Course Proposal

Set Theory and Applications

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KFUPM

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Chairman
Department of Mathematics and Statistics
KFUPM
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Dear Chairman

I am hoping that you will give your consideration to the following undergraduate course proposal in Set Theory. I would like the department to accept it under the MATH 499 topics course to be taught in semester 102. I propose to call it *Set Theory and Applications*. The course is targeted to mathematics majors, but will be accessible to anyone who has taken Math232 or the equivalent Computer Science course.

The course will build on the concepts introduced in Math 232 - sets, functions, relations, bijections and so on - and move on to develop the Cantorian theory of infinity. We will begin with the study of partial, linear and well-orders and move on to ordinal numbers, cardinal numbers and the Schroeder-Bernstein-Cantor Theorem. This should cover the first third of the course.

The middle of the course will include examples of transfinite induction and recursion, cardinal and ordinal arithmetic and the well-ordering principle. The formal axiomisation of mathematics via the Zermelo-Fraenkel axioms with the Axiom of Choice will be shown to the students, as well as the motivation behind these axioms as a response to Russell's Paradox. If time permits we might briefly cover Goodstein sequences.

The final third of the course will be dedicated to a proof of the Banach-Tarski Paradox. This theorem is a challenge for undergraduate students, but it is possible with preparation to present a rigourous and understandable exposition. The syllabus will include rigid transformations, the rotation group of the sphere, orbits and the application of the Axiom of Choice to complete the construction. None of these subjects in itself is too difficult for our students and I expect with preparation they will be able to grasp the argument in essence and in detail.

I have proposed calling this course Set Theory with Applications because in addition to the purely theoretical development of the subject that lays a foundation for further study in Logic, it also attempts to provide examples of the use Cantor's theory of infinity to other other branches of mathematics. In particular, Goodstein sequences and the Banach Tarski Paradox are particularly significant to Geometry and Number Theory, as well as being of general interest to most mathematically minded people. Our examination of the Axiom of Choice will help to frame students' future mathematical studies in a context of issues in the foundations of mathematics, its uses in mathematics and its controversial status.

I think this proposed course offers our students a combination of formal rigour as well as interesting and challenging theorems to ponder. As well as developing their ability to do formal proofs in the axiomatic part of the course, the students will be encouraged to broaden their understanding about the possibilities of mathematical argument. They will see that mathematics can analyse and reason about actual infinite objects, and that geometric and other naïve intuitions often fail when formal mathematical argumentation is applied to them.

I have developed this course at the direct and unsolicited request of two mathematics majors who were my students in Math232. I taught the students in different semesters but both approached me independently to teach a further course in set theory. Both have reaffirmed multiple times their intention to take the course if offered in semester 102. Furthermore one of the students has expressed his desire to do a PhD in mathematics.

This will not be one of the easier courses in the undergraduate program, but I believe that I can make it one of the most interesting.

Thank you for your consideration of this proposal.

Regards,

Stephen Binns
Department of Mathematics and Statistics
KFUPM

Proposed Course Specification:
Set Theory and Applications

0. Department	Department of Mathematics and Statistics
A. Course Identification and General Information	
1. Course title	Set Theory and Applications
2. Credit hours	3
3. Program	BS Mathematics
4. Faculty	Dr Stephen Binns
5. Level	4th semester and above
6. Prerequisites	MATH 232 or ICS 254
7. Corequisites	Nil
8. Location	Main campus
B. Aims and Objectives	
1. Summary of main learning outcomes	An introduction to the modern mathematical theory of sets and infinity. The students will be exposed to the basic concepts and techniques needed to continue with study of logic and set theory. Students who continue with mathematics will gain a richer and more subtle understanding of the foundations of their subject. Students from other disciplines will gain from further exposure to the most abstract thought.
2. Course development plans	The course will offered to those students taking a BS in mathematics or computer science. It is hoped that it will form a basis for possible graduate work in Logic or Set Theory and that it will form part of a collection of courses in foundational mathematics that can be offered to undergraduate students when scheduling allows. It is possible for this course to be extended and deepened to form the basis of a graduate course in Set Theory or Logic.

C. Course Description	
Proposed Bulletin description: The course will consist of the basics of mathematical set theory including well-orderings, cardinality and transfinite arithmetic. Foundational axioms are investigated and The Banach-Tarski paradox is proved. Philosophical issues surrounding the Axiom of Choice are investigated	
1. Topics to be covered	See attached syllabus.
2. Course Components	The course will consist of 3 lectures per week, two major exams and a final exam.
3. Additional private study or learning hours	The students will be expected to spend an average of 3 hours per week on homework. In addition there will be two projects to be completed in the semester. Approximately 5 hours to be spent on each project.
4. Development of Learning Outcomes in Domains of Learning	
a. Knowledge	
(i) Knowledge to be acquired	Basics of the mathematical theory of sets. Introduction to the Cantorian conception of infinity. An appreciation for the need for formalisation and a detailed understanding of the axiomisation of mathematics. An understanding of the Banach-Tarski paradox and other problematic consequences of the Axiom of Choice.
(ii) Teaching strategies	Lectures will be the primary method of instruction. Each new topic will be motivated at first asking an open-ended question. The students will propose answers which will be critiqued by the instructor and the class. An introduction will then be given to the standard mathematical analysis of the question. More details will be added in subsequent lectures and questions arising from the analysis will be raised and dealt with. Along the way students will be expected to answer basic questions based on the mathematical analysis
(iii) Methods of assessment	The assessment will be based on exams, projects and quizzes. The projects will aim to evaluate students' understanding as well as give them an opportunity to investigate more thoroughly issues arising from the material presented in lectures. Quizzes will test that students' current knowledge of the material, and exams will test the students' overall understanding of the material. All evaluation methods will test both understanding and problem-solving ability.

