Embeddings into the Medvedev and Muchnik lattices of Π^0_1 classes

Stephen Binns
Stephen G. Simpson

Department of Mathematics
Pennsylvania State University

December 8, 2004

Abstract

Let $\mathcal{P}_w$ and $\mathcal{P}_M$ be the lattices of Π^0_1 subsets of 2^ω under Muchnik and Medvedev reducibility respectively. We show that any countable distributive lattice can be lattice-embedded into $\mathcal{P}_w$ below any non-zero element. We also show that other natural examples of countable lattices can be similarly embedded into $\mathcal{P}_M$.

1 Introduction

The concepts of *Medvedev-* and *Muchnik- reducibility* have been defined and investigated in [**?**], [**?**], [**?**] and [**?**]. A set, $A \subseteq \omega^\omega$, is Medvedev reducible to $B \subseteq \omega^\omega$, (written $A \leqslant_M B$) if there exists some recursive fuctional, $\Phi : B \to A$. That is, if there exists a recursive function, $\{e\}$, such that $\{e\}^f \in A$ for all $f \in B$. Muchnik reducibility is a non-uniform version of Medvedev reducibility - A is said to be Muchnik reducible to B ($A \leqslant_w B$) if for each $f \in B$, there is a recursive functional, Φ, such that $\Phi(f) \in A$. In this paper we will restrict these reducibilities to the class of non-empty Π^0_1 subsets of 2^ω. $P \subseteq 2^\omega$ is a Π^0_1 class if there is some recursive relation, $R \subseteq \omega \times 2^\omega$ such that

$$f \in P \leftrightarrow \forall n \; R(n, f).$$

Π^0_1 classes have an alternative characterisation which is both instructive and useful: P is a Π^0_1 class if and only if P is the set of (infinite) paths through some recursive binary tree.

Two Π^0_1 classes, P and Q are Medvedev (Muchnik) equivalent, $A \equiv_M B$ ($A \equiv_w B$) if $A \leqslant_M B$ and $B \leqslant_M A$ ($A \leqslant_w B$ and $B \leqslant_w A$) and the set of equivalence classes (Medvedev (Muchnik) *degrees*) with the induced partial order forms a distributuve lattice with a top and bottom element, (see [**?**]). These lattices will be denoted $\mathcal{P}_M$ and $\mathcal{P}_w$ respectively. The top element is the Medvedev (Muchnik) degree of the set of completions of Peano Arithmetic, and the bottom element is the Medvedev (Muchnik) degree of any Π^0_1 set containing a recursive element.

Introductions to, and some basic results about $\mathcal{P}_M$ and $\mathcal{P}_w$ can be found in [**?**], [**?**], [**?**], [**?**], [**?**] and [**?**].

In this paper we prove the existence of certain sublattices of these lattices. Our results are, in essence as follows:

1. The free countable distributive lattice, $FD(\omega)$, can be embedded into $\mathcal{P}_M$.
2. The free countable Boolean Algebra, $FB(\omega)$, can be embedded into $\mathcal{P}_w$.
3. The lattice of finite (co-finite) subsets of ω can be embedded into $\mathcal{P}_M$.

Here, and in the rest of the paper, an "embedding" is a lattice embedding. Result 1 implies (but is not equivalent to the fact) that every finite lattice can be embedded into $\mathcal{P}_M$, as every such lattice can be embedded into $FD(\omega)$. Result 2 is as general as possible, as every countable distributive lattice is embeddable into $FB(\omega)$.

Result 3 is not implied by result 1 as neither of these lattices are embeddable into $FD(\omega)$. We will reference these lattice-thoretical results in the relevant sections.

In simple extentions, all of our results are relativised in the sense that the embeddings can be made below any non-minimum degree.

The paper is in four sections. Section 2 consists of two priority arguments. These construct Π^0_1 sets that have certain useful independence properties. Both build on the constructions in [**?**], and use a Sacks preservation argument (see [**?**], Chapter VII.3). The second argument is only sketched. If, at first, the reader wishes only to skim this section and accept Theorems 2.1 and 2.7, he or she should still find Sections 3 and 4 completely accessible.

