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THE MEDVEDEV AND MUCHNIK LATTICES OF Π^0_1 CLASSES

A Thesis in

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by

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Abstract

This thesis contains four chapters including the Introduction. It is an analysis of the structure of the Medvedev and Muchnik lattices of non-empty Π^0_1 classes of 2^ω. These two structures are denoted $\mathcal{P}_M$ and $\mathcal{P}_w$ respectively. $\mathcal{P}_M$ and $\mathcal{P}_w$ are countable, distributive lattices each with a maximum and minimum element.

Chapter 2 has been accepted for publication in Mathematical Logic Quarterly [4]. Its main result is a proof that every finite distributive lattice can be embedded into $\mathcal{P}_M$ and $\mathcal{P}_w$. Further, given any special Π^0_1 subset $P \subseteq 2^\omega$, any finite distributive lattice can be embedded into $\mathcal{P}_M$ and $\mathcal{P}_w$ with its maximal element going to P. As a corollary, any non-zero Muchnik or Medvedev degree is the least upper bound of two strictly lower degrees. The result can be extended further for $\mathcal{P}_M$. We show that given any $P >_M Q$, any finite distributive lattice can be embedded between P and Q with its top element going to P.

The second part of Chapter 2 deals with a model theoretic consequence of these theorems. Here it is shown that both $\mathcal{P}_M$ and $\mathcal{P}_w$ have decidable $\exists$-theories.

Chapter 3 also deals with lattice embeddings, but this time of countable lattices. We use similar techniques to those used in [17]. The two main countable lattices that we deal with are $FD(\omega)$ and $FB(\omega)$ - the free distributive lattice on $\aleph_0$ generators and the free Boolean algebra on $\aleph_0$ generators respectively. It is shown that, in $\mathcal{P}_M$, $FD(\omega)$ can be embedded below any non-zero element of the lattice. Similarly, we show that in the Muchnik lattice, $FB(\omega)$ can be embedded below any non-zero element.

The second result is stronger as any countable distributive lattice can be embedded into $FB(\omega)$, and this is not the case for $FD(\omega)$. We do, however, show that the distributive lattice of finite (co-finite) subsets of ω can be embedded into $\mathcal{P}_M$ below any non-zero element.

Chapter 4 examines the relationship between certain structural properties of Π^0_1 classes and their Muchnik and Medvedev degrees. Two structural properties - *smallness* and *very smallness* - are defined and examined. We show that the class of Π^0_1 classes that contain a small (very small) subset forms a non-trivial proper prime ideal in $\mathcal{P}_w$ and $\mathcal{P}_M$. We also look at the relationship between smallness, very smallness and the well-studied property of thinness. We show there are thin sets that are not very small and vice-versa as well as other results of this sort.

My advisor for this dissertation was Stephen G. Simpson of The Pennsylvania State University.

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Acknowledgments

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Chapter 1

Introduction and Preliminaries

1.1 Introduction

Let ω denote the set of natural numbers. ω^ω and 2^ω are, respectively, the set of functions from ω to ω and the set of functions from ω to $\{0,1\}$ endowed with the product topology. $2^{<\omega}$ is the set of finite binary strings.

Definition 1.1.1. *A subset, P, of ω^ω is a Π^0_1 class if and only if it can be defined in the following way:*

$$f \in P \Leftrightarrow \forall n \in \omega \ R(f,n),$$

where $R(f,n)$ is some recursive predicate.

This dissertation is a contribution to the study of Π^0_1 subsets of 2^ω. These sets are an important field of study in recursion theory and have applications to recursive algebra, analysis, model theory and reverse mathematics, as well as the general areas of logic and the foundations of mathematics.

Part of their attractiveness is their ubiquitousness. They have characterisations as prime ideals of recursively enumerable (r.e.) commutative rings with unity, k-colourings of recursive graphs, graphs of recursively continuous functions, completions of logical theories and more (see [6]).

Other, useful and enlightening definitions are posssible. For example:

Theorem 1.1.2. *$P \subseteq \omega^\omega$ is a Π^0_1 class if and only if it is the set of infinite paths through some recursive tree.*

A *recursive tree* is a recursive set of strings of natural numbers, closed under taking initial segments. Another possible definition of Π^0_1 subsets of 2^ω is:

Theorem 1.1.3. *$P \subseteq 2^\omega$ is a non-empty Π^0_1 class if and only if it is the Stone space of some recursively presented, countably generated Boolean Algebra.*

Here a *recursively presented Boolean algebra* is a presentation of a Boolean algebra with a recursively enumerated set of relations - the quotient by an r.e. ideal, of the free Boolean algebra on countably many generators.

All of these characterisations will be used in the chapters that follow, but the main conception will be that of viewing Π^0_1 classes as paths through infinite trees. We can also characterise a Π^0_1 class a slightly different way. If $\omega^{<\omega}$ is the full tree of finite sequences of natural numbers, and $\sigma_0, \sigma_1, \sigma_2, \ldots$ is a (usually infinite) recursive sequence of strings of natural numbers, then the subset of ω^ω,

$$\omega^\omega \setminus \bigcup_i \{f : f \supset \sigma_i\},$$

is a Π^0_1 class.

The basic notions that we will use to study Π^0_1 classes will be those of Muchnik and Medvedev reducibility. These are ideas that apply to subsets of ω^ω in general, not just Π^0_1 classes, and they seek to generalise the well-studied idea of Turing reducibility.

$X \subseteq \omega$ is Turing reducible to $Y \subseteq \omega$ if there is a Turing machine which, given Y as an oracle, computes X. This idea can be naturally extended to subsets of ω^ω in at least two different ways:

- $A \subseteq \omega^\omega$ is *Medvedev reducible to* $B \subseteq \omega^\omega$ ($A \leqslant_M B$) if there is a recursive functional $\Phi : B \to A$. That is there is an oracle Turing machine, Φ, which, given an element of B as an oracle, computes an element of A.

- $A \subseteq \omega^\omega$ is *Muchnik reducible to* $B \subseteq \omega^\omega$ ($A \leqslant_w B$) if, for every $f \in B$, there is some oracle Turing machine that can use f to compute an element of A.

It is apparent from the definition that Medvedev reducibility is stronger than Muchnik reducibility, and in fact it is strictly stronger. The w subscript for Muchnik reducibility stands for "weak".

We write $P \equiv_w Q$ if and only if $P \geqslant_w Q$ and $Q \geqslant_w P$ and similarly for $\equiv_M$.

In [24] § 13.7, Rogers discusses Medvedev reduciblity in terms of *mass problems*. The idea is that a subset of ω^ω is the solution set of some mathematical problem. If problem A is Medvedev reducible to problem B, it means there is some uniform computable way to convert solutions of B to solutions of A. A would be Muchnik reducible to B in this context, if each solution of B contained enough information to compute a solution of A.

Rogers also suggests (§15.1 pg 343 [24]) that Medvedev reducibility be used to analyse the analytic heirarchy (of which the Π^0_1 classes form a subset) in much the same way that Turing reducibility has been used to investigate the arithmetical heirarchy. The idea has just recently resurfaced with Simpson suggesting in [13] (Aug. 13 1999) that

Medvedev and Muchnik reducibility be used to investigate the recursion theoretic nature of Π^0_1 classes -the analogy being made between the upper semi-lattice of r.e. degrees and the Muchnik degrees of Π^0_1 classes. Since then the subject has grown with work having been done by Simpson [27], [28]; Cenzer and Hinman; [5]; and Slaman [29]. This thesis is a continuation of the project.

Applying these ideas to Π^0_1 classes, a Medvedev reduction is a recursive (and necessarily continuous) transformation of one Π^0_1 class into another. This is essentially an algebraic concept. Just as every non-empty Π^0_1 subclass of 2^ω is the Stone space of a Boolean algebra, each Medvedev reduction is a recursive Boolean homomorphism of their duals. More precisely, there is a contravariant bijective functor between the categories of recursively presented Boolean algebras and recursive homomorphisms and the category of non-empty Π^0_1 classes and Medvedev reductions. The entire part of this thesis that deals with Medvedev reducibility can be regarded as a contribution to the study of recursively presented Boolean algebras and their homomorphisms.

Muchnik reducibility has a much different flavour. We are concerned here only with the Turing degrees of the elements of the Π^0_1 class. In this, it is more like traditional recursion theory. It may be thought that nothing is to be gained by restricting ourselves to Π^0_1 subsets of 2^ω as opposed to arbitrary subsets of ω^ω. However, earlier work by Jockusch and Soare [17] [16] and others, has shown that indeed significant things can be said about the Turing degrees of elements of Π^0_1 classes. Furthermore, Simpson's lemma 4.2.13 in Chapter 4 provides an interesting connection between the two types of reducibility and I predict that it will evolve into a significant linchpin of the subject.

1.2 Basic Theory and Notation

This is a summary of basis results in the field, contained in [27]. Both $\leqslant_M$ and $\leqslant_w$ are pre-orders on the class of subsets of ω^ω. Degree structures are induced in the same way as for the Turing degrees, viz.,

$$\deg_w(X) = \{Y : Y \equiv_w X\}$$

and similarly for $\deg_M(X)$. A canonical partial order on the degrees is then defined by

$$\deg_w(X) \geqslant \deg_w(Y) \text{ if and only if } X \geqslant_w Y$$

and likewise for the Medvedev degrees.

Let $\mathcal{P}_M$ and $\mathcal{P}_w$ denote the degree structures of the non-empty Π^0_1 subsets of 2^ω under Medvedev and Muchnik reducibility respectively. $\mathcal{P}_M$ and $\mathcal{P}_w$ form distributive lattices with maximum and minimum elements. If P and Q are non-empty Π^0_1 subsets of 2^ω, the join and meet of their degrees in both of these lattices are the respective degrees of:

$$P \vee Q = \{f \oplus g : f \in P \text{ and } g \in Q\},$$

and,

$$P \wedge Q = \{0^\frown f : f \in P\} \cup \{1^\frown f : f \in Q\},$$

where,

$$i^\frown f(n) = \begin{cases} i & \text{if } n = 0, \\ f(n-1) & \text{otherwise,} \end{cases}$$

and $f, g \in \omega^\omega$, then $f \oplus g$ is defined by:

$$f \oplus g(n) = \begin{cases} f(n/2) & \text{if } n \text{ is even} \\ g((n-1)/2) & \text{if } n \text{ is odd} \end{cases}$$

If A and B are any two subsets of ω, then the *separating class* of A and B, denoted $\mathcal{S}(A,B)$, is the set $\{X : X \supseteq A, \text{ and } X \cap B = \emptyset\}$. If A and B are r.e. then $\mathcal{S}(A,B)$ is a Π^0_1 class.

In both lattices, the separating class of $\{n : \{n\}(n)\downarrow = 0\}$ and $\{n : \{n\}(n)\downarrow = 1\}$ has maximum degree [27]. The class of all completions of Peano arithmetic also has maximum degree. Any subset of 2^ω with a recursive element is a representative of the minimum degree. A *special* Π^0_1 class is one that is non-empty and has no recursive element. Any recursively bounded Π^0_1 subset of ω^ω is recursively homeomorphic to (and therefore Medvedev and Muchnik equivalent to) a Π^0_1 subset of 2^ω, so everything that follows can be generalised to recursively bounded Π^0_1 subsets of ω^ω.

The Turing degrees of elements of Π^0_1 sets have been investigated in [17] and [16] where the term *special* Π^0_1 set is used. Special Π^0_1 sets are defined as being non-empty and having no recursive members. They will play an important role in this paper.

If $f \in \omega^\omega$, then f^- denotes the function, $n \mapsto f(n+1)$

For standard recursion-theoretic notation see [30] or [24] or [21].

DNR_k is the Π^0_1 class,

$$\{f : \forall n \in \omega\ f(n) < k \text{ and } f(n) \neq \{n\}(n)\}$$

The first important important theorem in the subject is Friedberg and Jockusch's proof [15] that $\mathrm{DNR}_{k+1} \equiv_w \mathrm{DNR}_k$, for all $k \geqslant 2$, but that $\mathrm{DNR}_2 >_M \mathrm{DNR}_3 >_M \mathrm{DNR}_4 \ldots$. This work was done before $\mathcal{P}_w$ and $\mathcal{P}_M$ were explicitly defined.

Simpson proves in [27] that any two Medvedev complete degrees are recursively homeomorphic. This result will be used in Chapter 4. Since then, Cenzer and Hinman have shown that the Medvedev lattice is dense, a result that I improve upon in Chapter 2 by using different methods. In an as yet unpublished paper [29], Simpson and Slaman show that every non-zero Muchnik degree contains more than one Medvedev degree and that there is no $\leqslant_M$-maximum in the class of Π^0_1 classes of positive measure.

Chapter 2

Splittings and Finite Embeddings

2.1 Splitting Theorems

Theorem 2.1.1. *Let R and T be any special Π^0_1 subsets of 2^ω. Then there exist two other (necessarily special) Π^0_1 subsets of 2^ω, R^0 and R^1, such that:*

i. $R^0, R^1 <_w R$,

ii. $R^0 \vee R^1 \equiv_w R$,

iii. $R^0, R^1 \not\geq_w T$.

The above also holds for the same R^0 and R^1 with $<_M$ and $\equiv_M$ replacing $<_w$ and $\equiv_w$.

The essence of the theorem is contained in the following lemma. The proof of Theorem 2.1.1 will come after the proof of the lemma.

Lemma 2.1.2. *Let P be any special Π^0_1 subset of 2^ω and A be any r.e. set. Then there exist r.e. sets, A^0 and A^1, such that:*

i. $A^0 \cup A^1 = A, \quad A^0 \cap A^1 = \emptyset$,

ii. for each $i \in \{0, 1\}$ and $f \in P$, $A^i \not\geq_T f$.

Letting $\langle .,. \rangle : \omega^2 \to \omega$ be a recursive bijection, we will explicitly construct each A^i to satisfy all of the following requirements:

$$\mathcal{R}_{\langle e,i \rangle} \equiv \{e\}^{A^i} \notin P.$$

Notation and Conventions

- If $P \subseteq 2^\omega$ is a given non-empty Π^0_1 class, $\langle P_s \rangle_{s\in\omega}$ will be a recursive sequence of nested clopen subsets of 2^ω such that $P = \bigcap_s P_s$.

- If P is a Π^0_1 class, let T_P be a fixed recursive binary tree such that P is exactly the set of paths through T_P. $T_{P,s}$ will be a uniformly recursive sequence of nested trees such that, for each s, the set of paths through $T_{P,s}$ is P_s.

- $u(A; i, m, s)$ is the maximum use made of $A \subseteq \omega$ in the computation $\{i\}^A_s(m)$. If $f \in 2^\omega$ then $u(A; A \oplus f, i, m, s)$ is the maximum use made of A in the computation $\{i\}^{A\oplus f}_s(m)$.

- $[n]$ is the set $\{0, 1, 2, \dots n-1\}$ and $\{i\}[n]$ is a partial sequence of length n. That is,

$$\{i\}[n](m) = \begin{cases} \{i\}(m) & \text{if } m < n \text{ and } \{i\}(m)\downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

To say $\{i\}[n] \in T_P$ is to say that for all $m < n$, $\{i\}(m)\downarrow$ and

$$\langle \{i\}(0), \{i\}(1), \dots \{i\}(n-1) \rangle \in T_P.$$

- $f|_u = f$ restricted to $[u]$. $A|_u = \chi_A|_u$.

- If $\tau \in 2^{<\omega}$, then $|\tau|$ is the length of τ.

The method we will use is very similar to that used to prove Sacks' Splitting Theorem for the r.e. degrees, and we will closely follow the exposition in Soare ([30] Theorem VII.3.2). Lemma 2.1.2 may also be seen as a strengthening of Theorem 2 in [16].

The Construction

Let P, A and i be as in Lemma 2.1.2 and we fix a recursive enumeration of A such that $A_{s+1} \setminus A_s$ has exactly one element for each s. For each i we will define a recursive sequence of finite sets, $\langle A^i_s \rangle_{s\in\omega}$, and A^i will then be $\bigcup_s A^i_s$.

Stage 0: $\quad A^i_0 = \emptyset$.

Stage s+1: Assume A^i_s has been defined. We can then make the following definitions:

Length-of-agreement functions:

$$l_s(e,i) := \max\{y : \{e\}^{A^i_s}_s[y] \in T_P\}.$$

Restraint functions:

$$r_s(e,i) := \max\{u(A^i_s; e, x, s) : x \leqslant l_s(e,i)\}.$$

Injury sets:

$$I_{\langle e,i\rangle} := \{x : \exists s\ x \in A^i_{s+1} \setminus A^i_s \text{ and } x \leqslant r_s(e,i)\}.$$

If $x \in A^i_{s+1} \setminus A^i_s$ and $x \leqslant r_s(e,i)$, we say $\mathcal{R}_{\langle e,i\rangle}$ *is injured at stage* $s+1$.

Let x be the unique element of $A_{s+1} \setminus A_s$. Choose the least $\langle e,i\rangle < s$ such that $x \leqslant r_s(e,i)$ and enumerate x into A^{1-i}_{s+1}. That is, let $A^{1-i}_{s+1} = A^{1-i}_s \cup \{x\}$. Set $A^i_{s+1} = A^i_s$ and say $\mathcal{R}_{\langle e,i\rangle}$ *receives attention at stage* $s+1$.

If there is no such $\langle e,i\rangle$, then enumerate x into A^0_{s+1} and leave A^1_s unchanged.

Lemma 2.1.3. *If* $\{e\}^{A^i} \in P$, *then* $\lim_s l_s(e,i) = \infty$.

Proof. Suppose $\{e\}^{A^i} \in P$ and let $n \in \omega$ be arbitrary. Then let $u = \max\{u(A^i; e, m) : m < n\}$ and now take s' so large that both the following hold:

i. $A^i_{s'}|_u = A^i|_u$,

ii. $\forall m < n\ \{e\}^{A^i}_{s'}(m)\downarrow$.

Then $\{e\}^{A^i_s}_s[n] = \{e\}^{A^i}[n] \in T_P$ and $l_s(e,i) \geqslant n$ for all $s \geqslant s'$. As n was arbitrary, the result follows.