C. Course Description cont.	
b. Cognitive skills	
(i) Cognitive skills to be developed	The ability to reason logically and precisely about a well-defined mathematical subject. The ability to express arguments in the language of mathematics. The ability to argue cogently with mathematical symbolism. The development of the ability to think informally but accurately about the fundamental intuitive concepts of sets and functions. The creation of a correct intuition about infinity.
(ii) Teaching strategies	The students will be expected to produce formal mathematical proofs. These will be corrected and returned to the student for improvement if necessary. It is important that the student learns to internalise the requirements of a correct mathematical argument, and feedback will be given through homework correction and review of quizzes and exams. Opportunity will be given for students to improve their homework grades by reviewing and correcting their own work.
(iii) Methods of assessment	Quizzes and exams will contain a mixture of questions designed to assess both general understanding and problem solving ability. True/false questions and short explanation questions will be used to evaluate a student's intuition and understanding. More detailed questions will be used to evaluate accuracy and logical argument.

c. Interpersonal skills and responsibility	Some group work may be assigned but in general the students will work individually.
(i) Skills to be developed	
(ii) Teaching strategies	
(iii) Methods of assessment	
(d) Communication Information technology and numerical skills	
(i) Skills to be developed	Professional communication of ideas and arguments.
(ii) Teaching strategies	A WEBCT page will be created with a chat-room to discuss homework and Projects
(iii) Methods of assessment	No direct assessment of these skills
(e) Psychomotor skills	Not Applicable
(i) Skills to be developed	
(ii) Teaching strategies	
(iii) Methods of assessment	
6. Schedule of assessment tasks	

D. Student Support

1. Availability of faculty for consultation and advice	The lecturer will be available for office hours on Sundays and Tuesdays, and on Saturdays, Mondays, and Wednesdays by appointment. After-hours help can be given via WebCT.

E. Learning Resources

1. Required texts	*The Joy of Sets* by Keith Devlin Springer 2e 1993
2. Essential references	
3. Recommended books and reference material	For the pure set-theoretical aspects of the course, a variety of extra books and material exist in the main library. Minimal use of these will be required as the proposed textbook is comprehensive. For the Banach-Tarski paradox, the students will be provided with notes. These will be written by the instructor and be taken from various sources.
4. Electronic materials	
5. Other materials	

F. Facilities required	
1. Accommodation	One lecture room
2. Computing resources	Projection facilities for electronic slides
3. Other resources	
G. Course evaluation and Improvement Processes	
1. Strategies for obtaining student feedback on quality of teaching.	Standard course evaluation form. Anonymous feedback form on WebCT. Detailed course evaluation by students.
2. Other strategies for evaluation of teaching	Review by the lecturer of the amount and nature of the material covered.
3. Processes for improvement of teaching	Attending advanced WebCT seminar to improve knowledge and delivery of on-line teaching possibilities.
4. Processes for verifying standards of student achievement	Comparison of grades with similar level mathematics courses.
5. Action planning for improvement	Examination of on-line student feedback will lead to a review of the amount of material covered and a resulting adjustment in future offerings of this course. In particular, special attention will be paid to the results of teaching the Banach-Tarski paradox as this is the most challenging part of the course.

Proposed Syllabus - Set Theory and Applications

Section numbers refer to the proposed textbook:
The Joy of Sets 2e by Keith Devlin.
Springer 1993.

Week	Topic	Section	Topic	Hours
1	Sets and Foundations	1.1-1.6	Unions, Intersections, Subsets, Empty Set. Relations and Functions.	3
2	Orders	1.7	Partial orders. Linear Orders and Well orders.	3
3	Well Orders	1.7	Initial Segments and the $\preceq$ relation. $\preceq$ as a well order. Ordinals.	3
4	Theory of Ordinals	1.7	Proofs of theorems establishing the basic properties of ordinals.	3
5	Cardinality	1.7	Cantor's Theorem Schroeder-Bernstein-Cantor Theorem Cardinals.	3
		First exam		
6	Formalisation of mathematics	2.1-2.2	Russell's Paradox and proper classes The axioms of ZFC. The cumulative hierarchy of sets $\mathbf{V}$	3
7	ZFC in detail	2.3	Axioms of Extensionality, Null set, Union Axioms of Infinity, Power Set, Subset Selection Foundation, Replacement and Choice	3
8	Transfinite Arguments	2.5	Transfinite Recursion & Inductionn Well ordering and Choice	3
9	Ordinal Arithmetic	3.1-3.5	Addition and multiplication of Ordinals Exponentiation Cantor Polynomials	3
10	Cardinal Arithmetic	3.6	Addition and multiplication of Cardinals Exponentiation & Goodstein sequences	3
		Second exam		
11	The Banach-Tarski Theorem	Notes	Statement and Response to the Theorem Rigid Transformations Finite Decompositions	3
12	Rotations in $\mathbb{R}^2$ and $\mathbb{R}^3$	Notes	Rotations of Circles and Spheres Matrix representation of rotations Groups and Actions	3
13	Hausdorff's Construction	Notes	Orbits Hausdorff's group Orbits of Hausdorff's Group	3
14	Proof of the theorem for spheres	Notes	Orbits of the Hausdorff Group The Choice function and main result Generalisations	3