Notation and Preliminaries

We will first establish some standard notation. σ, τ, ρ and λ will be used to represent binary strings and the length of σ will be written $|\sigma|$. $\{e\}_s^\sigma$ will denote the longest binary string, τ, such that $|\tau| \leqslant s$ and $\{e\}_s^\sigma(n) \downarrow = \tau(n)$ for all $n < |\tau|$. The empty string is denoted by $\langle\rangle$ and $\{e\}^\sigma$ is short for $\{e\}_{|\sigma|}^\sigma$. The restriction of σ to $\{0, 1, 2, \ldots, n-1\}$ is denoted $\sigma\big|_n$.

A *binary tree* is a subset of $2^{<\omega}$ that is closed under taking initial segments. If T is a binary tree, then $[T] \subseteq 2^\omega$ represents the set of infinite paths through T. If P is a Π_1^0 class then $\mathrm{Ext}(P) \subseteq 2^{<\omega}$ - the extendable nodes of P - is the set of strings, $\{\sigma : \exists f \in P\ \sigma \subset f\}$.

The following notation is introduced specifically for our purposes. Let $\mathcal{S}$ be the class of finite sequences of finite strings. The uppercase Greek letters, Σ, Γ and Λ will be used to represent elements of $\mathcal{S}$. For ease of notation, a sequence of strings will sometimes be indentified with its range, so that $\sigma \in \Sigma$ means $\sigma \in \mathrm{rng}(\Sigma)$; $\Sigma \subseteq \Gamma$ means Σ is a subsequence of Γ and $\sigma \in \Sigma \smallsetminus \Gamma$ that $\sigma \in \mathrm{rng}(\Sigma) \smallsetminus \mathrm{rng}(\Gamma)$. We will reserve the symbol Σ^m to mean the sequence of all binary strings of length m in lexicographical order.

If $\Sigma = \langle\sigma_i\rangle_{i=1}^n$ and $\Gamma = \langle\gamma_i\rangle_{i=1}^m$, we will say Σ *extends* Γ if $m = n$ and $\sigma_i \supseteq \gamma_i$ for all $i \leqslant n$. Σ *properly extends* Γ if, in addition, $\sigma_k \supsetneq \gamma_k$ for at least one $k \leqslant n$. If $f_1, f_2, \ldots f_n$ are elements of 2^ω, then $\langle f_1, f_2, \ldots f_n\rangle$ *extends* Σ is defined similarly.

If $\Sigma = \langle\sigma_i\rangle_{i=1}^n \subseteq \Sigma^m$ and $\sigma \in 2^{<\omega}$, we will make the following definitions:

- $\sigma^- \in 2^{<\omega}$ such that, for all $n < |\sigma| - 1$, $\sigma^-(n) = \sigma(n+1)$. For $f \in 2^\omega$, f^- is defined similarly.

- $\bigoplus \Sigma \in 2^{<\omega}$ such that,

$$\big[\bigoplus \Sigma\big](i) = \sigma_k(q),$$

where $i = nq + k - 1$, for some (necesarily unique) $k \leqslant n$ and q. That is,

$$\bigoplus \Sigma = \langle\sigma_1(0), \sigma_2(0) \ldots \sigma_n(0), \sigma_1(1), \sigma_2(1) \ldots \sigma_n(1) \ldots\ldots \sigma_n(m-1)\rangle.$$

- If $\langle f_i\rangle_{i=1}^n$ is a sequence of elements of 2^ω. Then $\bigoplus_{i=1}^n f_i \in 2^\omega$ is defined to be such that, for all i,

$$[\bigoplus_{i=1}^{n} f_i](i) = f_k(q),$$

where, as before, $i = nq + k - 1$.

• For an arbitrary $\Gamma = \langle \gamma_i \rangle_{i=1}^n \in \mathcal{S}$ (with the γ_i of possibly different lengths), we define,

$$\bigoplus \Gamma = \bigoplus_{i=1}^{n} \gamma_i\big|_l,$$

where $l = \min\{|\gamma_i| : 1 \leqslant i \leqslant n\}$.