□

Lemma 2.1.4. *For all* $e \in \omega$ *and* $i \in \{0,1\}$,

I. $I_{\langle e,i\rangle}$ *is finite,*

II. $\{e\}^{A^i} \notin P$,

III. $r(e,i) := \lim_s r_s(e,i)$ *exists and is finite.*

Proof. Take any $e \in \omega$ and $i \in \{0,1\}$. As induction hypothesis assume I., II., and III. hold for all $\langle e',i'\rangle < \langle e,i\rangle$.

I. By III. we can choose t and r such that for all $\langle e',i'\rangle < \langle e,i\rangle$ and $s \geqslant t$, $r_s(e',i') = r(e',i')$ and $r > r(e',i')$. Now take $v > t$ such that $A_v|_r = A|_r$. So $\mathcal{R}_{\langle e,i\rangle}$ cannot be injured after stage v and I. holds for $\langle e,i\rangle$.

II. Assume $\{e\}^{A^i} \in P$. To get a contradiction we will construct a recursive path $f \in P$. Let s' be such that $\mathcal{R}_{\langle e,i\rangle}$ is never injured after stage s'. Fix any $n \in \omega$ and we will recursively compute $f(n)$. Using I., $\lim_s \ l_s(e,i) = \infty$ so choose the least $s = s(n) > s'$ such that $l_s(e,i) > n$. If x is enumerated into A^i after stage s, then it must be greater than $u(A^i_s; e, n, s)$. So $\{e\}^{A^i_s}_s(n) = \{e\}^{A^i}(n)$. Set $f(n)$ equal to $\{e\}^{A^i_s}_s(n)$ for all $n \in \omega$. s is clearly a recursive function of n, so f itself is recursive and an element of P.

III. Let n be maximum such that $\{e\}^{A^i}[n] \in T_P$. Choose s' so large that for all $s \geqslant s'$,

i. $\{e\}^{A^i_s}_s[n] = \{e\}^{A^i}[n]$,

ii. $A^i_s|_u = A^i|_u$ where $u = \max\{u(A^i; e, m) : m < n\}$,

iii. $\mathcal{R}_{\langle e,i\rangle}$ is not injured at stage s.

If $\{e\}^{A^i_s}_s(n)\uparrow$ for all $s \geqslant s'$, then $u(A^i_s; e, n, s) = 0$ and $r_s(e,i) = r_{s'}(e,i)$ for all $s \geqslant s'$. So $\lim_s r_s(e,i)$ exists. On the other hand, suppose $\{e\}^{A^i_t}_t(n)\downarrow$ for some $t \geqslant s'$. If $x \in A^i \setminus A^i_t$ then $x \in A^i_{v+1} \setminus A^i_v$ for some $v \geqslant t$. As $\mathcal{R}_{\langle e,i\rangle}$ is not injured at any stage $s \geqslant t$, $x > r_{v+1}(e,i)$. But $r_{v+1}(e,i) = r_t(e,i)$ by conditions i. and ii. above. So $x > u(A^i_t; e, n, t)$ and the computation $\{e\}^{A^i_t}_t(n)$ is preserved forever. Therefore, for all $s \geqslant t$,

$$\{e\}^{A^i_s}_s[n+1] = \{e\}^{A^i}[n+1] \notin T_P.$$

So $l_s(e,i) = l_t(e,i) = n$ and $u(A^i_s; e, x, s) = u(A^i_t; e, x, t)$ for all $x \leqslant n$ and $s \geqslant t$. $r(e,i)$ then exists by the definition of $r_s(e,i)$. □

The construction makes it clear that $A = A^0 \cup A^1$ and $A^0 \cap A^1 = \emptyset$, so Lemma 2.1.2 follows immediately from Lemma 2.1.4. Now we are in a position to prove Theorem 2.1.1. We will prove the Medvedev and Muchnik cases simultaneously.

Proof of Theorem 2.1.1. Let R and T be given. Suppose A and B be r.e. sets such that $S = \mathcal{S}(A, B)$ is Medvedev (and therefore Muchnik) complete.

Take the P in Lemma 2.1.2 to be $R \wedge T$ and A^0 and A^1 be as in the same lemma. Let $S^i = \mathcal{S}(A^i, B)$ for each $i \in \{0, 1\}$. Note that if $A^0 \subseteq X \subseteq \overline{B}$ and $A^1 \subseteq Y \subseteq \overline{B}$, then $A \subseteq X \cup Y \subseteq \overline{B}$, so it is clear that $S \leqslant_M S^0 \vee S^1$ and therefore that $S \equiv_M S^0 \vee S^1$ and $S \equiv_w S^0 \vee S^1$.

Set $R^i = R \wedge S^i$. It is immediate that $R^i \leqslant_M R$ and $R^i \leqslant_w R$ and because $A^i \in S^i$, item ii. of Lemma 2.1.2 implies $S^i \ngeqslant_w R \wedge T$ (and $S^i \ngeqslant_M R \wedge T$). Therefore, $S^i \ngeqslant_w R$ ($S^i \ngeqslant_M R$). So in fact, $R^i <_w R$ ($R^i <_M R$) for each $i \in \{0, 1\}$. Now we can make the following calculation:

$$
\begin{aligned}
R^0 \vee R^1 &= (R \wedge S^0) \vee (R \wedge S^1) \\
&\equiv_M R \wedge (R \vee S^0) \wedge (R \vee S^1) \wedge S \\
&\equiv_M R \wedge (R \vee S^0) \wedge (R \vee S^1).
\end{aligned}
$$

But,

$$R \geqslant_M R \wedge (R \vee S^0) \wedge (R \vee S^1) \equiv_M R \vee (R \wedge S^0 \wedge S^1) \geqslant_M R,$$

so $R^0 \vee R^1 \equiv_M R$ and $R^0 \vee R^1 \equiv_w R$. This gives us the required splitting. Finally, $R^i \ngeqslant_w R \wedge T$ so $R^i \ngeqslant_w T$ ($R^i \ngeqslant_M T$) for each i.

□

Lemma 2.1.2 is true even when P is taken to be a Π^0_1 subset of ω^ω. This can be seen in two ways. First, the assumption of recursive boundedness is never used in the proof, so the generalisation follows immediately from the proof of the lemma. Second, via a theorem of Jockusch and Soare (Corollary 1.3, [16]) which states that for any special Π^0_2 class, P, there is a special, recursively bounded Π^0_1 class, Q, such that

$$\{\deg_T(f) : f \in Q\} \supseteq \{\deg_T(f) : f \in P\}.$$

In this more general form, the lemma implies Sacks' Splitting Theorem. Let C be any non-recursive Δ^0_2 set. Then $\{C\}$ is a special Π^0_2 class. Take Q as above and then Lemma 2.1.2 easily implies Sacks' theorem.

2.2 Dense Splitting

In the Medvedev case, we can improve Theorem 2.1.1 considerably by proving the following refinement of Lemma 2.1.2:

Lemma 2.2.1. *Let P and Q be non-empty Π^0_1 subsets of 2^ω such that $P >_M Q$, and let A be any r.e. set. Then there exist r.e. sets, A^0 and A^1, such that:*

i. $A^0 \cup A^1 = A, \quad A^0 \cap A^1 = \emptyset,$

ii. for each $i \in \{0,1\}, \quad \{A^i\} \vee Q \not\geq_M P.$

We will use this lemma as we used Lemma 2.1.2 - this time to prove that P can be split above Q. This is in contrast to the r.e. degrees, where Lachlan's "monster" theorem [18] states that such dense splitting fails.

The requirements for the construction will be:

$$\mathcal{R}^*_{\langle e,i\rangle} \equiv \{e\} : \{A^i\} \vee Q \nrightarrow P.$$

We will make similar definitions to before. The compactness of Π^0_1 subsets of 2^ω ensures that the following are well defined:

Length-of-agreement functions:

$$l^*_s(e,i) := \max\{y : \text{ for all } f \in Q_s,\ \{e\}^{A^i_s \oplus f}_s[y] \in T_P\}.$$

Restraint functions:

$$r^*_s(e,i) := \max\{u(A^i_s; A^i_s \oplus f, e, x, s) : x \leqslant l^*_s(e,i),\ f \in Q_s\}.$$

Injury sets:

$$I^*_{\langle e,i\rangle} := \{x : \exists s\ x \in A^i_{s+1} \smallsetminus A^i_s \text{ and } x \leqslant r^*_s(e,i)\}.$$

If $x \in A^i_{s+1} \smallsetminus A^i_s$ and $x \leqslant r^*_s(e,i)$, we say $\mathcal{R}^*_{\langle e,i\rangle}$ *is injured at stage* $s+1$.

Note that $l^*_s(e,i)$ and $r^*_s(e,i)$ are recursive in e, i and s.

Let x be the unique element of $A_{s+1} \smallsetminus A_s$. Choose the least $\langle e,i\rangle < s$ such that $x \leqslant r^*_s(e,i)$ and enumerate x into A^{1-i}_{s+1}.

If there is no such $\langle e,i\rangle$, then enumerate x into A^0_{s+1}.

Lemma 2.2.2. *If* $\{e\} : \{A^i\} \vee Q \to P$*, then* $\lim_s l^*_s(e,i) = \infty$*.*

Proof. Suppose $\{e\} : \{A^i\} \vee Q \to P$ and let $n \in \omega$ be arbitrary. Then let:

$$u = \max\{u(f; A^i \oplus f, e, m) : m < n,\ f \in Q\},$$

(again this exists by compactness)

$$v = \max\{u(A^i; A^i \oplus f|_{u+1}, e, m) : m < n,\ f \in Q\},$$

$$w = \text{ least } k,\ A^i_k|_{v+1} = A^i|_{v+1},$$

$$t = \text{ least } k,\ \{\tau \in T_{Q,k} :\ |\tau| = u+1\} = \{\tau \in T_Q :\ |\tau| = u+1\}.$$

Then for all $s \geqslant \max\{w,t\}$ such that $\{e\}_s^{A_s^i \oplus f}(m) \downarrow$ for all $m < n$, we have, $\{e\}_s^{A_s^i \oplus f}[n] = \{e\}^{A^i \oplus f}[n] \in T_P$ for all $f \in Q_s$. That is $l_s^*(e,i) \geqslant n$ and, as n was arbitrary, $\lim_s l_s(e,i) = \infty$.

□

Lemma 2.2.3. *For all $e \in \omega$ and $i \in \{0,1\}$,*

I. $I^*_{\langle e,i\rangle}$ *is finite,*

II. $\{e\} : \{A^i\} \vee Q \nrightarrow P$,

III. $r^*(e,i) := \lim_s r_s^*(e,i)$ *exists and is finite.*

Proof. Take any $e \in \omega$ and $i \in \{0,1\}$. As induction hypothesis assume I., II., and III. hold for all $\langle e',i'\rangle < \langle e,i\rangle$.

I. By III. we can choose t and r such that for all $\langle e',i'\rangle < \langle e,i\rangle$ and $s \geqslant t$, $r_s(e',i') = r(e',i')$ and $r > r(e',i')$. Now take $v > t$ such that $A_v|_r = A|_r$. So $\mathcal{R}^*_{\langle e,i\rangle}$ cannot be injured after stage v and I. holds for $\langle e,i\rangle$.

II. Assume $\{e\}^{A^i \oplus f} \in P$ for all $f \in Q$. Fix any $n \in \omega$. Using I., let s' be such that $\mathcal{R}^*_{\langle e,i\rangle}$ is never injured after stage s'. $\lim_s l_s^*(e,i) = \infty$, so choose the least $s = s(n) > s'$ such that $l_s^*(e,i) > n$. If x is enumerated into A^i after stage s, then it must be greater than $u(A_s^i; A_s^i \oplus f, e, n, s)$ for all $f \in Q$. So $\{e\}_s^{A_s^i \oplus f}(n) = \{e\}^{A^i \oplus f}(n)$ for all $f \in Q$. s is a recursive function of n, so $f \mapsto \{e\}_s^{A_s^i \oplus f}$ describes a recursive functional from Q into P, contradicting the fact that $P >_M Q$.

III. Let n be maximum such that for all $f \in Q$, $\{e\}^{A^i \oplus f}[n] \in T_P$. Using the compactness of Q, choose s' so large that for all $s \geqslant s'$,

i. $\{e\}_s^{A_s^i \oplus f}[n] = \{e\}^{A^i \oplus f}[n]$, for all $f \in Q$,

ii.$A_s^i|_u = A^i|_u$ where $u = \max\{u(A^i; A^i \oplus f, e, m) : m < n, f \in Q\}$

iii. $\mathcal{R}^*_{\langle e,i \rangle}$ is not injured at stage s.

If $\{e\}_s^{A_s^i \oplus f}(n) \uparrow$ for all $s \geqslant s'$ and $f \in Q$, then $u(A_s^i; A_s^i \oplus f, e, n, s) = 0$ and $r_s^*(e,i) = r_{s'}^*(e,i)$ for all $s \geqslant s'$. So $\lim_s r_s^*(e,i)$ exists. On the other hand, suppose $\{e\}_t^{A_t^i \oplus f}(n) \downarrow$ for some $t \geqslant s'$ and $f \in Q$. As before, $\mathcal{R}^*_{\langle e,i \rangle}$ is not injured at any stage $\geqslant s'$, so the computation is preserved forever. Therefore $l_s^*(e,i) = n$ for all $s \geqslant t$ also as before.

By compactness, there is a v such that for all $f \in Q$, $x \leqslant n$ and $s \geqslant t$,

$$\begin{aligned} \{e\}_s^{A_s^i \oplus f}(x) &\simeq \{e\}_t^{A_t^i \oplus f}(x) \\ &\simeq \{e\}_t^{A_t^i \oplus f|_v}(x). \end{aligned}$$

Let $k \geqslant t$ be a stage when $\{f|_v : f \in Q_k\} = \{f|_v : f \in Q\}$ and then for all $s \geqslant k$, $f \in Q_s$ and $x \leqslant n$, $u(A_s^i; A_s^i \oplus f, e, x, s) = u(A_k^i; A_k^i \oplus f, e, x, k)$ and $l_s^*(e,i) = n$. Finally we have, for all $s \geqslant k$,

$$\begin{aligned} r_s^*(e,i) &= \max\{u(A_s^i; A_s^i \oplus f, e, x, s) : x \leqslant l_s^*(e,i),\ f \in Q_s\} \\ &= \max\{u(A_k^i; A_k^i \oplus \tau, e, x, k) : x \leqslant n,\ \tau \in T_Q,\ |\tau| = v\} \end{aligned}$$

which is the maximum of a fixed finite set. Therefore $\lim_s r_s^*(e,i)$ exists and is finite.

□

This also concludes the proof of Lemma 2.2.1, the main purpose of which is to prove the following "dense splitting" theorem.

Theorem 2.2.4. *For any three non-empty* Π^0_1 *subsets of* 2^ω*,* R*,* Q *and* T *such that* $R >_M Q$*, and* $R \wedge T >_M Q$*, there exist two other* Π^0_1 *subsets of* 2^ω*,* R^0 *and* R^1 *such that:*

i. $R^0, R^1 <_M R$,

ii. $R^0 \vee R^1 \equiv_M R$,

iii. $R^0, R^1 >_M Q$,

iv. $R^0, R^1 \not\geqslant_M T$

Proof. As in Theorem 2.1.1, let $S = \mathcal{S}(A, B)$ be Medvedev complete. Take $R \wedge T$ to be the P of Lemma 2.2.1. Let A^0 and A^1 be as in Lemma 2.2.1, and $S^i = \mathcal{S}(A^i, B)$ for $i \in \{0, 1\}$. Set $R^i = R \wedge (S^i \vee Q)$. $R^i \leqslant_M R$, and as $A^i \in S^i$, Lemma 2.2.1 implies $S^i \vee Q \not\geqslant_M R$. So $R^i <_M R$. Also,

$$
\begin{aligned}
R^0 \vee R^1 &= (R \wedge (Q \vee S^0)) \vee (R \wedge (Q \vee S^1)) \\
&\equiv_M R \wedge (Q \vee S^0 \vee S^1) \\
&\equiv_M R
\end{aligned}
$$

As R^0 and R^1 must be Medvedev incomparable, and $R^i \not\geqslant_M T$ for each i, the theorem follows. □

Theorem 2.2.4 implies immediately the density of $\mathcal{P}_M$. The proof given here, however, is significantly different from the ones given in [5] and [3].

Theorems 2.1.1 and 2.2.4 can be extended even further to a "generalised splitting" theorem and a "generalised dense splitting" theorem respectively:

Theorem 2.2.5. *Let P be any special Π^0_1 subset of 2^ω and $\mathcal{L}$ be any finite distributive lattice. Then there is a lattice embedding of $\mathcal{L}$ into $\mathcal{P}_w$ sending the maximum element of $\mathcal{L}$ to the Muchnik degree of P.*

Theorem 2.2.6. *Given Π^0_1 subsets of 2^ω, $P >_M Q$, and any finite distributive lattice, $\mathcal{L}$, there is a lattice embedding of $\mathcal{L}$ into $\mathcal{P}_M$ between P and Q taking the maximum element of $\mathcal{L}$ to the Medvedev degree of P.*

Theorems 2.2.5 and 2.2.6 can be easily extended to include a condition on T similar to the ones in Theorems 2.1.1 and 2.2.4. These extended theorems then have Theorems 2.1.1 and 2.2.4 as corollaries if $\mathcal{L}$ is taken to be the four element diamond lattice. The proofs of Theorems 2.2.5 and 2.2.6 will use the following lattice-theoretic lemma.

Lemma 2.2.7. *Every finite distributive lattice can be lattice-embedded into a free finite distributive lattice, in a way that preserves the maximum element.*

Proof. Let $FD(m)$ be the free distributive lattice with m generators and let $\mathcal{B}_n$ denote the lattice of subsets of $N = \{0, 1, 2, \ldots, n-1\}$ under $\cup$ and $\cap$. Let $\mathcal{L}$ be a distributive lattice with operations $\vee$ and $\wedge$.

