = \int F_{em}^2 d\mathbf{x}$. We now write down the equations $\frac{\partial Q}{\partial c_i} = 0, i = 2, 4$. and solve for c_2 and c_4 .

One then obtains the following first order approximation of c_2 and c_4 :

$$c_2 = \frac{a\beta - \gamma b}{\alpha\beta - \gamma^2} \quad (82)$$

and

$$c_4 = \frac{\alpha b - \gamma a}{\alpha\beta - \gamma^2} \quad (83)$$

c_2 gives a measure for strength of the fluctuations of η . Let the kurtosis K given by the following expression:

$$K = \frac{c_4}{c_2^2} \quad (84)$$

If $K = 0$, then there is no jump, the stochastic dynamic is purely diffusive. If $0 < |K| \ll 1$, there are some jumps but the stochastic dynamic is predominantly diffusive. If $|K| \gg 1$, then the stochastic dynamic is predominantly ruled by jumps.

8 Appendix A

For the convenience of the reader we collect some graph-theoretic notions which have been used in this work.

Let V be a finite set and

$$E = \{e = \{v, w\}; v, w \in V\}.$$

The elements of V are called vertices, which are of different types in our case we have the roots \times , the inner vertices \bullet , the empty vertices \circ and the leaves of type one \otimes . Such vertices are labeled by arguments in Γ . We have another type of vertex, called the leaf of type two, \diamond , which are in L_δ .

The elements of E are called edges, i.e. lines connecting exactly two vertices, and we say that an edge $e = \{v, w\}$ joins v and w . Thus we can define a graph

$$\mathcal{G} = \mathcal{G}(V)$$

with the vertex set V as a family of pairings

$$E(\mathcal{G}) \subseteq \{e = \{a, b\}, a, b \in V\}.$$

For a given graph \mathcal{G} we note $E(\mathcal{G})$ (resp. $V(\mathcal{G}) = V$) the set of edges (resp. vertices) of \mathcal{G} .

A graph H is called a subgraph of the graph \mathcal{G} and we write $H \subset \mathcal{G}$ when the vertex set $V(H)$ of H is contained in the vertex set $V(\mathcal{G})$ of \mathcal{G} and all edges of H are edges in \mathcal{G} .

For $v_1, v_n \in V(\mathcal{G})$ we define a $v_1 - v_n$ walk on \mathcal{G} as a sequence of vertices and edges $W = (v_1, e_1, v_2, e_2, \dots, v_n)$ such that

$$e_i = \{v_i, v_{i+1}\} \in E(\mathcal{G})$$

if $v_1 = v_n$ we say that W is a closed walk on \mathcal{G} , this closed walk is said to be a cycle when $e_i \neq e_j, \forall i, j \in \{1, \dots, n-1\}$.

degree of a vertex.

Let \mathcal{G} be a graph and $v \in V(\mathcal{G})$, the degree of v , noted by $p(v)$ is defined as :

$$p(v) = \#\{e \in E(\mathcal{G}) : e = \{v, w\}, w \in V\} + \#\{e \in E(\mathcal{G}) : e = \{v\}\}$$

connected graph.

A graph \mathcal{G} is called connected if there is a $v - w$ walk for all $v, w \in V(\mathcal{G})$, otherwise \mathcal{G} is disconnected.

rooted tree.

a rooted tree is a pair (T, v) such that T is a tree and v a vertex of T with $p(v) = 1$. v is called the root of the tree T .

leaf.

Let (T, v) be a rooted tree, then any vertex w of T such that $p(w) = 1$, $w \neq v$, is called a leaf of T .

We note $E(T)$, $V(T)$ and $L(T)$ respectively the sets of edges, vertices and leaves of the rooted tree T .

rooted subtree.

A rooted subtree of the rooted tree (T, x) is a pair (S, v) where S is a subgraph of T and $v \in V(T)$ such that $\forall w \in V(S)$ we have $v \leq w$ and $p(v) = 1$.

A **cut**, $\mathcal{C}(T)$ of the rooted tree T in the vertex v , directly connected to the root by an edge $e = \{x, v\}$, is the uniquely defined collections $\{T_1, \dots, T_n\}$ of rooted subtrees of T with root v such that $\bigcup_{j=1}^n L(T_j) = L(T)$.

Attachment of rooted trees.

Let $x \in \Gamma$ and $(T_1, y) \dots (T_n, y)$ be a collections of rooted trees with $V(T_i)$ and $E(T_i)$ design respectively the set of vertices and edges of the i -th rooted tree. We define an attached rooted tree (T, x) by :

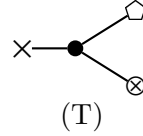
$V(T) = \bigcup_{i=1}^n V(T_i) \cup \{x\}$ and $E(T) = \bigcup_{i=1}^n E(T_i) \cup \{x, y\}$ respectively the set of vertices and edges of T .

Rooted trees with two types of leaves.

Let (T, x) be a rooted tree and $L(T)$ be the set of the leaves of T , (T, x) is said to be a rooted tree with two types of leaves if and only if :

$$L(T) = L_1(T) \cup L_2(T) \text{ and } l \in L(T) \setminus L_1(T) \implies l \in L_2(T).$$

We said that $L_1(T)$ is the set of the leaves of type one of T and $L_2(T)$ is the set of the leaves of type two of T .



In this graph (T) the inner vertices are **distinguishable** and have **non distinguishable** legs : Let x be the root of the tree T and we consider the edges $e_1 = \{y_1, z_1\}$ and $e_2 = \{y_1, z_2\}$, the value of the tree T [cf. Def.(3.1)], is the same when we permute the edges i.e when $e_1 = \{y_1, z_2\}$ and $e_2 = \{y_1, z_1\}$.

In this Parisi-Wu graph (\mathcal{G}), with one inner vertex y_1 and two roots x_1, x_2 , the empty vertex have **non distinguishable** legs .

Let z be the empty vertex and $e_1 = \{y_1, z\}$, $e_2 = \{y_2, z\}$ be the edges connected to the empty vertex.

The value of the generalized Parisi-Wu graph (\mathcal{G}), [cf. Def.(4.4)] is the same when we permute the edges i.e when $e_1 = \{y_2, z\}$ and $e_2 = \{y_1, z\}$.

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