$\bigoplus$ is not associative but it does have the useful property that if $\langle f_1, f_2, \dots f_n \rangle$ extends $\Sigma \subseteq \Sigma^m$, then $\bigoplus_{i=1}^n f_i \supset \bigoplus \Sigma$. If no confusion can result, we will write $\bigoplus f_i$ for $\bigoplus_{i=1}^n f_i$.

2 Two Constructions

Theorem 2.1. *For any special Π_1^0 set, P, there is a Π_1^0 set, Q, with the properties, for all sequences, $\langle f_i \rangle_{i=1}^n \subset Q$,*

I. $\forall f \in Q \smallsetminus \langle f_i \rangle_{i=1}^n,\ f \not\leqslant_T \bigoplus f_i$,
II. $\forall f \in P,\ f \not\leqslant_T \bigoplus f_i$

Proof. The proof will closely follow the proof of Theorem 4.7 in [**?**]. A recursive sequence, $\langle \psi_s \rangle_{s\in\omega}$, of recursive functions from $2^{<\omega}$ to $2^{<\omega}$ will be constructed with the properties that, for all $\sigma \in 2^{<\omega}$ and $s \in \omega$,

1. $\psi_s(\sigma^\frown\langle 0 \rangle)$ and $\psi_s(\sigma^\frown\langle 1 \rangle)$ are incompatible extensions of $\psi_s(\sigma)$,
2. $\text{range}(\psi_{s+1}) \subseteq \text{range}(\psi_s)$,
3. $\psi(\sigma) = \lim_t \psi_t(\sigma)$ exists.

Each ψ_s determines a recursive tree, namely,

$$T_s = \{\tau :\ \text{for some } \sigma,\ \psi_s(\sigma) \supseteq \tau\}.$$

The required Q will then be $\bigcap_{s\in\omega}[T_s]$. Q will be non-empty as $\langle [T_s] \rangle_{s\in\omega}$ is a nested sequence of closed subsets of 2^ω. It will be a Π_1^0 set because,

$$f \in Q \equiv \forall s f \in [T_s] \equiv \forall s \forall n \exists \sigma [|\sigma| \leqslant n \wedge \psi_s(\sigma) \subset f],$$

and $\exists \sigma [|\sigma| \leqslant n \wedge \psi_s(\sigma) \subset f]$ is a recursive predicate.

Each ψ_s will induce a mapping, $\Psi_s : \mathcal{S} \to \mathcal{S}$, defined by

$$\Psi_s(\Gamma) = \langle \psi_s(\gamma_i) \rangle_{i=1}^n,$$

where $\Gamma = \langle \gamma_i \rangle_{i=1}^n$. When it is proved that $\psi(\sigma)$ exists for all σ, it will be clear that $\Psi(\Sigma) = \lim_s \Psi_s(\Sigma)$ exists for all $\Sigma \in \mathcal{S}$.

We will define $\langle \psi_s \rangle_{s\in\omega}$ so that, for every $m \in \omega$, $\Gamma \subseteq \Sigma^m$ and $e \leqslant m$, Q satisfies the requirements:

$P^m_{\Gamma,e} \equiv$ for all $\langle f_i \rangle_{i=1}^n$ extending $\Psi(\Gamma)$, $\{e\}^{\bigoplus f_i} \notin P$,

$R^m_{\Gamma,e} \equiv$ for all $\langle f_i \rangle_{i=1}^n$ extending $\Psi(\Gamma)$, and for all $\sigma \in \Sigma^m \smallsetminus \Gamma$, $\{e\}^{\bigoplus f_i} \not\supset \psi(\sigma)$.

The P requirements guarantees that Q has property II. of the theorem, and the R requirements guarantee property I. The set of requirements can be ordered lexicographically, first on m, then on e and finally with the conventions that, for all m, and $\Gamma, \Gamma' \in \Sigma^m$,

i. $P^m_{\Gamma,e}$ precededs $R^m_{\Gamma',e}$ and,

ii. $P^m_{\Gamma,e}$ precedes $P^m_{\Gamma',e}$ and $R^m_{\Gamma,e}$ precedes $R^m_{\Gamma',e}$ whenever Γ precedes Γ' in the lexicographical ordering on Σ^m.