First observe that, using a representation theorem for finite distributive lattices (Theorem II.1.9 [14]), $\mathcal{L}$ can be represented as a sublattice of $\mathcal{B}_n$ for some n (in fact n is the number of join-irreducible elements of $\mathcal{L}$) and that the maximum element of $\mathcal{L}$ is represented by N - the maximum element of $\mathcal{B}_n$. So it is enough to embed $\mathcal{B}_n$ into $FD(n)$ preserving the maximum element. We will constuct an embedding, $\epsilon : \mathcal{B}_n \hookrightarrow FD(n)$, which preserves the least element of $\mathcal{B}_n$. As both $\mathcal{B}_n$ and $FD(n)$ are self dual, it is easy to convert this to an embedding that preserves the maximum.

Let $FD(n)$ be freely generated by $Y = \{y_0, y_1, \dots y_{n-1}\}$ and let $\widehat{y_i}$ denote $\bigwedge_{j\neq i} y_j$. If $Z \subseteq N$, we define,

$$\epsilon(Z) = \begin{cases} \bigvee_{i\in Z} \widehat{y_i} & \text{if } Z \neq \emptyset \\ \bigwedge_{i\in N} y_i & \text{if } Z = \emptyset \end{cases}$$

$\bigwedge_{i\in N} y_i$ is the minimum of $FD(n)$ so ϵ preserves the minimum. It is also clear that $\epsilon(Z_1 \cup Z_2) = \epsilon(Z_1) \vee \epsilon(Z_2)$. To see that ϵ preserves meets, note that $\widehat{y_i} \wedge \widehat{y_j} = \bigwedge_{i\in N} y_i$ if $i \neq j$ and that the distributive laws then give,

$$\bigvee_{i\in Z_1} \widehat{y_i} \wedge \bigvee_{i\in Z_2} \widehat{y_i} = \bigvee_{i\in Z_1\cap Z_2} \widehat{y_i}.$$

The proof that ϵ is one-to-one is also straightforward - if $\epsilon(X) = \epsilon(Y)$ and $k \in X \smallsetminus Y$ then,

$$\widehat{y_k} \leqslant \bigvee_{i\in X} \widehat{y_i} = \bigvee_{i\in Y} \widehat{y_i} \leqslant y_k,$$

contradicting freeness (see Theorem II.2.3 in [14]).

$\square$

The proofs of Theorems 2.2.5 and 2.2.6 now proceed as before. First, analogues of Lemmas 2.1.2 and 2.2.1 are established (Lemmas 2.2.8 and 2.2.9) and then Theorems 2.2.5 and 2.2.6 follow.

Lemma 2.2.8. *Let P be any special Π^0_1 subset of 2^ω and A be any r.e. set. Then there exist r.e. sets, A^i, $0 \leqslant i \leqslant n-1$, such that:*

i. $\{A^i : 0 \leqslant i \leqslant n-1\}$ *forms a partition of A,*

ii. for each $i \in \{0, 1, \ldots n-1\}$ *and* $f \in P$, $\bigoplus_{j \neq i} A^j \not\geqslant_T f$.

Proof. (*sketch*)

The proof will be virtually the same as Lemma 2.1.2. The requirements will be:

$$\mathcal{R}_{\langle e,i \rangle} \equiv \{e\} : \{\bigoplus_{j \neq i} A^j\} \nrightarrow P,$$

and corresponding changes are made to the definitions of the length-of-agreement function, restraint function and injury set. To construct the partition, one takes the least $\langle e, i \rangle < s$ such that $x \leqslant r_s(e,i)$ and enumerates x into A^i_{s+1} (or A^0_{s+1} if no such $\langle e, i \rangle$ exists).

□

Now Theorem 2.2.5 can be proved.

Proof. (Theorem 2.2.5) The lemma is sufficient to prove that $FD(n)$ can be embedded into $\mathcal{L}_w$ below P with the top element going to P. In fact we show that $\{P \wedge S^i : 0 \leqslant i \leqslant n-1\}$ freely generates $FD(n)$ where, as before, $S^i = \mathcal{S}(A^i, B)$. To do this, it is sufficient to show that for all non-empty $I \subsetneq \{0, 1, 2 \ldots n-1\}$,

$$P \wedge \bigvee_{i \in I} S^i \not\geqslant_w P \wedge \bigwedge_{i \notin I} S^i,$$

(again use Theorem II.2.3 in [14]). Fix I as above. The requirements imply that $\{\bigoplus_{i \in I} A^i\} \not\geqslant_w P$ as I is a proper subset of $\{0, 1, 2 \ldots n-1\}$. But if $\{\bigoplus_{i \in I} A^i\} \geqslant_w \bigwedge_{i \notin I} S^i$, then $\{\bigoplus_{i \in I} A^i\} \geqslant_w S^j$ for some $j \notin I$. This implies

$$\{\bigoplus_{i \neq j} A^i\} \geqslant_w \bigvee_{i<n} S^i \equiv_w \mathcal{S}(A, B) \geqslant_w P,$$

contradicting $\mathcal{R}_{\langle e,j \rangle}$. Therefore $\{\bigoplus_{i \in I} A^i\} \not\geqslant_w P \wedge \bigwedge_{i \notin I} S^i$ and so $P \wedge \bigvee_{i \in I} S^i \not\geqslant_w P \wedge \bigwedge_{i \notin I} S^i$, as required. The top element of $FD(n)$ is $P \wedge \bigvee_{i<n} S^i \equiv_w P$. Lemma 2.2.7 then completes the proof.

□

To prove Theorem 2.2.6 we need the following slightly more complex lemma.

Lemma 2.2.9. *Let P and Q be non-empty Π^0_1 subsets of 2^ω such that $P >_M Q$, and let A be any r.e. set. Then there exist r.e. sets, A^i, $0 \leqslant i \leqslant n-1$, such that:*

i. $\{A^i : 0 \leqslant i \leqslant n-1\}$ forms a partition of A,

ii. for each non-empty $J \subsetneq \{0, 1, \ldots n-1\}$,

$$\{\bigoplus_{i \in J} A^i\} \vee Q \not\geqslant_M P \wedge \bigwedge_{i \notin J} S^i.$$

Proof. (sketch) Let $A^J = \bigoplus_{i\in J} A^i$ and $A^J_s = \bigoplus_{i\in J} A^i_s$. Let T^J be a recursive tree whose set of paths is $P \wedge \bigwedge_{i\notin J} S^i$ and T^J_s be a recursive tree whose set of paths is $P_s \wedge \bigwedge_{i\notin J} S^i_s$. The requirements for the construction are:

$$\mathcal{R}_{\langle e,J\rangle} \equiv \{e\} : \{A^J\} \vee Q \nrightarrow P \wedge \bigwedge_{i\notin J} S^i.$$

The length-of-agreement function, restraint function and injury sets are:

$$l_s(e,J) := \max\{y : \text{ for all } f \in Q_s,\ \{e\}_s^{A^J_s \oplus f}[y] \in T^J_s\},$$

$$r_s(e,J) := \max\{u(A^i_s; A^J_s \oplus f, e, x, s) : i \in J,\ x \leqslant l_s(e,J),\ f \in Q_s\},$$

$$I_{\langle e,J\rangle} := \{x : \exists s\ \exists i \in J\ x \in A^i_{s+1} \smallsetminus A^i_s \text{ and } x \leqslant r_s(e,J)\}.$$

As before, to construct the partition, at stage s, one takes the least $\langle e,J\rangle < s$ such that $x \leqslant r_s(e,J)$ and the least $i \notin J$ and enumerates x into A^i_{s+1} (or into A^0_{s+1} if no such $\langle e,J\rangle$ exists). The equivalents of Lemmas 2.2.2 and 2.2.3 are then proved in the same way. □

Proof. (Theorem 2.2.6.) It will be shown that $\{(P \wedge S^i) \vee Q : 0 \leqslant i \leqslant n-1\}$ generates $FD(n)$ above Q. Straightforward manipulations show that P is the top element of this copy of $FD(n)$. Let J be a non-empty, proper subset of $\{0,1,2,\ldots n-1\}$. Then,

$$\begin{aligned}
& Q \vee \{A^J\} && \not\geqslant_M && P \wedge \textstyle\bigwedge_{i\notin J} S^i \\
\Rightarrow\quad & Q \vee \textstyle\bigvee_{i\in J} S^i && \not\geqslant_M && P \wedge \textstyle\bigwedge_{i\notin J} S^i \\
\Rightarrow\quad & \textstyle\bigvee_{i\in J} Q \vee S^i && \not\geqslant_M && \textstyle\bigwedge_{i\notin J} P \wedge S^i \\
\Rightarrow\quad & \textstyle\bigvee_{i\in J}(P \wedge S^i) \vee Q && \not\geqslant_M && \textstyle\bigwedge_{i\notin J}(P \wedge S^i) \vee Q.
\end{aligned}$$

Applying Theorem II.2.3 in [14] again is then enough to finish the proof. □

2.3 The ∃-theories of $\mathcal{P}_w$ and $\mathcal{P}_M$

Definition 2.3.1. *If L' is a first-order language in the predicate calculus and $\mathcal{M}$ is an L'-structure, then the* ∃-theory *of $\mathcal{M}$ in L' is the set of all L'-sentences of the form $\exists x_1 x_2 \ldots x_n \phi$ (where ϕ is a quantifier-free formula) that are true in $\mathcal{M}$. If $\mathcal{M} \models \exists x_1 x_2 \ldots x_n \phi$, then ϕ is said to be* satisfiable *in $\mathcal{M}$. An ∃-theory is* decidable *if the set of Gödel numbers of its elements is recursive.*

The main theorem to be proved in this section is:

Theorem 2.3.2. *The ∃-theories of $\mathcal{P}_w$ and $\mathcal{P}_M$ in the language $\langle \wedge, \vee, \leqslant, =, \mathbf{0}, \mathbf{1} \rangle$ are identical and decidable.*

What follows is a proof only that the ∃-theory of $\mathcal{P}_w$ in the language $\langle \wedge, \vee, =, \mathbf{0}, \mathbf{1} \rangle$ is decidable. The proof of the $\mathcal{P}_M$ case will be the same and it will be clear that the decision procedure for the ∃-theory of $\mathcal{P}_M$ is identical to the decision procedure for the ∃-theory of $\mathcal{P}_w$ - implying that their ∃-theories are the same. $\leqslant$ can be defined in terms of $\wedge$ and $=$ so Theorem 2.3.2 will follow.

In order to avoid confusion between propositional connectives and lattice operations we will use $\cdot$ and $+$ for the lattice operations $\wedge$ and $\vee$. $\prod$ and $\sum$ will be used to denote general products and sums.

Let L_{01} be the language $\langle \cdot, +, =, \mathbf{0}, \mathbf{1} \rangle$ with intended interpretation in $\mathcal{P}_w$ as $\wedge$, $\vee$, $=$ and the minimum and maximum elements of $\mathcal{P}_w$ respectively. The languages $L = \langle \cdot, +, = \rangle$ and $L_1 = \langle \cdot, +, =, \mathbf{1} \rangle$ will be restrictions of L_{01}. Two L_{01}-terms, σ and

τ, with free variables among $x_1, x_2 \dots x_n$ are said to be *equivalent (over $\mathcal{P}_w$)* if $\mathcal{P}_w \models \forall x_1 x_2 \dots x_n(\tau = \sigma)$. Two formulas, ψ and ϕ, with free variables among $x_1, x_2, \dots x_n$ are *equivalent (over $\mathcal{P}_w$)* if $\mathcal{P}_w \models \forall x_1 x_2 \dots x_n(\phi \leftrightarrow \psi)$.

Lemma 2.3.3. *The $\exists$-theory of $\mathcal{P}_w$ in L is decidable.*

Proof. One can argue from Theorem 2.2.5 that a quantifier-free L-formula, ψ, is satisfiable in $\mathcal{P}_w$ if and only if it is satisfiable in some finite distributive lattice. As there are only finitely many distributive lattices of any given finite size, determining if ψ is satisfiable in a distibutive lattice of size $m \in \mathbb{N}$ is a finite task. To decide, then, if ψ is satisfiable in $\mathcal{P}_w$ it is enough to compute, uniformly in ψ, an m such that if ψ is satisfiable in some distributive lattice, it is satisfiable in a distributive lattice of size at most m. We do this now. m will depend only on the number of free variables in ψ.

Suppose ψ is as above with free variables x_1, x_2, $\dots x_n$. Then ψ is equivalent to a formula of the form:

$$\bigvee_{i \in I} [\bigwedge_{j \in J_i} (\tau_{ij} = \sigma_{ij}) \wedge \bigwedge_{\overline{j} \in \overline{J_i}} (\tau_{i\overline{j}} \neq \sigma_{i\overline{j}})],$$

where $\tau_{ij}, \sigma_{ij}, \tau_{i\overline{j}}$ and $\sigma_{i\overline{j}}$ are L-terms and I, J_i and $\overline{J}_i$ are finite sets. If it is decidable whether or not each disjunct of ψ is satisfiable in $\mathcal{P}_w$, then it is decidable if ψ is satisfiable. So without losing generality, we can assume ψ is of the form:

$$\bigwedge_{j \in J} (\tau_j = \sigma_j) \wedge \bigwedge_{\overline{j} \in \overline{J}} (\tau_{\overline{j}} \neq \sigma_{\overline{j}}),$$

As before, let $FD(n)$ denote the free distributive lattice on n generators. If $\{\tau_k = \sigma_k : k \leqslant m\}$ is a finite set of lattice relations on $FD(n)$, then we can form the quotient lattice, $\{[\sigma] : \sigma \in FD(n)\}$, where $[\sigma] = [\tau]$ if and only if σ can be transformed formally into τ by applications of the axioms of distributive lattices and substitutions described by the relations. The lattice operations on the quotient lattice are then canonically induced. The claim is that if ψ is satisfiable in some lattice, then it is satifiable in the quotient of $FD(n)$ by $\{\tau_j = \sigma_j : j \in J\}$.

To see this, note that $\bigwedge_{j \in J}(\tau_j = \sigma_j)$ is satisfiable in this quotient lattice, and if some subformula of ψ of the form $\tau_{\overline{j}} \neq \sigma_{\overline{j}}$ were *not* satisfied in the quotient lattice, then $\tau_{\overline{j}}$ could be transformed into $\sigma_{\overline{j}}$ by applications of distributive laws and the relations $\{\tau_j = \sigma_j : j \in J\}$. But this could be done in *any* distributive lattice satisfying $\{\tau_j = \sigma_j : j \in J\}$ and so ψ would not be satisfiable in any distributive lattice. Therefore, if ψ is satisfiable in some distributive lattice, it is satisfiable in the quotient of $FD(n)$ by $\{\tau_j = \sigma_j : j \in J\}$.

The cardinality of the quotient lattice is less than the cardinality of $FD(n)$ which is bounded by $2^{2^n - 2}$ (Theorem II.2.1(iii) [14]). So this is the required m. □

Lemma 2.3.4. *The* $\exists$*-theory of* $\mathcal{P}_w$ *in* L_1 *is decidable.*

Proof. Let ψ be a quantifier-free L_1-formula with $x_1, x_2, \ldots x_n$ its free variables. As above, we can assume ψ is of the form:

$$\bigwedge_{j \in J} (\tau_j = \sigma_j) \wedge \bigwedge_{\overline{j} \in \overline{J}} (\tau_{\overline{j}} \neq \sigma_{\overline{j}}).$$

Every such L_1-formula can be transformed using standard manipulations into an equivalent one of the form:

$$\bigwedge_{k \in K} (\nu_k = \mathbf{1}) \wedge \bigwedge_{\overline{k} \in \overline{K}} (\nu_{\overline{k}} \neq \mathbf{1}) \wedge \phi$$

where ϕ is a quantifier-free L-formula, ν_k and $\nu_{\overline{k}}$ are L-terms, and K and $\overline{K}$ are finite index sets. Let ψ^* be an L-formula formed from ψ by replacing every occurrence of $\mathbf{1}$ by $\sum_{i \leqslant n} x_i$. The claim is that ψ is satisfiable in $\mathcal{P}_w$ if and only if ψ^* is. Lemma 2.3.3 then gives the required result.

Suppose ψ^* is satisfiable in $\mathcal{P}_w$. Then it is satisfiable in some quotient, $\mathcal{L}$, of $FD(n)$. The element $\sum_{i \leqslant n}[x_i]$ is the maximum of $\mathcal{L}$ and by Theorem 2.2.5 we can embed $\mathcal{L}$ into $\mathcal{P}_w$ with $\sum_{i \leqslant n}[x_i]$ mapping to $\mathbf{1}$. So $\psi^* \wedge \sum_{i \leqslant n} x_i = \mathbf{1}$ is satisfiable in $\mathcal{P}_w$ and therefore so is ψ.

Conversely, suppose ψ is satisfied in $\mathcal{P}_w$ by a given assignment of variables. There are two cases based on the form of ψ.

Case 1. $K = \emptyset$. Let ϕ be satisfiable in some finite distributive lattice, $\mathcal{L}$, and let $\mathbf{p}$ be an intermediate element of $\mathcal{P}_w$. Then $\mathcal{L}$ can be embedded into $\mathcal{P}_w$ below $\mathbf{p}$ (Theorem 2.2.5). Under the induced assignment of variables, $\nu_{\overline{k}} \neq \mathbf{1}$ is satisfied for all $\overline{k} \in \overline{K}$. So ψ^* is satisfiable.

Case 2. $K \neq \emptyset$. $\nu_k = \mathbf{1}$ formally implies $\sum_{i \leqslant n} x_i = \mathbf{1}$. So any assignment of variables that satisfies $\nu_k = \mathbf{1}$ will satisfy $\sum_{i \leqslant n} x_i = \mathbf{1}$. This also means that for all $\overline{k} \in \overline{K}$, $\nu_{\overline{k}} \neq \sum_{i \leqslant n} x_i$ under the given assignment. So ψ^* is satisfiable in $\mathcal{P}_w$. □

Theorem 2.3.5. *The ∃-theory of $\mathcal{P}_w$ in L_{01} is decidable.*

Proof. An effective procedure will be described that, given a quantifier-free formula, ψ, of L_{01}, will produce a quantifier-free formula, ψ_1, of L_1 which is satisfiable in $\mathcal{P}_w$ if and only if ψ is. Lemma 2.3.4 will then complete the proof.