Priority is given to the requirements in reverse lexicographical order so that reqirement S_0 has higher priority than requirement S_1 is it precedes it in the ordering. $P^m_{\Gamma,e}$ is said to be *satisfied at stage s* if,

$$\{e\}^{\bigoplus \Psi_s(\Gamma)} \notin T_P,$$

and $R^m_{\Gamma,e}$ *is satisfied at stage s* if, for all $\sigma \in \Sigma^m \smallsetminus \Gamma$,

$$\{e\}^{\bigoplus \Psi_s(\Gamma)} \not\supseteq \psi_s(\sigma).$$

We now define ψ_s as follows:

Stage $s = 0$: $\psi_0(\sigma) = \sigma$ for all $\sigma \in 2^\omega$.

Stage s+1:

We say $P^m_{\Gamma,e}$ *requires attention at stage* $s+1$ if $P^m_{\Gamma,e}$ is not satisfied at stage $s+1$ and there is a $\Lambda = \langle \lambda_i \rangle_{i=1}^n$ properly extending Γ such that $\max\{|\lambda_j| : \lambda_j \in \Lambda\} \leqslant s+1$ and,

i. $\{e\}^{\bigoplus \Psi_s(\Lambda)} \in T_P$,

ii. $\{e\}^{\bigoplus \Psi_s(\Lambda)} \supsetneq \{e\}^{\bigoplus \Psi_s(\Gamma)}$.

We say $R^m_{\Gamma,e}$ *requires attention at stage* $s+1$ if $R^m_{\Gamma,e}$ is not satisfied at stage $s+1$ and there is a $\Lambda = \langle \lambda_i \rangle_{i=1}^n$, properly extending Γ, such that $\max\{|\lambda_j| : \lambda_j \in \Lambda\} \leqslant s+1$ and,

$$\{e\}^{\bigoplus \Psi_s(\Lambda)} \supseteq \psi_s(\sigma^\frown \langle x \rangle), \text{ for some } x \in \{0,1\} \text{ and } \sigma \in \Sigma^m \smallsetminus \Gamma.$$

If $P^m_{\Gamma,e}$ has priority greater than the priority of $P^s_{\Sigma^s,s}$ and is the highest priority requirement requiring attention at stage $s+1$, let Λ witness this fact and define,

$$\psi_{s+1}(\nu) = \begin{cases} \psi_s(\lambda_i^\frown \nu') & \text{if } \nu = \gamma_i^\frown \nu' \text{ for some } \gamma_i \in \Gamma \\ \psi_s(\nu) & \text{if } \nu \not\supseteq \gamma_i \text{ for any } \gamma_i \in \Gamma. \end{cases}$$

If $R^X_{m,e}$ has priority greater than the priority of $P^s_{\Sigma^s,s}$ and is the highest priority requirement requiring attention at stage $s+1$, let Λ, σ and x witness this and define,

$$\psi_{s+1}(\nu) = \begin{cases} \psi_s(\lambda_i^\frown \nu') & \text{if } \nu = \gamma_i^\frown \nu' \text{ for some } \gamma_i \in \Gamma, \\ \psi_s(\sigma^\frown \langle 1-x \rangle^\frown \nu') & \text{if } \nu = \sigma^\frown \nu', \\ \psi_s(\nu) & \text{if } \nu \not\supseteq \tau \text{ for any } \tau \in \Gamma \cup \{\sigma\}\ . \end{cases}$$

If no requirement of priority greater than the priority of $P^s_{\Sigma^s,s}$ requires attention at stage $s+1$, then let $\psi_{s+1} = \psi_s$.

The following lemmas establish the theorem.

Lemma 2.2. *For any requirement, S, there is a stage, s_0, such that S does not require attention at any stage $t > s_0$.*

Proof. Assume not and let S be the highest priority requirement requiring attention infinitely often. If $S = P^m_{\Gamma,e}$, then let t be a stage such that $P^m_{\Gamma,e}$ has priority greater than $P^t_{\Sigma^t,t}$ and such that all higher priority requirements are satisfied for all stages $\geqslant t$. Let $s_1, s_2, s_3, \ldots$ be an infinite increasing sequence of stages greater than t at which S requires attention. At each of these stages S will be the highest priority requirement requiring attention and so $s_1, s_2, s_3, \ldots$ will generate a recursive sequence,

$$\{e\}^{\bigoplus \Psi_{s_1}(\Gamma)} \subsetneq \{e\}^{\bigoplus \Psi_{s_2}(\Gamma)} \subsetneq \{e\}^{\bigoplus \Psi_{s_3}(\Gamma)} \ldots,$$

of elements of T_P. But then $\bigcup_i \{e\}^{\bigoplus \Psi_{s_i}(\Gamma)}$ is a recursive path through T_P, contradicting the original assumption that P is special.

Next suppose $S = R^m_{\Gamma,e}$. If t is such that the priority of $R^m_{\Gamma,e}$ is greater than $P^t_{\Sigma^t,t}$, all higher priority requirements are permanently satisfied at stage t, and S requires attention at stage t, then S will be satisfied at stage $t+1$. Suppose, at some stage $u > t$, a lower priority requirement, T, requires attention. If $T = P^{m'}_{\Lambda,e'}$ or $T = R^{m'}_{\Lambda,e'}$ with $m' > m$, and any Λ and e', then $\Psi_{u+1}(\Gamma) = \Psi_u(\Gamma)$ and S will remain satisfied at stage $u+1$. If $T = R^m_{\Lambda,e'}$ or $T = P^m_{\Lambda,e'}$, then $\Psi_{u+1}(\Gamma) \supseteq \Psi_u(\Gamma)$ and so S will remain satisfied at stage $u+1$. We then argue by induction that S will remain satisfied, and hence not require attention, at all stages $u \geqslant t$, contradicting the assumption. $\square$

Lemma 2.3. *$\psi(\sigma) = \lim_s \psi_s(\sigma)$ exists for all σ.*

Proof. Let $\sigma \in 2^{<\omega}$ be arbitrary. By Lemma 2.2, there exists a stage, t, such that for all $m \leqslant |\sigma|$, and all $\Gamma \subseteq \Sigma^m$, the requirements $R^m_{\Gamma,e}$ and $P^m_{\Gamma,e}$ do not require attention after stage t. Then $\psi_{t_1}(\sigma) = \psi_{t_2}(\sigma)$ for all $t_1, t_2 > t$. $\square$

Lemma 2.4. *If $m \in \omega$, $e \leqslant m$ and $\Gamma \subseteq \Sigma^m$ are such that $\{e\}^{\bigoplus \Psi(\Gamma)} \in T_P$, then there does not exist a Λ properly extending Γ such that $\{e\}^{\bigoplus \Psi(\Lambda)} \in T_P$ and $\{e\}^{\bigoplus \Psi(\Lambda)} \supsetneq \{e\}^{\bigoplus \Psi(\Gamma)}$.*

Proof. Suppose such a Λ existed for m, e and Γ. Take t so large that $\Psi_t(\Gamma) = \Psi(\Gamma)$ and $\Psi_t(\Lambda) = \Psi(\Lambda)$. Then,

$$\{e\}^{\bigoplus \Psi_t(\Lambda)} = \{e\}^{\bigoplus \Psi(\Lambda)} \supsetneq \{e\}^{\bigoplus \Psi(\Gamma)} = \{e\}^{\bigoplus \Psi_t(\Gamma)},$$

and so, at some stage $u \geqslant t$, $P^m_{\Gamma,e}$ would be the highest priority requirement requiring attention, implying,

$$\{e\}^{\bigoplus \Psi_{u+1}(\Gamma)} \supsetneq \{e\}^{\bigoplus \Psi_u(\Gamma)} = \{e\}^{\bigoplus \Psi_t(\Gamma)} = \{e\}^{\bigoplus \Psi(\Gamma)},$$

contradicting the fact that $\Psi_{u+1}(\Gamma) = \Psi(\Gamma)$. $\square$

Lemma 2.5. *If $\langle f_i \rangle_{i=1}^n \subseteq Q$ then, for all $f \in P$, $f \not\leqslant_T \bigoplus f_i$.*