Suppose ψ is as above with free variables $x_1, x_2, \ldots x_n$. As before, we can assume ψ is of the form:

$$\bigwedge_{j \in J} (\tau_j = \sigma_j) \wedge \bigwedge_{\overline{j} \in \overline{J}} (\tau_{\overline{j}} \neq \sigma_{\overline{j}}), \tag{2.1}$$

for some finite sets, J and $\overline{J}$. ψ is then equivalent to a formula of the form

$$\bigwedge_{k \in K} (\nu_k = \mathbf{0}) \wedge \bigwedge_{\overline{k} \in \overline{K}} (\nu_{\overline{k}} \neq \mathbf{0}) \wedge \phi, \tag{2.2}$$

where K and $\overline{K}$ and are finite sets, ϕ is a quantifier-free L_1-formula and ν_k and $\nu_{\overline{k}}$ are L-terms.

Case 1. $K = \emptyset$. Suppose ϕ is satisfiable in the finite lattice, $\mathcal{L}$. The proof of Lemma 2.2.5 describes an embedding of $\mathcal{L}$ into $\mathcal{P}_w$ strictly above $\mathbf{0}$. So $\nu_{\overline{k}} \neq \mathbf{0}$ will be satisfied for all $\overline{k} \in \overline{K}$ by such an embedding. So ψ is satisfiable in $\mathcal{P}_w$ if and only if ϕ is.

Case 2. $K \neq \emptyset$. For each $k \in K$, ν_k is equivalent to $\sum_{s \in S} \prod_{t \in T_s} y_{st}$ where $y_{st} \in \{x_1, x_2, \ldots x_n\}$ and T_s and S are some finite index sets. Using the fact that $\mathcal{P}_w \models \forall x, y[x \cdot y = \mathbf{0} \leftrightarrow (x = \mathbf{0} \vee y = \mathbf{0})]$, we can calculate that $\nu_k = \mathbf{0}$ is equivalent to $\bigwedge_{s \in S} \bigvee_{t \in T_s} (y_{st} = \mathbf{0})$. So ψ is equivalent to a formula of the form

$$\bigwedge_{m \in M} \bigvee_{p \in P_m} (y_{mp} = \mathbf{0}) \wedge \bigwedge_{\overline{k} \in \overline{K}} (\nu_{\overline{k}} \neq \mathbf{0}) \wedge \phi. \tag{2.3}$$

Putting this in disjunctive normal form, and re-indexing appropriately, we get something of the form

$$\bigvee_{u \in U} [\bigwedge_{v \in V_u} (y_{uv} = \mathbf{0}) \wedge \bigwedge_{\overline{k} \in \overline{K}} (\nu_{\overline{k}} \neq \mathbf{0}) \wedge \phi]. \tag{2.4}$$

Again it is enough to decide the satisfiablity of each disjunct, so we assume ψ is equivalent to a formula of the form

$$\bigwedge_{v \in V} (y_v = \mathbf{0}) \wedge \bigwedge_{\overline{k} \in \overline{K}} (\nu_{\overline{k}} \neq \mathbf{0}) \wedge \phi. \tag{2.5}$$

Let ψ^* be the formula obtained by replacing, for all $v \in V$, each occurrence of y_v with $\mathbf{0}$. ψ^* is satisfiable if and only if ψ is, and ψ^* is equivalent to a formula of the same form as Equation (2.1) but with strictly fewer variables.

By iterating the above process we get, finally, either $\mathbf{0} = \mathbf{0}$ or a formula to which Case 1 applies. $\square$

Chapter 3

Embeddings of Countable Lattices

3.1 Introduction

In this chapter we prove that certain countable lattices can be embedded into $\mathcal{P}_w$ and $\mathcal{P}_M$. The results are as follows:

1. The free distributive lattice on ω many generators, $FD(\omega)$, can be embedded into $\mathcal{P}_M$ below any special Π^0_1 class. This is proved in Section 3.3

2. The free Boolean algebra on ω generators, $FB(\omega)$, can be embedded into $\mathcal{P}_w$ below any special Π^0_1 class.

3. If $\mathcal{L}_1$ is the lattice of cofinite subsets of ω and $\mathcal{L}_2$ is the lattice of finite subsets of ω, then $\mathcal{L}_1 \times \mathcal{L}_2$ can be embedded into $\mathcal{P}_M$ below any special Π^0_1 class. Results 2 and 3 are proved in Section 3.4

Result 2 is as good as possible, as every countable distributive lattice embeds into $FB(\omega)$. Result 1 is not as general as there are countable distributive lattices that do not embed into $FD(\omega)$, in fact $\mathcal{L}_1 \times \mathcal{L}_2$ is one such lattice. This is an immediate consequence of Theorem 4.6 in [2]. In this paper, Balbes proves that in any free distributive lattice there does not exist an infinite sequence of elements, $\langle a_i \rangle$ such that $a_i \wedge a_j = \mathbf{0}$ for all $i \neq j$. In $\mathcal{L}_2$, however, the sequence $\{\{n\} : n \in \omega\}$ clearly has this property. So $\mathcal{L}_2$ can not be embedded into $FD(\omega)$. Despite this, we conjecture that $FB(\omega)$ is in fact embeddable into $\mathcal{P}_M$ below any special Π^0_1 class.

This chapter is in four sections. Section 3.2 consists of two priority arguments. These construct Π^0_1 sets that have certain useful independence properties. Both build on the constructions in [17], and use a Sacks preservation argument (see [30], Chapter VII.3). The second argument is only sketched. If, at first, the reader wishes only to skim this section and accept Theorems 3.2.1 and 3.2.7, he or she should still find Sections 3.3 and 3.4 completely accessible.

Notation and Preliminaries

We will first establish some notation. As before, σ, τ, ρ and λ will be used to represent binary strings and the length of σ will be written $|\sigma|$. $\{e\}^\sigma_s$ will denote the longest binary string, τ, such that $|\tau| \leqslant s$ and $\{e\}^\sigma_s(n) \downarrow = \tau(n)$ for all $n < |\tau|$. The empty string is denoted by $\langle\rangle$ and $\{e\}^\sigma$ is short for $\{e\}^\sigma_{|\sigma|}$. The restriction of σ to $\{0, 1, 2, \ldots, n-1\}$ is denoted $\sigma\big|_n$.

Let $\mathcal{S}$ be the class of finite sequences of finite strings. The uppercase Greek letters, Σ, Γ and Λ will be used to represent elements of $\mathcal{S}$. For ease of notation sometimes a sequence of strings will be indentified with its range, so that $\sigma \in \Sigma$ means $\sigma \in$ rng(Σ); $\Sigma \subseteq \Gamma$ means Σ is a subsequence of Γ and $\sigma \in \Sigma \smallsetminus \Gamma$ that $\sigma \in \text{rng}(\Sigma) \smallsetminus \text{rng}(\Gamma)$. We will reserve the symbol Σ^m to mean the sequence of all binary strings of length m in lexicographical order.

If $\Sigma = \langle\sigma_i\rangle^n_{i=1}$ and $\Gamma = \langle\gamma_i\rangle^m_{i=1}$, we will say Σ *extends* Γ if $m = n$ and $\sigma_i \supseteq \gamma_i$ for all $i \leqslant n$. Σ *properly extends* Γ if, in addition, $\sigma_k \supsetneq \gamma_k$ for at least one $k \leqslant n$. If $f_1, f_2, \ldots f_n$ are elements of 2^ω, then $\langle f_1, f_2, \ldots f_n\rangle$ *extends* Σ is defined similarly.

If $\Sigma = \langle\sigma_i\rangle^n_{i=1} \subseteq \Sigma^m$ and $\sigma \in 2^{<\omega}$, we will make the following definitions:

• $\sigma^- \in 2^{<\omega}$ such that, for all n, $\sigma^-(n) = \sigma(n+1)$.

• $\bigoplus \Sigma \in 2^{<\omega}$ such that,

$$[\bigoplus \Sigma](i) = \sigma_k(q),$$

where $i = nq + k - 1$, for some (necesarily unique) $k \leqslant n$ and q.

• If $\langle f_i \rangle_{i=1}^n$ is a sequence of elements of 2^ω. Then $\bigoplus_{i=1}^n f_i \in 2^\omega$ is defined to be such that, for all i,

$$[\bigoplus_{i=1}^n f_i](i) = f_k(q),$$

where, as before, $i = nq + k - 1$.

• For an arbitrary $\Gamma = \langle \gamma_i \rangle_{i=1}^n \in \mathcal{S}$ (with the γ_i of possibly different lengths), we define,

$$\bigoplus \Gamma = \bigoplus_{i=1}^n \gamma_i|_l,$$

where $l = \min\{|\gamma_i| : 1 \leqslant i \leqslant n\}$.

$\bigoplus$ is not associative but it does have the useful property that if $\langle f_1, f_2, \dots f_n \rangle$ extends $\Sigma \subseteq \Sigma^m$, then $\bigoplus_{i=1}^n f_i \supset \bigoplus \Sigma$. If no confusion can result, we will write $\bigoplus f_i$ for $\bigoplus_{i=1}^n f_i$.

3.2 Two Constructions

Theorem 3.2.1. *For any special Π_1^0 set, P, there is a Π_1^0 set, Q, with the properties, for all sequences, $\langle f_i \rangle_{i=1}^n \subset Q$,*

I. $\forall f \in Q \setminus \langle f_i \rangle_{i=1}^n,\ f \not\leqslant_T \bigoplus f_i$, *II.* $\forall f \in P,\ f \not\leqslant_T \bigoplus f_i$

Proof. The proof will closely follow the proof of Theorem 4.7 in [17]. A recursive sequence, $\langle \psi_s \rangle_{s\in\omega}$, of recursive functions from $2^{<\omega}$ to $2^{<\omega}$ will be constructed with the properties that, for all $\sigma \in 2^{<\omega}$ and $s \in \omega$,

1. $\psi_s(\sigma^\frown\langle 0\rangle)$ and $\psi_s(\sigma^\frown\langle 1\rangle)$ are incompatible extensions of $\psi_s(\sigma)$,
2. $\text{range}(\psi_{s+1}) \subseteq \text{range}(\psi_s)$,
3. $\psi(\sigma) = \lim_t \psi_t(\sigma)$ exists.

Each ψ_s determines a recursive tree, namely,

$$T_s = \{\tau : \text{ for some } \sigma,\ \psi_s(\sigma) \supseteq \tau\}.$$

The required Q will then be $\bigcap_{s\in\omega}[T_s]$. Q will be non-empty as $\langle [T_s]\rangle_{s\in\omega}$ is a nested sequence of closed subsets of 2^ω. It will be a Π^0_1 set because,

$$f \in Q \equiv \forall s f \in [T_s] \equiv \forall s \forall n \exists \sigma[|\sigma| \leqslant n \wedge \psi_s(\sigma) \subset f],$$

and $\exists\sigma[|\sigma| \leqslant n \wedge \psi_s(\sigma) \subsetneq f]$ is a recursive predicate.

Each ψ_s will induce a mapping, $\Psi_s : \mathcal{S} \to \mathcal{S}$, defined by

$$\Psi_s(\Gamma) = \langle \psi_s(\gamma_i)\rangle_{i=1}^n,$$

where $\Gamma = \langle \gamma_i \rangle_{i=1}^n$. When it is proved that $\psi(\sigma)$ exists for all σ, it will be clear that $\Psi(\Sigma) = \lim_s \Psi_s(\Sigma)$ exists for all $\Sigma \in \mathcal{S}$.

We will define $\langle \psi_s \rangle_{s\in\omega}$ so that, for every $m \in \omega$, $\Gamma \subseteq \Sigma^m$ and $e \leqslant m$, Q satisfies the requirements:

$$P^m_{\Gamma,e} \equiv \text{ for all } \langle f_i \rangle_{i=1}^n \text{ extending } \Psi(\Gamma),\ \{e\}^{\bigoplus f_i} \notin P,$$

$$R^m_{\Gamma,e} \equiv \text{for all } \langle f_i \rangle_{i=1}^n \text{ extending } \Psi(\Gamma), \text{ and for all } \sigma \in \Sigma^m \smallsetminus \Gamma,\ \{e\}^{\bigoplus f_i} \not\supset \psi(\sigma).$$

The P requirements guarantees that Q has property II. of the theorem, and the R requirements guarantee property I. The set of requirements can be ordered lexicographically, first on m, then on e and finally with the conventions that, for all m, and $\Gamma, \Gamma' \in \Sigma^m$,

i. $P^m_{\Gamma,e}$ precedes $R^m_{\Gamma',e}$ and,

ii. $P^m_{\Gamma,e}$ precedes $P^m_{\Gamma',e}$ and $R^m_{\Gamma,e}$ precedes $R^m_{\Gamma',e}$ whenever Γ precedes Γ' in the lexicographical ordering on Σ^m.

Priority is given to the requirements in reverse lexicographical order.

$P^m_{\Gamma,e}$ is said to be *satisfied at stage* s if,

$$\{e\}^{\bigoplus \Psi_s(\Gamma)} \notin T_P,$$

and $R^m_{\Gamma,e}$ *is satisfied at stage* s if, for all $\sigma \in \Sigma^m \smallsetminus \Gamma$,

$$\{e\}^{\bigoplus \Psi_s(\Gamma)} \not\supseteq \psi_s(\sigma).$$

We now define ψ_s as follows:

Stage $s = 0$: $\psi_0(\sigma) = \sigma$ for all $\sigma \in 2^\omega$.

Stage s+1:

We say $P^m_{\Gamma,e}$ *requires attention at stage* $s+1$ if $P^m_{\Gamma,e}$ is not satisfied at stage $s+1$ and there is a $\Lambda = \langle\lambda_i\rangle_{i=1}^n$ properly extending Γ such that $\max\{|\lambda_j| : \lambda_j \in \Lambda\} \leqslant s+1$ and,

i. $\{e\}^{\bigoplus \Psi_s(\Lambda)} \in T_P$,

ii. $\{e\}^{\bigoplus \Psi_s(\Lambda)} \supsetneq \{e\}^{\bigoplus \Psi_s(\Gamma)}$.

We say $R^m_{\Gamma,e}$ *requires attention at stage* $s+1$ if $R^m_{\Gamma,e}$ is not satisfied at stage $s+1$ and there is a $\Lambda = \langle\lambda_i\rangle_{i=1}^n$, properly extending Γ, such that $\max\{|\lambda_j| : \lambda_j \in \Lambda\} \leqslant s+1$ and,

$$\{e\}^{\bigoplus \Psi_s(\Gamma)} \supseteq \psi_s(\sigma^\frown\langle x\rangle),\ \text{for some } x \in \{0,1\} \text{ and } \sigma \in \Sigma^m \setminus \Gamma.$$

If $P^m_{\Gamma,e}$ has priority greater than the priority of $P^s_{\Sigma^s,s}$ and is the highest priority requirement requiring attention at stage $s+1$, let Λ witness this fact and define,

$$\psi_{s+1}(\nu) = \begin{cases} \psi_s(\lambda_i^\frown\nu') & \text{if } \nu = \gamma_i^\frown\nu' \text{ for some } \gamma_i \in \Gamma \\ \psi_s(\nu) & \text{if } \nu \not\supseteq \gamma_i \text{ for any } \gamma_i \in \Gamma. \end{cases}$$

If $R^X_{m,e}$ has priority greater than the priority of $P^s_{\Sigma^s,s}$ and is the highest priority requirement requiring attention at stage $s+1$, let Λ, σ and x witness this and define,

$$\psi_{s+1}(\nu) = \begin{cases} \psi_s(\lambda_i^\frown \nu') & \text{if } \nu = \gamma_i^\frown \nu' \text{ for some } \gamma_i \in \Gamma, \\ \psi_s(\sigma^\frown\langle 1-x\rangle^\frown \nu') & \text{if } \nu = \sigma^\frown \nu', \\ \psi_s(\nu) & \text{if } \nu \not\supseteq \tau \text{ for any } \tau \in \Gamma \cup \{\sigma\} . \end{cases}$$

If no requirement of priority $\geqslant$ the priority of $P^s_{\Sigma^s,s}$ requires attention at stage $s+1$, then let $\psi_{s+1} = \psi_s$.

The following lemmas establish the theorem.

Lemma 3.2.2. *For any requirement, S, there is a stage, s_0, such that S does not require attention at any stage $t > s_0$.*

Proof. Assume not and let S be the highest priority requirement requiring attention infinitely often. If $S = P^m_{\Gamma,e}$, then let t be a stage such that $P^m_{\Gamma,e}$ has priority greater than $P^t_{\Sigma^t,t}$ and such that all higher priority requirements are satisfied for all stages $\geqslant t$. Let $s_1, s_2, s_3, \ldots$ be an infinite increasing sequence of stages greater than t at which S requires attention. At each of these stages S will be the highest priority requirement requiring attention and so $s_1, s_2, s_3, \ldots$ will generate a recursive sequence,

$$\{e\}^{\bigoplus \Psi_{s_1}(\Gamma)} \subsetneq \{e\}^{\bigoplus \Psi_{s_2}(\Gamma)} \subsetneq \{e\}^{\bigoplus \Psi_{s_3}(\Gamma)} \ldots,$$

of elements of T_P. But then $\bigcup_i \{e\}^{\bigoplus \Psi_{s_i}(\Gamma)}$ is a recursive path through T_P, contradicting the original assumption that P is special.

Next suppose $S = R^m_{\Gamma,e}$. If t is such that the priority of $R^m_{\Gamma,e}$ is greater than $P^t_{\Sigma^t,t}$; all higher priority requirements are permanently satisfied at stage t; and S requires attention at stage t, then S will be satisfied at stage $t+1$. Suppose, at some stage $u > t$, a lower priority requirement, T, requires attention. If $T = P^{m'}_{\Lambda,e'}$ or $T = R^{m'}_{\Lambda,e'}$ with $m' > m$, and any Λ and e', then $\Psi_{u+1}(\Gamma) = \Psi_u(\Gamma)$ and S will remain satisfied at stage $u+1$. If $T = R^m_{\Lambda,e'}$ or $T = P^m_{\Lambda,e'}$, then $\Psi_{u+1}(\Gamma) \supseteq \Psi_u(\Gamma)$ and so S will remain satisfied at stage $u+1$. We then argue by induction that S will remain satisfied, and hence not require attention, at all stages $u \geqslant t$, contradicting the assumption.