Proof. We can assume without losing generality that $\langle f_i \rangle_{i=1}^n$ is in lexicographic order. Suppose the lemma is false and let $\{e\}^{\bigoplus f_i} \in P$. Let $m \in \omega$ and $\Gamma \subseteq \Sigma^m$ be such that,

i. $e \leqslant m$,

ii. $\langle f_i\rangle_{i=1}^n$ extends $\Psi(\Gamma)$

Such a Γ can be found because $\langle f_i\rangle_{i=1}^n$ is in lexicographic order. But $\{e\}^{\Psi(\Gamma)} \in T_P$, so there must be a $\Lambda \supsetneq \Gamma$ such that $\{e\}^{\bigoplus \Psi(\Lambda)} \supsetneq \{e\}^{\bigoplus \Psi(\Gamma)}$, contradicting Lemma 2.4.

$\square$

Lemma 2.6. *For all $\langle f_i\rangle_{i=1}^n \subseteq Q$ and all $f \in Q \setminus \langle f_i\rangle_{i=1}^n$,*

$$f \not\leqslant_T \bigoplus f_i$$

Proof. Suppose not and let $\{e\}^{\bigoplus f_i} = f \in Q$. Let $m \in \omega$, $\Gamma \subseteq \Sigma^m$ and $\sigma \in \Sigma^m \setminus \Gamma$ be such that,

i. $e \leqslant m$,
ii. $\langle f_i\rangle_{i=1}^n$ extends $\Psi(\Gamma)$,
iii. $f \supset \psi(\sigma)$,

(again we are assuming $\langle f_i\rangle_{i=1}^n$ is in lexicographic order). Let t be such that $\Psi_u(\Gamma) = \Psi(\Gamma)$ and $\psi_u(\sigma^\frown\langle x\rangle) = \psi(\sigma^\frown\langle x\rangle)$ for all $u \geqslant t$ and $x \in \{0,1\}$. By the supposition, there must be a stage, $s \geqslant t$ and a Λ extending Γ such that

$$\{e\}^{\Psi_s(\Lambda)} \supseteq \psi_s(\sigma^\frown\langle x\rangle) \text{ for some } x \in \{0,1\}.$$

So there will be a stage, $v \geqslant s$, at which $R^m_{\Gamma,e}$ requires attention and is, in fact, the highest priority requirement requiring attention. But then,

$$\Psi_{v+1}(\Gamma) \neq \Psi_v(\Gamma) = \Psi(\Gamma),$$

contradicting the fact that $v \geqslant u$.

$\square$

Theorem 2.1 Lemmas 2.5 and 2.6 prove that Q has properties I. and II. as required. $\square$

Theorem 2.7. *Given any special Π^0_1 set, P, there is an infinite recursive sequence of Π^0_1 sets, $\langle Q_i : i \in \omega\rangle$, with the properties, for all $i, j \in \omega$ such that $i \neq j$,*

I. $\forall f \in Q_i\ \forall g \in Q_j\ f \not\leqslant_T g$,
II. $\forall f \in Q_i\ \forall g \in P\ g \not\leqslant_T f$.

Proof. (*sketch*)
A recursive sequence of recursive functions, $\psi^i : 2^{<\omega} \to 2^{<\omega}$, is constructed, the range of each function is the tree T_i and then Q_i will be $[T_i]$. Each ψ^i is constructed as the limit of a recursive sequence of recursive functions, $\langle\psi^i_s\rangle_s$ and will be defined so that, for every $m \in \omega$, ψ^i satisfies the requirements:

$$\text{for all } e \leqslant m;\ j \leqslant m;\ \sigma \in \Sigma^m \text{ and for all } f \text{ extending } \psi^i(\sigma),$$
$$P^m \equiv \quad \{e\}^f \notin P,$$
$$R^m \equiv \quad j \neq i \Rightarrow \{e\}^f \not\supseteq \psi^j(\sigma).$$

These requirements are then further specified by indexing them according to i, j, σ and e (bounded as above), and an exhaustive priority ordering is given to them. The same method as in Theorem 2 is then used to ensure all are satisfied. If at any stage of construction an R^m requirement is the highest priority requirement requiring attention then the requirement is satisfied (permanently) at the next stage.

If at some stage of construction a P^m requirement will be the highest priority requirement requiring attention and then the function being constructed is adapted to keep the requirement unsatisfied (as per Sacks' preservation strategy, see [?] Chapter VII.3). An (non-constructive) argument is then made to show that this strategy will eventually fail (because P has no recursive elements) and P_m will eventually be satisfied. These are essentially the arguments of Lemmas 2.5 and 2.6.

□

3 $FD(\omega) \hookrightarrow \mathcal{P}_M$

Theorem 3.1. *Given any special Π^0_1 class, P, $FD(\omega)$ can be embedded into $\mathcal{P}_M$ below P.*

Proof. Let P be any special Π^0_1 class and suppose Q and ψ are as in Theorem 2.1. Let $\{\sigma_i : i \in \omega\}$ be a set of binary strings defined by:

i. $|\sigma_i| = i + 1$,

ii. $\sigma_i(n) = \begin{cases} 1 & \text{if } n = i, \\ 0 & \text{otherwise.} \end{cases}$

Then $\{\sigma_i : i \in \omega\}$ is a pairwise incomparable set of strings and hence so is $\{\psi(\sigma_i) : i \in \omega\}$. Denote by Q_i the set of elements of Q extending $\psi(\sigma_i)$, and let $P_i = P \wedge Q_i$. The set $\{P_i : i \in \omega\}$ then generates a sublattice of $\mathcal{P}_M$ strictly below P. To see this note that if X is a non-empty finite subset of ω,

$$\bigvee_{i \in X} P_i <_M P,$$

because $\bigvee_{i \in X} P_i \leqslant_M P$, and if $\bigvee_{i \in X} P_i \geqslant_M P$ then $P \wedge \bigvee_{i \in X} Q_i \geqslant_M P$ and some element of $\bigvee_{i \in X} Q_i$ would compute an element of P, contradicting property II. of Theorem 2.1. This is enough to show that all elements of the generated sublattice are strictly below P.

We will use a standard lattice theoretical result - Theorem II.2.3 in [**?**] - to show that the lattice generated by the P_i's is free. If X and X' are finite subsets of ω, then,

$$\begin{array}{rrcl} & \bigwedge_{i \in X} P_i & \leqslant_M & \bigvee_{j \in X'} P_j, \\ \Rightarrow & P \wedge \bigwedge_{i \in X} Q_i & \leqslant_M & P \wedge \bigvee_{j \in X'} Q_j, \\ \Rightarrow & P \wedge \bigwedge_{i \in X} Q_i & \leqslant_M & \bigvee_{j \in X'} Q_j, \end{array}$$

so if $\bigoplus_{j \in X'} f_j \in \bigvee_{j \in X'} Q_j$, then there is a $g \in P \vee \bigwedge_{i \in X} Q_i$ such that $g \leqslant_T \bigoplus_{j \in X'} f_j$. Therefore, $g^- \leqslant_T \bigoplus_{j \in X'} f_j$ where $g^- \in P$ or $g^- \in \bigwedge_{i \in X} Q_i$. But $g^- \notin P$ by property II. of Theorem 2.1. And if $j \notin X$ then $g^- \notin \bigwedge_{i \in X} Q_i$ by property I. of Theorem 2.1. Therefore, $j \in X$ and $X \cap X' \neq \emptyset$ as required by Theorem II.2.3 in [**?**]. □

Corollary 3.2. *Every finite distributive lattice can be embedded into* $\mathcal{P}_M$.

Proof. This follows immediately from Theorem 3.1 and the fact that every finite distributive lattice is embeddable in $FD(\omega)$. This seems to have first been observed by Simpson. The proof is presented in [**?**], along with a different proof of this corollary. □

4 $FB(\omega) \hookrightarrow \mathcal{P}_w$

In the section we give the second principal embedding theorem - that the free Boolean algebra on ω generators, $FB(\omega)$, is embeddable into $\mathcal{P}_w$, the lattice of Muchnik degrees. We represent $FB(\omega)$ as an algebra of recursive sets and then give an explicit embedding into $\mathcal{P}_w$.