□

Lemma 3.2.3. *$\psi(\sigma) = \lim_s \psi_s(\sigma)$ exists for all σ.*

Proof. Let $\sigma \in 2^{<\omega}$ be arbitrary. By Lemma 3.2.2, there exists a stage, t, such that for all $m \leqslant |\sigma|$, and all $\Gamma \subseteq \Sigma^m$, the requirements $R^m_{\Gamma,e}$ and $P^m_{\Gamma,e}$ do not require attention after stage t. Then $\psi_{t_1}(\sigma) = \psi_{t_2}(\sigma)$ for all $t_1, t_2 > t$. □

Lemma 3.2.4. *If $m \in \omega,\ e \leqslant m$ and $\Gamma \subseteq \Sigma^m$ are such that $\{e\}^{\bigoplus \Psi(\Gamma)} \in T_P$, then there does not exist a Λ properly extending Γ such that $\{e\}^{\bigoplus \Psi(\Lambda)} \in T_P$ and $\{e\}^{\bigoplus \Psi(\Lambda)} \supsetneq \{e\}^{\bigoplus \Psi(\Gamma)}$.*

Proof. Suppose such a Λ existed for m, e and Γ. Take t so large that $\Psi_t(\Gamma) = \Psi(\Gamma)$ and $\Psi_t(\Lambda) = \Psi(\Lambda)$. Then,

$$\{e\}^{\bigoplus \Psi_t(\Lambda)} = \{e\}^{\bigoplus \Psi(\Lambda)} \supsetneq \{e\}^{\bigoplus \Psi(\Gamma)} = \{e\}^{\bigoplus \Psi_t(\Gamma)},$$

and so, at some stage $u \geqslant t$, $P^m_{\Gamma,e}$ would be the highest priority requirement requiring attention, implying,

$$\{e\}^{\bigoplus \Psi_{u+1}(\Gamma)} \supsetneq \{e\}^{\bigoplus \Psi_u(\Gamma)} = \{e\}^{\bigoplus \Psi_t(\Gamma)} = \{e\}^{\bigoplus \Psi(\Gamma)},$$

contradicting the fact that $\Psi_{u+1}(\Gamma) = \Psi(\Gamma)$. □

Lemma 3.2.5. *If* $\langle f_i \rangle_{i=1}^n \subseteq Q$ *then, for all* $f \in P$, $f \not\leqslant_T \bigoplus f_i$.

Proof. We can assume without losing generality that $\langle f_i \rangle_{i=1}^n$ is in lexicographic order. Suppose the lemma is false and let $\{e\}^{\bigoplus f_i} \in P$. Let $m \in \omega$ and $\Gamma \subseteq \Sigma^m$ be such that,

i. $m \geqslant e$,

ii. $\langle f_i \rangle_{i=1}^n$ extends $\psi(\Gamma)$

Such a Γ can be found because $\langle f_i \rangle_{i=1}^n$ is in lexicographic order. But $\{e\}^{\Psi(\Gamma)} \in T_P$, so there must be a $\Lambda \supsetneq \Gamma$ such that $\{e\}^{\bigoplus \Psi(\Lambda)} \supsetneq \{e\}^{\bigoplus \Psi(\Gamma)}$, contradicting Lemma 3.2.4.

□

Lemma 3.2.6. *For all* $\langle f_i \rangle_{i=1}^n \subseteq Q$ *and all* $f \in Q \setminus \langle f_i \rangle_{i=1}^n$,

$$f \not\leqslant_T \bigoplus f_i$$

Proof. Suppose not and let $\{e\}^{\bigoplus f_i} = f \in Q$. Let $m \in \omega$, $\Gamma \subseteq \Sigma^m$ and $\sigma \in \Sigma^m \setminus \Gamma$ be such that,

i. $m \geqslant e$,

ii. $\langle f_i \rangle_{i=1}^n$ extends $\psi(\Gamma)$,

iii. $f \supset \psi(\sigma)$,

(again we are assuming $\langle f_i \rangle_{i=1}^n$ is in lexicographic order). Let t be such that $\Psi_u(\Gamma) = \Psi(\Gamma)$ and $\psi_u(\sigma^\frown\langle x\rangle) = \psi(\sigma^\frown\langle x\rangle)$ for all $u \geqslant t$ and $x \in \{0,1\}$. By the supposition, there must be a stage, $s \geqslant t$ and a Λ extending Γ such that

$$\{e\}^{\Psi_s(\Lambda)} \supseteq \psi_s(\sigma^\frown\langle x\rangle) \text{ for some } x \in \{0,1\}.$$

So there will be a stage, $v \geqslant s$, at which $R^m_{\Gamma,e}$ requires attention and is, in fact, the highest priority requirement requiring attention. But then,

$$\Psi_{v+1}(\Gamma) \neq \Psi_v(\Gamma) = \Psi(\Gamma),$$

contradicting the fact that $v \geqslant u$.

□

Theorem 3.2.1 Lemmas 3.2.5 and 3.2.6 prove that Q has properties I. and II. as required. □

Theorem 3.2.7. *Given any special Π^0_1 set, P, there is an infinite recursive sequence of Π^0_1 sets, $\langle Q_i : i \in \omega\rangle$, with the properties, for all $i, j \in \omega$ such that $i \neq j$,*

I. $\forall f \in Q_i\ \forall g \in Q_j\ f \nleqslant_T g$,

II. $\forall f \in Q_i\ \forall g \in P\ g \nleqslant_T f$.

Proof. (*sketch*)

A recursive sequence of recursive functions, $\psi^i : 2^{<\omega} \to 2^{<\omega}$, is constructed, the range of each function is the tree T_i and then Q_i will be $[T_i]$. Each ψ^i is constructed as the limit of a recursive sequence of recursive functions, $\langle\psi^i_s\rangle_{s<\omega}$ and will be defined so that, for every $m \in \omega$, ψ^i satisfies the requirements:

for all $e \leqslant m$; $j \leqslant m$; $\sigma \in \Sigma^m$ and for all f extending $\psi^i(\sigma)$,

$P^m \equiv \quad \{e\}^f \notin P,$

$R^m \equiv \quad j \neq i \Rightarrow \{e\}^f \not\supseteq \psi^j(\sigma).$

These requirements are then further specified by indexing them according to i, j, σ and e (bounded as above), and an exhaustive priority ordering is given to them. The same method as in Theorem 3.2 is then used to ensure all are satisfied. If at any stage of construction an R_m requirement is the highest priority requirement requiring attention then the requirement is satisfied (permanently) at the next stage.

If at some stage of construction a P_m requirement will be the highest priority requirement requiring attention and then the function being constructed is adapted to keep the requirement unsatisfied (as per Sacks' preservation strategy, see [30] Chapter VII.3). An (non-constructive) argument is then made to show that this strategy will eventually fail (because P has no recursive elements) and P_m will eventually be satisfied. These are essentially the arguments of Lemmas 3.2.5 and 3.2.6.

□

3.3 $FD(\omega) \hookrightarrow \mathcal{P}_M$

Theorem 3.3.1. *Given any special Π^0_1 class, P, $FD(\omega)$ can be embedded into $\mathcal{P}_M$ below P.*

Proof. Let P be any special Π^0_1 class and suppose Q and ψ are as in Theorem 3.2.1. Let $\{\sigma_i : i \in \omega\}$ be a set of binary strings defined by:

i. $|\sigma_i| = i + 1$,

ii. $\sigma_i(n) = \begin{cases} 1 & \text{if } n = i, \\ 0 & \text{otherwise.} \end{cases}$

Then $\{\sigma_i : i \in \omega\}$ is a pairwise incomparable set of strings and hence so is $\{\psi(\sigma_i) : i \in \omega\}$. Denote by Q_i the set of elements of Q extending $\psi(\sigma_i)$, and let $P_i = P \wedge Q_i$. The set $\{P_i : i \in \omega\}$ then generates a sublattice of $\mathcal{P}_M$ strictly below P. To see this note that if X is a non-empty finite subset of ω,

$$\bigvee_{i \in X} P_i <_M P,$$

because $\bigvee_{i \in X} P_i \leqslant_M P$, and if $\bigvee_{i \in X} P_i \geqslant_M P$ then $P \wedge \bigvee_{i \in X} Q_i \geqslant_M P$ and some element of $\bigvee_{i \in X} Q_i$ would compute an element of P, contradicting property II. of Theorem 3.2.1. This is enough to show that all elements of the generated sublattice are strictly below P.

As in Chapter One, we will use Theorem II.2.3 in [14] to show that the lattice generated by the $P_i's$ is free. If X and X' are finite subsets of ω, then,

$$\begin{aligned} \textstyle\bigwedge_{i \in X} P_i &\leqslant_M \textstyle\bigvee_{j \in X'} P_j, \\ \Rightarrow \quad P \wedge \textstyle\bigwedge_{i \in X} Q_i &\leqslant_M P \wedge \textstyle\bigvee_{j \in X'} Q_j, \\ \Rightarrow \quad P \wedge \textstyle\bigwedge_{i \in X} Q_i &\leqslant_M \textstyle\bigvee_{j \in X'} Q_j, \end{aligned}$$

so if $\bigoplus_{j\in X'} f_j \in \bigvee_{j\in X'} Q_j$, then there is a $g \in P \vee \bigwedge_{i\in X} Q_i$ such that $g \leqslant_T \bigoplus_{j\in X'} f_j$. Therefore, $g^- \leqslant_T \bigoplus_{j\in X'} f_j$ where $g^- \in P$ or $g^- \in \bigwedge_{i\in X} Q_i$. But $g^- \notin P$ by property II. of Theorem 3.2.1. And if $j \notin X$ then $g^- \notin \bigwedge_{i\in X} Q_i$ by property I. of Theorem 3.2.1. Therefore, $j \in X$ and $X \cap X' \neq \emptyset$ as required by Theorem II.2.3 in [14]. □

3.4 $FB(\omega) \hookrightarrow \mathcal{P}_w$

In the section we give the second principal embedding theorem - that the free Boolean algebra on ω generators, $FB(\omega)$, is embeddable into $\mathcal{P}_w$, the lattice of Muchnik degrees. We represent $FB(\omega)$ as an algebra of recursive sets and then give an explicit embedding into $\mathcal{P}_w$. As before, the argument will use Π^0_1 sets constucted using a priority argument. This time on those Π^0_1 sets of Theorem 3.2.7. Then we show that all countable distributive lattices embed into $FB(\omega)$. Finally we establish result 3.

We will require two constructions given by the following definitions. Let $\emptyset \neq A \subseteq \omega$ be recursive and let $\langle P_i : i \in \omega\rangle$ be a recursive sequence of Π^0_1 sets. Let $(\cdot,\cdot) : \omega\times\omega \to \omega$ be a recursive coding bijection.

Definition 3.4.1. *If $f \in 2^\omega$, we define $(f)_i \in 2^\omega$ by,*

$$(f)_i(n) = f((i, n)),$$

and then the recursive product of $\langle P_i : i \in A\rangle$, *denoted $\bigwedge_{i\in A} P_i$, is given by,*

$$f \in \bigvee_{i\in A} P_i \Leftrightarrow (f)_i \in P_i \text{ for all } i \in A.$$

Notice that $\bigwedge_{i \in A} P_i$ is a Π^0_1 set as,

$$x \in \bigvee_{i \in A} P_i \equiv \forall i \forall n(\; i \in A \Rightarrow \; R_i(n,x)),$$

where $\langle R_i : i \in \omega \rangle$ is the recursive sequence of recursive predicates that defines $\langle P_i : i \in \omega \rangle$.

We will now define a recursive sum. Let A and $\langle P_i : i \in \omega \rangle$ be as above and, for each $i \in \omega$, let T_i be a recursive tree such that $[T_i] = P_i$. If T is a recursive tree such that $[T] = \mathrm{DNR}_2$ (or any Medvedev complete Π^0_1 class), then let $\langle \sigma_j : j \in \omega \rangle$ be the sequence, in lexicographical order, of all binary strings such that $\sigma_j \in T$ but $\sigma_j^\frown \langle x \rangle \notin T$ for any $x \in \{0,1\}$. The sequence will be infinite as $[T]$ has no recursive element. Define,

$$T^* = T \cup \{\sigma_i^\frown \tau : i \in A, \; \tau \in T_i\}.$$

Definition 3.4.2. *The* recursive sum *of* $\langle P_i : i \in A \rangle$*, denoted* $\bigwedge_{i \in A} P_i$*, is* $[T^*]$*, the set of paths through* T^**.*

Note that if A is finite, the recursive sum and product are Medvedev equivalent to the standard, lattice-theoretic sum and product respectively, allowing us some ambiguity of notation. However, it is not to be assumed that these constructions are necessarily the greatest lower or least upper bounds when A is infinite. Indeed, this may not even be the case if A is recursive.

Now let $\langle Q_i : i \in \omega \rangle$ be as in Theorem 3.2.7 (with P arbitrary). Define,

$$\widehat{Q}_i = \bigwedge_{j \neq i} Q_j,$$

and, for any recursive, non-empty set, A, let,

$$\widehat{Q}(A) = \bigvee_{i \in A} \widehat{Q}_i.$$

Lemma 3.4.3. *If $A, B \neq \emptyset$ and $A \neq B$, then $\widehat{Q}(A) \not\equiv_w \widehat{Q}(B)$ (and therefore $\widehat{Q}(A) \not\equiv_M \widehat{Q}(B)$).*

Proof. Suppose that A and B are as above and that, without losing generality, $j \in B \setminus A$. Choose any $x \in Q_j$ and define $\overline{x}$ by,

$$\left(\overline{x}\right)_i = \sigma_j^\frown x \text{ for all } i \in \omega.$$

Then $\overline{x} \in \widehat{Q}(A)$ as $\sigma_j^\frown x \in \widehat{Q}_i$ for all $i \neq j$ and, in particular, for all $i \in A$. Now let $y \in \widehat{Q}_j$ be arbitrary. There are two cases.

Case 1. $y = \sigma_i^\frown z$ for some $i \neq j$ and $z \in Q_i$. Then,

$$y \equiv_T z \not\leqslant_T x \equiv_T \overline{x},$$

($z \not\leqslant_T x$ as $z \in Q_i$ and $x \in Q_j$, with $i \neq j$).

Case 2. $y \in [T]$, where $[T]$ is the Medvedev complete Π^0_1 class used in the construction of the recursive sum. Then for any $i \in \omega$, there is a $z \in Q_i$ such that $y \geqslant_T z$. We choose some $i \neq j$, and then fix z. If $\overline{x} \geqslant_T y$, we would have,

$$Q_j \ni x \equiv_T \overline{x} \geqslant_T y \geqslant_T z \in Q_i, \text{ with } i \neq j,$$

contrary to construction of $\langle Q_i : i \in \omega \rangle$.

Therefore, in both cases we have $y \nleqslant_T \overline{x}$. As y was arbitrary, $\widehat{Q}_j \nleqslant_w \widehat{Q}(A)$. But $\widehat{Q}_j \leqslant_w \widehat{Q}(B)$ via the map $x \mapsto (x)_j$ so it must be that $\widehat{Q}(B) \nleqslant_w \widehat{Q}(A)$ and therefore that $\widehat{Q}(B) \not\equiv_w \widehat{Q}(A)$, as required. □

Lemma 3.4.4. *If A and B are non-empty and recursive, then,*

$$\widehat{Q}(A \cup B) \equiv_M \widehat{Q}(A) \vee \widehat{Q}(B).$$

Proof.

$$\begin{aligned}
\widehat{Q}(A \cup B) &= \{x : \forall i \in A \cup B,\ (x)_i \in \widehat{Q}_i\}, \\
&= \{x : \forall i \in A,\ (x)_i \in \widehat{Q}_i\} \cap \{x : \forall i \in B,\ (x)_i \in \widehat{Q}_i\}, \\
&= \widehat{Q}(A) \cap \widehat{Q}(B).
\end{aligned}$$

So, $x \mapsto x \oplus x$, is a map from $\widehat{Q}(A \cup B)$ to $\widehat{Q}(A) \vee \widehat{Q}(B)$, and therefore, $\widehat{Q}(A \cup B) \geqslant_M \widehat{Q}(A) \vee \widehat{Q}(B)$. Conversely, let $x \oplus y \in \widehat{Q}(A) \vee \widehat{Q}(B)$. Define, $z \in 2^\omega$ by,

$$\left(z\right)_i = \begin{cases} \left(x\right)_i & \text{if } i \in A \\ \\ \left(y\right)_i & \text{if } i \in \omega \smallsetminus A. \end{cases}$$

Then $z \leqslant_T x \oplus y$ and for all $i \in A \cup B$, $\left(z\right)_i \in \widehat{Q}_i$, so $z \in \widehat{Q}(A \cup B)$. Therefore, $\widehat{Q}(A \cup B) \leqslant_M \widehat{Q}(A) \vee \widehat{Q}(B)$ as required. □

Lemma 3.4.5. *If A and B are recursive and $A \cap B \neq \emptyset$, then,*

$$\widehat{Q}(A \cap B) \equiv_w \widehat{Q}(A) \wedge \widehat{Q}(B).$$

Proof. First, $\widehat{Q}(A \cap B) \leqslant_w \widehat{Q}(A) \wedge \widehat{Q}(B)$ (in fact, $\leqslant_M$). If $x \in \widehat{Q}(A) \wedge \widehat{Q}(B)$, then define $z \in \widehat{Q}(A \cap B)$ by,

$$\left(z\right)_i = \left(x^-\right)_i \text{ for all } i \in \omega.$$

If $\left(x\right)_i(0) = 0$, then, for all $i \in A$, $\left(z\right)_i \in \widehat{Q}_i$, and, a fortiori, for all $i \in A \cap B$, $\left(z\right)_i \in \widehat{Q}_i$. So $z \in \widehat{Q}(A \cap B)$. There is a similar argument if $\left(x\right)_i(0) = 1$.

Next, $\widehat{Q}(A \cap B) \geqslant_w \widehat{Q}(A) \wedge \widehat{Q}(B)$. Modulo the following two claims, the argument will be:

$$
\begin{aligned}
\widehat{Q}(A \cap B) &= \bigvee_{i \in A \cap B} \widehat{Q}_i, \\
&\geqslant_w \bigvee_{i \in A} \bigvee_{j \in B} \widehat{Q}_i \wedge \widehat{Q}_j \quad \text{(in fact, } \geqslant_M\text{)} \textit{ Claim 1}, \\
&\geqslant_w \bigvee_{i \in A} \widehat{Q}_i \wedge \bigvee_{j \in B} \widehat{Q}_j \quad \textit{Claim 2}, \\
&= \widehat{Q}(A) \wedge \widehat{Q}(B).
\end{aligned}
$$

Proving the *Claims* :

Claim 1. Let $x \in \bigvee_{i \in A \cap B} \widehat{Q}_i$ and take any $k \in A \cap B$. So $(x)_k \in \widehat{Q}_k$. We define (recursively in x) $z \in \bigvee_{i \in A} \bigvee_{j \in B} \widehat{Q}_i \wedge \widehat{Q}_j$ by defining $\big((z)_i\big)_j$ for all $i, j \in \omega$, such that,

$$\big((z)_i\big)_j \in \widehat{Q}_i \wedge \widehat{Q}_j \text{ for all } i \in A \text{ and } j \in B.$$

To this end, let,

$$
\big((z)_i\big)_j = \begin{cases}
\langle 0 \rangle^\frown (x)_i & \text{if } i = j, \\
\langle 0 \rangle^\frown (x)_k & \text{if } i \neq j \text{ and } (x)_k \not\supseteq \sigma_i, \\
\langle 1 \rangle^\frown (x)_k & \text{if } i \neq j \text{ and } (x)_k \supseteq \sigma_i.
\end{cases}
$$

So, suppose that $i \in A$ and $j \in B$. If $i = j$, then $i \in A \cap B$ and $\big((z)_i\big)_j = \langle 0 \rangle^\frown (x)_i \in \widehat{Q}_i \wedge \widehat{Q}_j$. If $i \neq j$ and $(x)_k \not\supseteq \sigma_i$, then $(x)_k \in \widehat{Q}_i$, and $\big((z)_i\big)_j = \langle 0 \rangle^\frown (x)_k \in \widehat{Q}_i \wedge \widehat{Q}_j$. If $i \neq j$ and $(x)_k \supseteq \sigma_i$, then $(x)_k \in \widehat{Q}_j$ and $\big((z)_i\big)_j = \langle 1 \rangle^\frown (x)_k \in \widehat{Q}_i \wedge \widehat{Q}_j$. These three cases are exhaustive and so *Claim 1* is established. Note that the above is a uniform procedure for computing z from an arbitrary x, and so the stronger, Medvedev reducibility has been shown.

Claim 2. Let $x \in \bigvee_{i\in A}\bigvee_{j\in B}\widehat{Q}_i \wedge \widehat{Q}_j$. We will construct $z \leqslant_T x$ such that $z \in \bigvee_{i\in A}\widehat{Q}_i \wedge \bigvee_{j\in B}\widehat{Q}_j$. There are two cases.

Case 1. $\exists i \in A \smallsetminus B\ \forall j \in B \smallsetminus A\ \ \big((x)_i\big)_j(0) = 1.$

Fix such an i, set $z(0) = 1$ and let,

$$\left(z^-\right)_k = \begin{cases} \big((x)_i\big)^-_k & \text{if } k \notin A \cap B, \\ \big((x)_k\big)^-_k & \text{if } k \in A \cap B. \end{cases}$$

Then, if $k \in B \smallsetminus A$, $\left(z^-\right)_k = \big((x)_i\big)^-_k \in \widehat{Q}_k$ and if $k \in B \cap A$, $\left(z^-\right)_k = \big((x)_k\big)^-_k \in \widehat{Q}_k$. So, for all $k \in B$, $\left(z^-\right)_k \in \widehat{Q}_k$, giving $z^- \in \bigvee_{j\in B}\widehat{Q}_j$ and $z \in \bigvee_{i\in A}\widehat{Q}_i \wedge \bigvee_{j\in B}\widehat{Q}_j$.

Case 2. $\forall i \in A \smallsetminus B\ \exists j \in B \smallsetminus A\ \ \big((x)_i\big)_j(0) = 0.$

Let $z(0) = 0$ and define,

$$f(i) = \begin{cases} \text{the least such } j & \text{if } i \in A \smallsetminus B, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \leqslant_T x$, and $\big((x)_i\big)^-_{f(i)} \in \widehat{Q}_i$ for all $i \in A \smallsetminus B$. We can then define,

$$\left(z^-\right)_k = \begin{cases} \big((x)_k\big)^-_{f(k)} & \text{if } k \notin A \cap B, \\ \big((x)_k\big)^-_k & \text{if } k \in A \cap B. \end{cases}$$

As above we have $\left(z^{-}\right)_k \in \widehat{Q}_k$, if $k \in A \cap B$ and if $k \in A \setminus B$ then $\left(z^{-}\right)_k = \left((x)_k\right)^{-}_{f(k)} \in \widehat{Q}_k$. So $z^{-} \in \bigvee_{i \in A} \widehat{Q}_i$, and $z \in \bigvee_{i \in A} \widehat{Q}_i \wedge \bigvee_{j \in B} \widehat{Q}_j$, as required.

□

We would like to improve Lemma 3.4.5 by showing that $\widehat{Q}(A \cap B) \equiv_M \widehat{Q}(A) \wedge \widehat{Q}(B)$, but the division into cases in the proof of *Claim 2* is non-effective and we have only been able to show the weaker result. However, we can improve the result under the stricter conditions of the following lemma.

Lemma 3.4.6. *If the symmetric difference of two recursive sets,*

$$A \triangle B = A \setminus B \cup B \setminus A,$$

is finite, then,

$$\widehat{Q}(A \cap B) \equiv_M \widehat{Q}(A) \wedge \widehat{Q}(B).$$

Proof. The proof is identical with the proof of 3.4.5 noting that in the proof of Lemma *Claim 2* the division into two cases is now effective as both $A \setminus B$ and $B \setminus A$ are finite. □

We are now in a position to prove the theorem in the title of the section.

Theorem 3.4.7. *The free Boolean algebra on countably many generators, $FB(\omega)$, is embeddable into $\mathcal{P}_w$.*

Proof. Consider the mapping $A \mapsto \widehat{Q}(A)$. Lemmas 3.4.3, 3.4.4 and 3.4.5 prove that this is an embedding of the lattice of non-empty, recursive subsets of ω under $\cap$ and $\cup$ into

$\mathcal{P}_w$. So to prove the theorem it is sufficient to show that $FB(\omega)$ can be represented by a collection of non-empty, recursive subsets of ω.

To this end let p_j be the j^{th} prime number and let $B_j = \{p_j \cdot n : n \in \omega\}$. Define $\widetilde{B_j} = (\omega \setminus B_j) \cup \{0\}$. The set $\{B_j : j \in \omega\}$ generates a distributive lattice under operations of intersection and union. Further, this lattice can be extended to a Boolean algebra with **1** represented by ω, **0** represented by $\{0\}$ and $\widetilde{B_j}$ the Boolean complement of B_j. It would, perhaps, seem more natural to have $\emptyset$ as the minimum element and $\omega \setminus B_j$ as the Boolean complement, however the text definition ensures that each element of the Boolean algebra is non-empty. This Boolean algebra is in fact free and therefore a representation of $FB(\omega)$. To show this it is sufficient to show (Exercise II.3.43 [14]) that for all finite $X, Y \subseteq \omega$,

$$\bigcap_{i \in X} B_i \subseteq \bigcup_{j \in Y} B_j \Rightarrow X \cap Y \neq \emptyset.$$

But this is easily seen as $\prod_{i \in X} p_i \in \bigcap_{i \in X} B_i$ and so, if the antecedent holds, $\prod_{i \in X} p_i \in B_j$ for some $j \in Y$. By primality, this means $p_j = p_i$ for some $i \in X$, giving $X \cap Y \neq \emptyset$. □

Corollary 3.4.8. *$FB(\omega)$ can be embedded into $\mathcal{P}_w$ below any given special Π^0_1 set, P.*

Proof. Let such a P be given and let $\langle Q_i : i \in \omega \rangle$ be as in Theorem 3.2.7. The required embedding will be,

$$A \mapsto P \wedge \widehat{Q}(A).$$

The fact that this is a homomorphism follows from the lattice theoretic identities:

$$(P \wedge \bigwedge_{i \in A} \widehat{Q}_i) \wedge (P \wedge \bigwedge_{i \in B} \widehat{Q}_i) = P \wedge (\bigwedge_{i \in A} \widehat{Q}_i \wedge \bigwedge_{i \in B} \widehat{Q}_i),$$

and,

$$(P \times \bigwedge_{i \in A} \widehat{Q}_i) \wedge (P \times \bigwedge_{i \in B} \widehat{Q}_i) = P \times (\bigwedge_{i \in A} \widehat{Q}_i \wedge \bigwedge_{i \in B} \widehat{Q}_i),$$

and the fact that $A \mapsto \widehat{Q}(A)$ describes a homomorphism. To see that it's an embedding, suppose that $A \neq B$ and take $j \in B \setminus A$, $x \in Q_j$ and $\overline{x} \in \widehat{Q}(A)$ as in the proof of Lemma 3.4.3. Let $\overline{x}_1 = \langle 1 \rangle^\frown \overline{x} \in P \wedge \widehat{Q}(A)$. Suppose that there is a $y \in P \wedge \widehat{Q}(B)$ such that $y \leqslant_T \overline{x}_1$. By the proof of Lemma 3.4.3 we know that $y^- \notin \widehat{Q}(B)$ (or else $\overline{x} \equiv_T \overline{x}_1 \geqslant_T y \equiv_T y^- \in \widehat{Q}(B)$, contradiction). But, if $y^- \in P$, then,

$$P \ni y^- \leqslant_T \overline{x}_1 \equiv_T x \in Q_j,$$

contrary to the construction of $\langle Q_i : i \in \omega \rangle$. So there is no $y \in P \wedge \widehat{Q}(B)$, such that $y \leqslant_T \overline{x}_1$. Therefore, $P \wedge \widehat{Q}(B) \nleqslant_M P \wedge \widehat{Q}(A)$, as required. □

Theorem 3.4.9. *Every countable distributive lattice can be embedded into $\mathcal{P}_w$ below any given special Π^0_1 set.*

We show that every countable distributive lattice embeds into $FB(\omega)$ and then apply Theorem 3.4.7. All the lattice theoretical background can be found in [14] or [20]. Every countable distributive lattice can be embedded into a countable Boolean algebra so it is sufficient to show that every countable Boolean algebra can be embedded into $FB(\omega)$.

It is most convenient here to work with the dual space of $FB(\omega)$. Stone duality gives a contravariant functor from the category of closed subspaces of 2^ω and continuous maps to the category of Boolean Algebras and Boolean homomorphisms. Such a functor will take 2^ω to $FB(\omega)$ and continuous surjections to Boolean injections. So it is enough (in fact equivalent) to prove the following theorem (attributed to Sierpiński in [25] page 46):

Theorem 3.4.10. *For every closed subset, T, of 2^ω, there exists a continuous surjection,*

$$\psi : 2^\omega \longrightarrow T.$$

Proof. Let $\text{Ext}(T) = \{\sigma \in 2^{<\omega} : \exists f \in T \; f \supset \sigma\}$. We will define a continuous surjection, $\phi : 2^{<\omega} \longrightarrow \text{Ext}(P)$, which will then induce the required map on 2^ω. Let

$$\phi(\langle\rangle) = \langle\rangle,$$

$$\phi(\sigma^\frown\langle i\rangle) = \begin{cases} \phi(\sigma)^\frown\langle i\rangle & \text{if } \phi(\sigma)^\frown\langle i\rangle \in \text{Ext}(P) \\ \phi(\sigma)^\frown\langle 1-i\rangle & \text{otherwise.} \end{cases}$$

It is straightforward to see that this is a continuous surjection. It is in fact a retract ([25] page 46) of 2^ω. $\square$

The next theorem is result 3 of page 30.

Theorem 3.4.11. *Let $\mathcal{L}_1$ ($\mathcal{L}_2$) be the lattice of finite (co-finite) subsets of ω under $\cap$ and $\cup$. Then, for any special Π^0_1 set, P, there is an embedding of $\mathcal{L}_1 \times \mathcal{L}_2$ into $\mathcal{P}_M$ below P.*

Proof. Let E be any infinite, co-infinite recursive subset of ω (for example the even numbers). Let $\mathcal{K}$ be the distributive lattice $\{X \subseteq \omega : X \triangle E$ is finite $\}$ with the operations of $\cap$ and $\cup$. Then $\mathcal{K} \simeq \mathcal{L}_1 \times \mathcal{L}_2$ (represent $\mathcal{L}_1$ by finite subsets of odd numbers and $\mathcal{L}_2$ by (relatively) co-finite sets of even numbers and the isomorphism is witnessed by $(X, Y) \mapsto X \cup Y$). The symmetric difference of any two elements of $\mathcal{K}$ is finite so Lemmas 3.4.3, 3.4.4, 3.4.6 and the proof of Corollary 3.4.8 give the result. □

Corollary 3.4.12. *$\mathcal{L}_1$ and $\mathcal{L}_2$ are embeddable in $\mathcal{P}_M$ below any special Π^0_1 set.*

Proof. Immediate, as $\mathcal{L}_1$ and $\mathcal{L}_2$ are sublattices of $\mathcal{K}$, above. □

Chapter 4

Small Π^0_1 Classes

4.1 Introduction

In this chapter we will investigate the relationship between structural properties of special Π^0_1 classes and their Muchnik and Medvedev degrees. An attempt is made to define notions that will guarantee Muchnik and Medvedev incompleteness. A lot of what is done is informed by Post's effort [22] to construct a non-zero r.e. degree strictly below $\mathbf{0}'$. Post's attempt was ultimately unsuccessful and the construction of such a degree needed more sophisticated methods. A discussion of these issues can be found in [30] chapter V, or [24] §9.7.

Perhaps surprisingly, Post's methods are more conducive to solving the corresponding problem in the Medvedev and Muchnik lattices. This is already known, as the notion of thinness of a special Π^0_1 class is a structural property that guarantees both Muchnik and Medvedev incompleteness. Here we define two new properties also guaranteeing incompleteness and having properties not shared by thin Π^0_1 classes. Both of these properties relate to the "size" of a Π^0_1 class.

First we will introduce some notation that will be useful.

Notation:

$\|X\| =$ the cardinality of X

If $f \in 2^\omega$, $f[n] = \langle f(0), f(1), \dots f(n-1)\rangle$

If $P \subseteq 2^\omega$, $P[n] = \{f[n] : f \in P\}$

If $X \subseteq 2^{<\omega}$, $X[n] = \{\sigma \in X : |\sigma| = n\}$

If $P \subseteq 2^\omega$, $P[<n] = \{f[m] : m < n,\ f \in P\}$

Similarly for $P[\leqslant n]$, $X[<n]$ and $X[\leqslant n]$.

$\{e\}^\tau[n]$ is a partial sequence from $\{0, 1, \dots n-1\}$ to ω. $\{e\}^\tau[n] \in T \subseteq 2^{<\omega}$ implies $\{e\}^\tau(m)\downarrow$ for all $m < n$.

$|\{e\}^\tau| = \max\{k : \forall m < k,\ \{e\}^\tau(m)\downarrow\}$.

4.2 Small Π^0_1 classes

Definition 4.2.1. *$P \subseteq \omega^\omega$ is* small *if it is non-empty, closed and if there is no recursive function, g, such that for all n, $\|P[g(n)]\| \geqslant n$.*

Notice that any finite subset of ω^ω is small. In fact, one way to think of smallness is to say that a closed subset of ω^ω is small exactly when there is no recursive function witnessing its infinitude. It will be shown that the property of smallness is invariant under recursive homeomorphisms, and therefore has a certain robustness. Rather than arbitrary small subsets of ω^ω, we will primarily be concerned with small recursively bounded Π^0_1 subsets of ω^ω. In fact, as Corollary 4.2.11 will show, we can concentrate on small Π^0_1 subsets of 2^ω.

Theorem 4.2.2. *All Medvedev (and therefore Muchnik) degrees have a representative that is not small.*

Proof. For any r.b. Π^0_1 class $P \subseteq \omega^\omega$, $P \vee 2^\omega$ is never small because for all n,

$$\|P \vee 2^\omega[2n]\| = \|P[n]\| \cdot \|2^\omega[n]\| \geqslant 2^n \geqslant n.$$

□

Theorem 4.2.3. *DNR_2 is not small.*

Proof. Let $\langle e_i \rangle_{i \geqslant 0}$ be a recursive sequence of indices for the empty function. For all $\sigma \in \mathrm{DNR}_2[e_i]$, $\sigma^\frown\langle 0\rangle$ and $\sigma^\frown\langle 1\rangle$ are in $\mathrm{DNR}_2[e_i + 1]$. Arguing by induction, and using the fact that $\mathrm{DNR}_2[n]$ is increasing in n we have $\|\mathrm{DNR}_2[e_i]\| \geqslant 2^i$ for all i. If $h(m) =$ least k such that $2^k \geqslant m$, then

$$\|\mathrm{DNR}_2[e_{h(m)}]\| \geqslant 2^{h(m)} \geqslant m$$

for all $m \in \omega$. $m \mapsto e_{h(m)}$ is clearly a recursive function, so DNR_2 is not small. □

Theorem 4.2.4. *A small Π^0_1 class with no recursive path exists.*

Proof. If A is hypersimple and A^0 and A^1 are disjoint r.e. sets such that $A^0 \cup A^1 = A$, then we claim that $S = \mathcal{S}(A^0, A^1)$ is small. Suppose S were not small, witnessed by the recursive function, g. S branches at level n (that is, $S[n+1] > S[n]$) precisely when $n \in \overline{A}$. For such an n, $\|S[n+1]\| = 2\|S[n]\|$. So the principal function of $\overline{X}$, p, has the property that $S[p(n)] = 2^n$. But $\|S[g(2^n+1)]\| \geqslant 2^n + 1$. So the function, $n \mapsto g(2^n+1)$ is a recursive function dominating p, contradicting the fact that A is hypersimple.

A^0 and A^1 can be constructed to ensure $\mathcal{S}(A^0, A^1)$ has no recursive element. □

Theorem 4.2.5. *If P and Q are small Π^0_1 subsets of ω^ω, then $P \wedge Q$ is small.*

Proof. Suppose $P \wedge Q$ were not small and let g be a recursive function such that $\|P \wedge Q[g(n)]\| \geqslant n$ for all n. We can take g to be strictly positive. By the definition of $\wedge$, for all $n > 0$, $\|P \wedge Q[n]\| = \|P[n-1]\| + \|Q[n-1]\|$ so, for all n, $\|P[g(n)-1]\| \geqslant n/2$ or $\|Q[g(n)-1]\| \geqslant n/2$. The set $\{n : \|P[g(n)-1]\| < n/2\}$ is r.e. as P is a Π^0_1 class and it is infinite as P is small. So it has an infinite recursive subset, Y. Therefore, for all $y \in Y$, $\|Q[g(y)-1]\| \geqslant y/2$. If $h(n) =$ least $y \in Y$ $y \geqslant 2n$, then

$$\forall n\ \|Q[g(h(n))-1]\| \geqslant \frac{h(n)}{2} \geqslant n,$$

contradicting the smallness of Q. □

Theorem 4.2.6. *If P and Q are small, then so is $P \vee Q$.*

Proof. The proof is very similar to the proof of Theorem 4.2.5. Assume not and let g be such that $\|P \vee Q[g(n)]\| \geqslant n$ for all n. The function $n \mapsto \|P[n]\|$ is increasing in n so we also have $\|P \vee Q[2g(n)]\| \geqslant n$. Using the definition of $\vee$, $\|P \vee Q[2n]\| = \|P[n]\| \cdot \|Q[n]\|$ so, for all n, $\|P[g(n)]\| \geqslant \sqrt{n}$ or $\|Q[g(n)]\| \geqslant \sqrt{n}$. Again, the set $\{n : \|P[g(n)]\| < \sqrt{n}\ \}$ is r.e. and infinite. The proof is then similar to Theorem 4.2.5 □

Theorem 4.2.7. *For every small P, there exists a small Q, such that $2^\omega <_M Q <_M P$. This is also true with $<_w$ substituting for $<_M$.*

Proof. For this we will construct a small Π^0_1 class, T, using a construction similar to the one used in Theorem 3.2.7. T will have the property that for all $f \in T$ and $g \in P$, $f \not\geqslant_T g$. We can also ensure that T has no recursive element. The construction of

Theorem 3.2.7 will clearly be sufficient for this. We only need to introduce requirements that ensure S is small. These are as follows.

$$S_e \equiv \{e\}(e)\downarrow \Rightarrow \|S[\{e\}(e)]\| < e.$$

S_e will *require attention* at stage s if $\|T_s[\{e\}(e)]\| \geqslant e$ and $\{e\}_s(e)\downarrow$. To ensure that each requirement gets satisfied, we wait for a stage, s, such that $\{e\}_s(e)\downarrow$ and such that S_e is the highest priority requirement requiring attention. To satisfy S_e, we take the least number, k, such that $2^k < e$, and i be the least number such that for all τ of length $k+i$, $|\psi_s(\tau)| > \{e\}(e)$. If we let 0^i denote the string of i zeroes, we define,

$$\psi_{s+1}(\nu) = \begin{cases} \psi_s(\sigma^\frown 0^{i\frown}\nu') & \text{if } \nu = \sigma^\frown\nu' \text{ and } |\sigma| = k \\ \psi_s(\nu) & \text{if } |\nu| < k \end{cases}$$

As before, T will be $\bigcap_e T_e$. Each requirement will be satisfied for all time after receiving attention, so this construction will result in a small Π^0_1 class with the required properties. Also $T \wedge P$ will be small and,

$$P >_M T \wedge P >_M >_M 2^\omega,$$

as required.

□

Theorem 4.2.8. *Let P and Q be Π^0_1 subsets of ω^ω. If P is r.b. and small, and if $\{e\} : P \to Q$ is a recursive surjection, then Q is small.*

Proof. Suppose P, Q and $\{e\}$ are as stated. Let $\langle T_s\rangle_s$ be a recursive sequence of recursive trees such that $\bigcap_s T_s = \mathrm{Ext}(P)$. Let s and l be recursive functions such that for all n

$$\forall\tau \in T_{s(n)}[l(n)],\ |\{e\}^{\tau}_{s(n)}| \geqslant n.$$

To see that such an l and s exist, notice that there is a k such that $\forall\tau\in P[k],\ |\{e\}^{\tau}| \geqslant n$, and a t such that $T_t[k] = P[k]$. As P is recursively bounded, given any n, a search will eventually find two numbers with the required property.

Now suppose Q isn't small, witnessed by the recursive function, g. For all n,

$$\forall\tau \in T_{s(g(n))}[l(g(n))],\ |\{e\}^{\tau}| \geqslant g(n).$$

As $\{e\}$ is onto,

$$\forall\sigma \in Q[g(n)]\ \exists\tau \in P[l(g(n))]\ \{e\}^{\tau} \supseteq \sigma.$$

Therefore,

$$\|P[l(g(n))]\| \geqslant \|Q[g(n)]\| \geqslant n.$$

$l(g(n))$ is recursive so this contradicticts the smallness of P. ☐

Corollary 4.2.9. *If $P \geqslant_M Q$ are Π^0_1 subsets of ω^ω, and if P is r.b. and contains a small Π^0_1 subset, then Q contains a small Π^0_1 subset.*

Proof. If $\{e\} : P \to Q$, and $S \subseteq P$ is Π^0_1 and small, then the theorem implies that the image of S under $\{e\}$ is a small Π^0_1 subset of Q. ☐

Corollary 4.2.10. *Smallness is preserved by recursive homeomorphisms.*

Corollary 4.2.11. *Any small r.b.* Π^0_1 *subset of* ω^ω *is recursively homeomorphic to a small* Π^0_1 *subset of* 2^ω*.*

Corollary 4.2.11 allows us to move from small r.b. Π^0_1 subsets of ω^ω to small Π^0_1 subsets of 2^ω without losing generality (up to recursive homeomorphism).

Corollary 4.2.12. *No Medvedev complete* Π^0_1 *subset of* 2^ω *has a small* Π^0_1 *subset.*

Proof. If some such Medvedev complete Π^0_1 class contained a small Π^0_1 subset, S, then S would also be Medvedev complete. But all Medvedev complete Π^0_1 subsets of 2^ω are recursively homeomorphic [27]. Therefore S would be recursively homeomorphic to DNR_2, which would then be small. But we have seen that DNR_2 is not small. □

The following observation by Simpson allows us to transfer a lot of these theorems to the Muchnik lattice. In this respect it is a central lemma in the subject.

Lemma 4.2.13. *(Simpson) If* $P, Q \subseteq 2^\omega$ *are* Π^0_1*, and if* $P \geqslant_w Q$*, then there exists a* Π^0_1 *class,* $P' \subseteq P$*, such that* $P' \geqslant_M Q$*.*

Proof. Let $f \in P$ be of hyperimmune-free degree. Such an f exists by the hyperimmune-free basis theorem, [17]. Then for some $g \in Q$, $f \geqslant_T g$. But the proof of Theorem VI.5.5 [21] (attributed to D.A. Martin) then implies $f \geqslant_{tt} g$. Proposition III.3.2 [21] (Trakhtenbrot, Nerode) then states we can find a total recursive functional, Φ, taking f to g. Then $\Phi^{-1}(Q) \cap P$ is a non-empty Π^0_1 subclass of P. This is the required P', as $\Phi(\Phi^{-1}(Q) \cap P) \subseteq Q$. □

Corollary 4.2.14. *No Muchnik complete* Π^0_1 *subset of* 2^ω *has a small* Π^0_1 *subset*

Proof. Suppose $S \subseteq 2^\omega$ is small, Π^0_1 and Muchnik complete. Then $S \geqslant_w \mathrm{DNR}_2$. By Lemma 4.2.13, there must be a Π^0_1, $S' \subseteq S$, such that $S' \geqslant_M \mathrm{DNR}_2$. As S' is necessarily small, its image under any recursive functional is also small, and so DNR_2 must have a small Π^0_1 subclass - contradicting Corollary 4.2.12. □

Lemma 4.2.13 also has corollaries for the study of $\mathcal{R}$ - the upper semi-lattice of r.e. Turing degrees:

Corollary 4.2.15. *For any hypersimple set, X, and any r.e. partition, $X_0 \cup X_1 = X$, there exists a separating set of X_0 and X_1 that is not of PA degree.*

Proof. If X is hypersimple then $\mathcal{S}(X_0, X_1)$ is small. By Corollary 4.2.14, it can not be Muchnik complete and so must contain an element not of PA degree. □

The following is a somewhat more general consequence of Lemma 4.2.13.

Corollary 4.2.16. *If $S \subseteq 2^\omega$ is a small Π^0_1 class and $P \subseteq 2^\omega$ is Π^0_1 with no small Π^0_1 subclass, then no hyperimmune-free element of S computes an element of P.*

Theorem 4.2.17. *The set of Medvedev degrees:*

$$\mathcal{I} = \{deg_M(P) : P \text{ has a small } \Pi^0_1 \text{ subset}\}$$

forms a (proper, nontrivial) prime ideal in $\mathcal{P}_M$.

Proof. First note that if $P \equiv_M Q$ and P has a small Π^0_1 subset then so does Q by Corollary 4.2.9, so in what follows we are free to choose arbitrary representatives of Medvedev degrees.

i. Suppose $\deg_M(P) \in \mathcal{I}$ and $Q \subseteq 2^\omega$ is a Π^0_1 class such that $P \geqslant_M Q$. Corollary 4.2.9 then implies $\deg_M(Q) \in \mathcal{I}$.

ii. If $\deg_M(P),\ \deg_M(Q) \in \mathcal{I}$ and $S_1 \subseteq P$ and $S_2 \subseteq Q$ are small, then $S_1 \vee S_2 \subseteq P \vee Q$ and by Theorem 4.2.6, $S_1 \vee S_2$ is small. So $\deg_M(P \vee Q) \in \mathcal{I}$.

iii. No Medvedev complete Π^0_1 class has a small Π^0_1 subset by Corollary 4.2.12, so $\mathcal{I}$ is proper.

iv. $\mathcal{I}$ is non-trivial by Theorem 4.2.4

v. Suppose $P \subseteq 2^\omega$ and $Q \subseteq 2^\omega$ are Π^0_1 and such that $\deg_M(P \wedge Q) \in \mathcal{I}$. If $S \subseteq P \wedge Q$ were small, then either $\{f : \langle 0 \rangle^\frown f \in S\} \cap P$ or $\{f : \langle 1 \rangle^\frown f \in S\} \cap P$ would be non-empty and consequently, small. So $\mathcal{I}$ is prime.

□

Using an argument similar to that used in Corollary 4.2.14, we can show that Theorem 4.2.17 is true in $\mathcal{P}_w$ as well.

Theorem 4.2.18. *The set of Muchnik degrees:*

$$\mathcal{J} = \{\mathit{deg}_w(P) : P \text{ has a small } \Pi^0_1 \text{ subset}\}$$

forms a (proper, nontrivial) prime ideal in $\mathcal{P}_w$.

Proof. ii, iv and v are proved exactly as in Theorem 4.2.17. iii follows from Corollary 4.2.14. For i, suppose $\deg_w(P) \in \mathcal{J}$ and $Q \leqslant_w P$. Let $S \subseteq P$ be Π^0_1 and small and let $f \in S$ be hyperimmune-free. As in Corollary 4.2.14, there is a total recursive functional,

Φ, such that $\Phi(f) \in Q$. Thus $\Phi[S] \cap Q$ is non-empty and therefore a small subset of Q. $\square$

We will now consider alternative characterisations of smallness for recursively bounded Π^0_1 classes.

Definition 4.2.19. *If $P \subseteq 2^\omega$ is Π^0_1, then let $\mathrm{Br}(P)$, the branching nodes of P, be the set*

$$\{\sigma \in \mathrm{Ext}(P) : \ \sigma^\frown\langle 0\rangle \in \mathrm{Ext}(P) \text{ and } \sigma^\frown\langle 1\rangle \in \mathrm{Ext}(P)\}.$$

Observation 4.2.20. $\|\mathrm{Br}(P)[< n]\| + 1 = \|P[n]\|$.

Proof. This is just a matter of counting. Each branching node below a given level of $\mathrm{Ext}(P)$ increases the number of extendible nodes at that level by one. $\square$

Theorem 4.2.21. *For any Π^0_1 class, $P \subseteq 2^\omega$, P is small if and only if $\mathrm{Br}(P)$ is h-immune.*

Proof. $\Rightarrow$) Assume $\mathrm{Br}(P)$ is not h-immune. Let $f(n)$ be a total recursive function and let $\langle D_{f(n)}\rangle_{n\geqslant 0}$ be a strong array such that $D_{f(n)} \cap \mathrm{Br}(P) \neq \emptyset$ for all $n \in \omega$. For all $n \in \omega$, define a total recursive function, g, by:

$$g(n) = \max\{|\sigma| : \sigma \in \bigcup_{i=0}^{n} D_{f(i)}\}.$$

Then for all $n \in \omega$, $\|\mathrm{Br}(P)[\leqslant g(n)]\| \geqslant n+1$. Therefore, by observation 4.2.20, for all n, $\|P[g(n)+1]\| = \|\mathrm{Br}(P)[\leqslant g(n)]\| + 1 \geqslant n+2 \geqslant n$. So P is not small.

$\Leftarrow$) Assume P is not small and the fact is witnessed by a strictly increasing, recursive function, h. We now construct the required strong array as follows: first define the recursive function:

$$\hat{h}(n) = \begin{cases} h(0) & \text{if } n = 0 \\ h(2^{\hat{h}(n-1)}) + 1 & \text{if } n \neq 0. \end{cases}.$$

The point of this definition is that, for all n,

$$\|P[\hat{h}(n+1)]\| \geqslant 2^{\hat{h}(n)} + 1 > \|P[\hat{h}(n)]\|,$$

and so there must be a $\sigma \in \mathrm{Br}(P)$ such that $\hat{h}(n) \leqslant |\sigma| < \hat{h}(n+1)$. Now define:

$$D_{f(n)} = \{\sigma : \hat{h}(n) \leqslant |\sigma| < \hat{h}(n+1)\}.$$

So $\langle D_{f(n)} \rangle_{n \geqslant 0}$ is a strong array and for each n, $D_{f(n)} \cap \mathrm{Br}(P) \neq \emptyset$. □

Notice that $\mathrm{Br}(P)$ is a co-r.e. set so that P is small if and only if $\overline{\mathrm{Br}(P)}$ is hypersimple. This observation allows us to apply knowledge of the algebraic structure of r.e. sets to the Medvedev and Muchnik lattices via the idea of branching sets.

Definition 4.2.22. *$n \in \omega$ is said to be a* branching level *of P if there exists a $\sigma \in \mathrm{Br}(P)$ such that $|\sigma| = n$. We denote the set of branching levels of P by* $\mathrm{Brl}(P)$

Theorem 4.2.23. *$P \subseteq 2^\omega$ is small if and only if $\overline{\mathrm{Brl}(P)}$ is hypersimple.*

Proof. Assume $\overline{\mathrm{Brl}(P)}$ is not hypersimple. Let $\langle D_{f(n)} \rangle$ be a disjoint strong array such that for all n, $D_{f(n)} \cap \mathrm{Brl}(P) \neq \emptyset$. Let $D_{g(n)} = \{\sigma \in 2^{<\omega} : |\sigma| \in D_{f(n)}\}$. Then $\langle D_{g(n)} \rangle$ forms a disjoint strong array and for all n, $D_{g(n)} \cap \mathrm{Br}(P) \neq \emptyset$.

Conversely, suppose $\langle D_{f(n)} \rangle$ is a disjoint strong array such that for all n, $D_{f(n)} \cap \mathrm{Br}(P) \neq \emptyset$. Let $D_{g(n)} = \{|\sigma| : \sigma \in D_{f(n)}\}$. $\langle D_{g(n)} \rangle$ is not a disjoint array but it can easily be made so. Let

$$h(n) = \text{ least } k \ \{|\sigma| : \sigma \in D_{f(k)}\} \cap \bigcup_{i=0}^{n-1} D_{g(i)} = \emptyset.$$

Then $\langle D_{f(h(n))} \rangle$ is the required disjoint strong array. □

4.3 Very Small Π^0_1 classes

The definition of smallness can be strengthened to define a proper subset of the set of small Π^0_1 classes. This new property will have much in common with smallness.

Definition 4.3.1. $P \subseteq \omega^\omega$ *is* very small *if it is non-empty, closed and the function* $n \mapsto$ *least* k *such that* $\|P[k]\| \geqslant n$ *dominates every recursive function.*

The similarity to smallness can be made more explicit by the observation that P is small if and only if the function $n \mapsto$ least k such that $\|P[k]\| \geqslant n$ is not dominated by any recursive function. This also proves that every very small class is small.

Now theorems analogous to Theorems 4.2.4 - 4.2.21 can be established.

Theorem 4.3.2. *A very small* Π^0_1 *class with no recursive path exists.*

Proof. Recall that an r.e. set, X, is *dense simple* if the principal function of its complement dominates every recursive function. Now, if A is dense simple and A^0 and A^1 are disjoint r.e. sets such that $A^0 \cup A^1 = A$, then $\mathcal{S}(A^0, A^1)$ is very small by an argument similar to 4.2.4. A^0 and A^1 can be constructed to ensure $\mathcal{S}(A^0, A^1)$ has no recursive element. □

Theorem 4.3.3. *If P and Q are very small Π^0_1 subsets of ω^ω, then $P \wedge Q$ is very small.*

Proof. Suppose $P \wedge Q$ is not very small. Let g be a recursive function such that $\|P \wedge Q[g(n)]\| \geqslant n$ for infinitely many n. Then, for infinitely many n, either $\|P[g(n) - 1]\| \geqslant n/2$ or $\|Q[g(n) - 1]\| \geqslant n/2$ (using the definition of $\wedge$). Therefore, either $\{n : \|P[g(n) - 1]\| \geqslant n/2\}$ or $\{n : \|Q[g(n) - 1]\| \geqslant n/2\}$ is infinite. Assume, without losing generality, that $\{n : \|Q[g(n) - 1]\| \geqslant n/2\}$ is infinite. Then either $\{2n : n \in \omega\} \cap \{n : \|Q[g(n) - 1]\| \geqslant n/2\}$ is infinite or $\{2n + 1 : n \in \omega\} \cap \{n : \|Q[g(n) - 1]\| \geqslant n/2\}$ is infinite. If the first case holds then, for infinitely many n, $\|Q[g(2n) - 1]\| \geqslant n$. If the second case holds then, for infinitely many n, $\|Q[g(2n + 1) - 1]\| \geqslant n + 1/2 \geqslant n$. In either case Q is not very small. □

Theorem 4.3.4. *If P and Q are very small, then so is $P \vee Q$.*

Proof. The proof imitates Theorem 4.3.3. Assume not and let g be such that $\|P \vee Q[g(n)]\| \geqslant n$ for infinitely many n. The function $n \mapsto \|P[n]\|$ is increasing in n so we also have $\|P \vee Q[2g(n)]\| \geqslant n$. Using the definition of $\vee$, for infinitely many n, either $\|Q[g(n)]\| \geqslant \sqrt{n}$ or $\|P[g(n)]\| \geqslant \sqrt{n}$. Assume as before, that $X = \{n : \|Q[g(n)]\| \geqslant \sqrt{n}\}$ is infinite. Let $\{n_0, n_1, n_2 \ldots\}$ be an infinite subset of ω such that for all i, there exists a $k \in X$ such that $n_i^2 \leqslant k < (n_i+1)^2$. Then for all i,

$$\begin{aligned} \|Q[g((n_i+1)^2)]\| &\geqslant \|Q[g(k)]\| \quad \text{for some } k \in X \\ &\geqslant \sqrt{k} \\ &\geqslant n_i. \end{aligned}$$

So there are infinitely many n such that $\|Q[g((n+1)^2)]\| \geqslant n$ and Q is not very small. □

Theorem 4.3.5. *For every very small $P \subseteq 2^\omega$, there exists a very small Q, such that $2^\omega <_M Q <_M P$. This is also true with $<_w$ substituting for $<_M$.*

Proof. We will use the same kind of construction as in Theorem 4.2.7. We will construct a Π^0_1 class, $V \subseteq 2^\omega$ and require that it has no recursive path and that no element of V computes an element of P. We then combine these requirements with the following to ensure that it's very small. This time the requirements will be indexed by a pair of numbers:

$$R_{\langle e,n\rangle} \equiv \{e\}(n)\downarrow \Rightarrow \|V[\{e\}(n)]\| < n$$

$R_{\langle e,n\rangle}$ *requires attention* at stage s if $\{e\}_s(n)\downarrow$ and $\|V_s[\{e\}_s(n)]\| \geqslant n$. Suppose $R_{\langle e,n\rangle}$ is the highest priority requirement requiring attention at stage s. Let l be the least natural number such that,

$$|\psi_s(\tau)| > \{e\}(n)$$

for all τ of length l. Let k be the greatest natural number strictly less that $\log_2(n)$. Now define,

$$\psi_{s+1}(\nu) = \begin{cases} \psi_s(\sigma^\frown 0^{l-k\frown}\nu') & \text{if } \nu = \sigma^\frown\nu \text{ and } |\sigma| = k, \\ \psi_s(\nu) & \text{if } |\nu| < k \end{cases}$$

Then at stage $s = 1$, $R_{\langle e,n\rangle}$ will be satisfied as the number of strings of length k is $2^k < n$. So if $|\sigma| = k$, $|\psi_{s+1}(\sigma)| > \{e\}(n)$ and so $\|V[\{e\}(n]\| < n$.

Each requirement is satisfied for all time after receiving attention once. $\lim_s(\sigma)$ exists for all σ as, for all k, there comes a stage, s, when $k + 1 < \log_2(n)$ for all $R_{\langle e,n\rangle}$ that are not satisfied at stage s. By this stage, $\psi(\sigma) = \psi_s(\sigma)$ for all σ of length k. □

Theorem 4.3.6. *Let P and Q be Π^0_1 subsets of ω^ω. If P is r.b. and very small, and if $\{e\} : P \to Q$ is a recursive surjection, then Q is very small.*

Proof. The proof is virtually identical to Theorem 4.2.8

□

Corollary 4.3.7. *If $P \geqslant_M Q$ are Π^0_1 subsets of ω^ω, and if P is r.b. and contains a very small Π^0_1 subset, then Q contains a very small Π^0_1 subset.*

Proof. See the proof of Corollary 4.2.9 □

Corollary 4.3.8. *Very smallness is preserved by recursive homeomorphisms.*

Corollary 4.3.9. *Any very small r.b. Π^0_1 subset of ω^ω is recursively homeomorphic to a very small Π^0_1 subset of 2^ω.*

Theorem 4.3.10. *The set of Medvedev degrees:*

$$\mathcal{K} = \{deg_M(P) : P \text{ has a very small } \Pi^0_1 \text{ subset}\}$$

forms a (proper, nontrivial) prime ideal in $\mathcal{P}_M$.

Proof. The proof of this is essentially the same as Theorem 4.2.17.

□

Theorem 4.3.11. *The set of Muchnik degrees:*

$$\mathcal{L} = \{deg_w(P) : P \text{ has a very small } \Pi^0_1 \text{ subset}\}$$

forms a (proper, nontrivial) prime ideal in $\mathcal{P}_w$.

Proof. See the proof of Theorem 4.2.18. □

Theorem 4.3.12. *For any Π^0_1 class, $P \subseteq 2^\omega$, P is very small if and only if $\overline{\mathrm{Br}(P)}$ is dense simple.*

Proof. It is convenient here to provide an alternative characterisation of dense simplicity.

Lemma 4.3.13. *An r.e. set is dense simple if and only if for all strong arrays, $\langle D_{f(n)} \rangle$,*

$$\{n : \|\overline{X} \cap \bigcup_{i=0}^{n} D_{f(i)}\| \geqslant n\} \text{ is finite.}$$

Proof. Suppose that for some recursive function, f, there are infinitely many n such that $\|\overline{X} \cap \bigcup_{i=0}^{n} D_{f(i)}\| \geqslant n$. Let $m(n) = \max(\bigcup_{i=0}^{n} D_{f(i)})$. Then for infinitely many n,

$$\|\{x : x \in \overline{X} \text{ and } x \leqslant m(n)\}\| \geqslant n.$$

Therefore, if $p_{\overline{X}}$ is the principal function of $\overline{X}$, $p_{\overline{X}}(n) \leqslant m(n)$ for infinitely many n. But m is recursive, so X is not dense simple.

Conversely, suppose there is a recursive function, ϕ, such that $p_{\overline{X}} \leqslant \phi(n)$ for infinitely many n. Let

$$D_{f(n)} = \begin{cases} [0, \phi(0)] & \text{if } n = 0 \\ (\phi(n-1), \phi(n)] & \text{otherwise,} \end{cases}$$

(where the notation $(a, b]$ represents the appropriate interval in ω). Then, whenever $p_{\overline{X}}(n) \leqslant \phi(n)$, we have $\|\overline{X} \cap \bigcup_{i=0}^{n} D_{f(i)}\| \geqslant n$.

$\square$

Now we complete the proof of the theorem. Suppose P is not very small and let g be recursive such that for infinitely many n, $\|P[g(n)]\| \geqslant n$. Let $g'(n) = g(n+1)$, so that, for all n,

$$\|P[g(n)]\| \geqslant n \Rightarrow \|P[g'(n)]\| \geqslant n + 1.$$

Therefore, for infinitely many n, $\|P[g'(n)]\| \geqslant n+1$. By Observation 4.2.20, it follows that $\|\mathrm{Br}[< g'(n)]\| \geqslant n$ for infinitely many n. Let

$$D_{f(n)} = \begin{cases} \{\sigma \in 2^{<\omega} : g'(n-1) \leqslant |\sigma| < g'(n)\} & \text{if } n \neq 0 \\ \{\sigma \in 2^{<\omega} : |\sigma| < g'(0)\} & \text{otherwise.} \end{cases}$$

Then, for infinitely many n,

$$\begin{aligned} \|\mathrm{Br}(P) \cap \textstyle\bigcup_{i=0}^{n} D_{f(i)}\| &= \|\mathrm{Br}[< g'(n)]\| \\ &= \|P[g'(n)]\| - 1 \\ &\geqslant n, \end{aligned}$$

and $\overline{\mathrm{Br}(P)}$ is not dense simple.

For the other direction, suppose $\langle D_{f(n)}\rangle$ is such that $\|\mathrm{Br}(P) \cap \bigcup_{i=0}^{n} D_{f(i)}\| \geqslant n$ for infinitely many n. Let $m(n) = \max(\bigcup_{i=0}^{n} D_{f(i)})$. Then, for infinitely many n, $\|\mathrm{Br}[\leqslant m(n)]\| \geqslant n$, which implies, using Observation 4.2.20, that $\|P[m(n)+1]\| \geqslant n$ and so P is not very small. □

Theorem 4.3.14. *P is very small if and only if $\overline{\mathrm{Brl}(P)}$ is dense simple.*

Proof. Similar to the proof of Theorem 4.2.23. If $\langle D_{f(n)}\rangle$ is a disjoint strong array such that, for infinitely many n, $\|\mathrm{Brl}(P) \cap \bigcup_{i=0}^{n} D_{f(i)}\| \geqslant n$ then define $\langle D_{g(n)}\rangle$ as in the first half of the proof of Theorem 4.2.23 and this disjoint strong array witnesses the fact that $\overline{\mathrm{Br}(P)}$ is not dense simple.

In the other direction, if $\langle D_{f(n)} \rangle$ is a disjoint strong array witnessing the fact that $\overline{\mathrm{Br}(P)}$ is not dense simple, then define $h(n)$ as in Theorem 4.2.23 and let

$$D_{g(n)} = \bigcup_{i=0}^{n} D_{f(h(i))} \setminus \bigcup_{i=0}^{n-1} D_{f(h(i))}.$$

$\langle D_{g(n)} \rangle$ then witnesses the fact that $\overline{\mathrm{Brl}(P)}$ is not dense simple.

□

Very smallness is a strictly stronger property than smallness as the next theorem shows. First we will need the following lemma.

Lemma 4.3.15. *(Lachlan [19] and Robinson [23]) There is a hypersimple set that has no dense simple superset.*

Robinson and Lachlan actually proved that there is an r-maximal set with no dense-simple superset, but as all r-maximal sets are hypersimple (see, for example [30], chapter X), the lemma follows.

Theorem 4.3.16. *There exists a small Π^0_1 subset of 2^ω that has no very small subset.*

Proof. Let X be hypersimple with no dense simple superset and let $X_0 \cup X_1 = X$ be any r.e. partition of X. We claim that $S = \mathcal{S}(X_0, X_1)$ is small with no very small Π^0_1 subclass.

We first observe that S is small as X is hypersimple. Suppose $V \subseteq S$ is a very small Π^0_1 subclass. Then, by Theorem 4.3.12, $\overline{\mathrm{Br}(P)}$ is dense simple. But $\mathrm{Br}(V) \subseteq \mathrm{Br}(S)$, so $\overline{\mathrm{Br}(V)} \supseteq \overline{\mathrm{Br}(S)}$. Therefore it is sufficient to show that $\overline{\mathrm{Br}(S)}$ has no dense simple superset.

Suppose $Y \supseteq \overline{\mathrm{Br}(S)}$ were dense simple. Define,

$$H = \{n : \forall\sigma\ |\sigma| = n \implies \sigma \in Y\}.$$

The claim is that H is a dense simple superset of X, contradicting our original assumption. It is clear from its definition that it is r.e.. Also note that if $m \in X$ then for all σ of length m, $\sigma \in \overline{\mathrm{Br}(S)} \subseteq Y$ so $X \subseteq H$. Also H is co-infinite as, if it were co-finite, Y would also be co-finite and not dense simple.

Suppose now that there were a disjoint strong array, $\langle D_{f(n)} \rangle_n$, such that for infinitely many n,

$$\|\overline{H} \cap \bigcup_{i=0}^{n} D_{f(i)}\| \geqslant n.$$

Define a recursive function $g(n)$ such that,

$$D_{g(n)} = \{\sigma : |\sigma| \in D_{f(n)}\}.$$

For any natural numbers, k and n, if $k \in \overline{H} \cap D_{f(n)}$ then there exists a $\sigma \in 2^{<\omega}$ such that $|\sigma| = k$, $\sigma \in \overline{Y}$ and $\sigma \in D_{g(n)}$. That is, $\sigma \in \overline{Y} \cap D_{g(n)}$. For any two distinct $k, k' \in \omega$, the required σ and σ' will have different lengths and therefore be distinct, so we get, for all n,

$$\|\overline{H} \cap D_{f(n)}\| \leqslant \|\overline{Y} \cap D_{g(n)}\|.$$

Therefore, for infinitely many n,

$$\|\overline{Y} \cap \bigcup_{i=0}^{n} D_{g(i)}\| \geqslant \|\overline{H} \cap \bigcup_{i=0}^{n} D_{f(i)}\| \geqslant n,$$

contradicting the fact that Y is dense simple. □

Corollary 4.3.17. *If P and V are Π^0_1 subsets of 2^ω such that V is very small, P has no small Π^0_1 subclass, and $P >_w V$, then there exists a Π^0_1 class, $Q \subseteq 2^\omega$ such that $V <_w Q <_w P$.*

Proof. Let S be small with no very small Π^0_1 subclass. Then we claim $V \vee (P \wedge S)$ is the required Q. $V \wedge S$ is small and so is not Muchnik reducible to P (using Lemma 4.2.13). Therefore $V \vee (P \wedge S) \equiv_w P \wedge (V \vee S) <_w P$. But also $V \not\geqslant_w (P \wedge S)$ as neither P nor S has a very small Π^0_1 subclass. Therefore $V \vee (P \wedge S) >_w V$. □

4.4 Small Π^0_1 classes, Measure, and Thinness

In this sections we compare smallness with the well-established concepts of measure and thinness.

μ will be the standard fair-coin measure on subsets of 2^ω. If $\sigma \in 2^{<\omega}$ then, μ_σ is μ relativised to $\{f \in 2^\omega : f \supset \sigma\}$. If P is a closed subset of 2^ω, the function, $n \mapsto \|P[n]\|/2^n$ is decreasing and $\mu(P) = \lim_n \|P[n]\|/2^n$.

Theorem 4.4.1. *If $P \subseteq 2^\omega$ is closed, and $\mu(P) > 0$, then P is not small.*

Proof. Choose some recursive $r \in \mathbb{R}$ such that $0 < r \leqslant \mu(P)$. Then for all n, $\|P[n]\| \geqslant r \cdot 2^n$ and, if $g(n) =$ least k, $k \geqslant \log_2(n/r)$, then $\|P[g(n)]\| \geqslant n$. □

A Π^0_1 class, P, is *thin* if every Π^0_1 subclass of P is the intersection of P with some clopen set. Equivalently, P is thin if and only if its lattice (under $\cap, \cup$) of Π^0_1 subclasses forms a Boolean algebra. The notion has been studied by Downey, Coles, Cholak and others in [7], [9], [10] and elsewhere. As both small and thin classes are "diminutive" in some sense, it is natural to ask at this stage how the notions of thinness and smallness relate to each other.

Theorem 4.4.2. *There exists a very small (and hence small) Π^0_1 class that is not thin.*

Proof. If V is any very small Π^0_1 class, then by Lemma 4.3.4, so is $V \vee V$. However $V \vee V$ is never thin as $\{f \oplus f : f \in V\}$ is a Π^0_1 subclass of $V \vee V$ that is not the intersection of V with any clopen set (it is easy to see its complement in V is not closed). $\square$

Theorem 4.4.3. *There is a thin Π^0_1 class that is not very small*

Proof. We first show that for any perfect (hence special) $\Pi^0_1 \subseteq 2^\omega$, $\mathrm{Ext}(P) \equiv_T \mathrm{Br}(P)$. One direction is clear because $\sigma \in \mathrm{Br}(P) \Leftrightarrow \sigma^\frown\langle 0\rangle, \sigma^\frown\langle 1\rangle \in \mathrm{Ext}(P)$. So $\mathrm{Br}(P) \leqslant_T \mathrm{Ext}(P)$. For the other direction, $\sigma \in \mathrm{Ext}(p) \Leftrightarrow \exists\tau \in \mathrm{Br}(P)\ \tau \supseteq \sigma$. So $\mathrm{Ext}(P)$ is r.e. in $\mathrm{Br}(P)$. But $\mathrm{Ext}(P)$ is a co r.e. set, so it is in fact, recursive in $\mathrm{Br}(P)$. That is, $\mathrm{Ext}(P) \leqslant_T \mathrm{Br}(P)$.

The rest of the proof follows from results in [9] about the Turing degree of the extendible nodes of thin Π^0_1 classes. In [9], Downey, Jockusch and Stob introduce a class of r.e. degrees called the anr degrees (later called anc degrees). They prove that there are thin separating classes whose extendible nodes are of anr degree (viz. the Π^0_1 sets associated with Martin Pour-el theories), and indeed that every anr degree contains

Ext(T) for some thin separating class Π^0_1 class, T. They also show in [9] that there are low anr degrees.

Let T be a thin separating Π^0_1 class such that Ext(T) is of low degree. Suppose T is very small. Then $\overline{\mathrm{Br}(T)}$ would be dense simple, and therefore of high degree. As $\overline{\mathrm{Br}(T)} \equiv_T \mathrm{Br}(T) \equiv_T \mathrm{Ext}(T)$, this is a contradiction. □

Theorem 4.4.4. *There exists a thin, very small* Π^0_1 *class*

Proof. This is just a matter of combining the requirements from theorem 4.3.5 with the requirements for thinness (see for example [7]).

□

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