As before, the argument will use Π^0_1 sets constucted using a priority argument. This time on those Π^0_1 sets of Theorem 2.7. Then we show that all countable distributive lattices embed into $FB(\omega)$. Finally we establish result 3 on page 2.

We will require two constructions given by the following definitions. Let $\emptyset \neq A \subseteq \omega$ be recursive and let $\langle P_i : i \in \omega \rangle$ be a recursive sequence of Π^0_1 sets. Let $(\cdot,\cdot) : \omega \times \omega \to \omega$ be a recursive bijection.

Definition 4.1. *If $x \in 2^\omega$, we define $\big(x\big)_i \in 2^\omega$ by,*

$$\big(x\big)_i(n) = x((i,n)),$$

and then the recursive join of $\langle P_i : i \in A \rangle$, *denoted $\bigvee_{i \in A} P_i$, is given by,*

$$x \in \bigvee_{i \in A} P_i \Leftrightarrow \big(x\big)_i \in P_i \textit{ for all } i \in A.$$

$\bigvee_{i \in A} P_i$ is clearly a Π^0_1 class as,

$$x \in \bigvee_{i \in A} P_i \equiv \forall i \ \ i \in A \Rightarrow \ \big(x\big)_i \in P_i.$$

Also, note that there is no restriction on $\big(x\big)_i$ if $i \notin A$.

We will now define a recursive meet. Let A and $\langle P_i : i \in \omega \rangle$ be as above and, for each $i \in \omega$, let T_i be a recursive tree such that $[T_i] = P_i$. If T is a recursive tree such that $[T] = \mathrm{DNR}_2$ (or any Medvedev complete Π^0_1 class), then let $\langle \sigma_j : j \in \omega \rangle$ be the sequence, in lexicographical order, of all binary strings such that $\sigma_j \in T$ but $\sigma_j^\frown \langle x \rangle \notin T$ for any $x \in \{0,1\}$. The sequence will be infinite as $[T]$ has no recursive element. Define,

$$T^* = T \cup \{\sigma_i^\frown \tau : i \in A,\ \tau \in T_i\}.$$

Definition 4.2. *The* recursive meet of $\langle P_i : i \in A \rangle$, *denoted $\bigwedge_{i \in A} P_i$, is $[T^*]$, the set of paths through T^*.*

Note that if A is finite, the recursive meet and join are Medvedev equivalent to the standard, lattice-theoretic meet and join respectively, allowing us some ambiguity of notation. However, it is not to be assumed that these constructions are necessarily the greatest lower or least upper bounds when A is infinite.

Now let $\langle Q_i : i \in \omega \rangle$ be as in Theorem 2.7 (with P arbitrary). Define,

$$\widehat{Q}_i = \bigwedge_{j \neq i} Q_j,$$

and, for any recursive, non-empty set, A, let,

$$\widehat{Q}(A) = \bigvee_{i \in A} \widehat{Q}_i.$$

Lemma 4.3. *If $A, B \neq \emptyset$ and $A \neq B$, then $\widehat{Q}(A) \not\equiv_w \widehat{Q}(B)$ (and therefore $\widehat{Q}(A) \not\equiv_M \widehat{Q}(B)$).*

Proof. Suppose that A and B are as above and that, without losing generality, $j \in B \setminus A$. Choose any $x \in Q_j$ and define $\bar{x}$ by,

$$\left(\bar{x}\right)_i = \sigma_j{}^\frown x \text{ for all } i \in \omega.$$

Then $\bar{x} \in \widehat{Q}(A)$ as $\sigma_j{}^\frown x \in \widehat{Q}_i$ for all $i \neq j$ and, in particular, for all $i \in A$. Now let $y \in \widehat{Q}_j$ be arbitrary. There are two cases.

Case 1. $y = \sigma_i{}^\frown z$ for some $i \neq j$ and $z \in Q_i$. Then,

$$y \equiv_T z \not\leqslant_T x \equiv_T \bar{x},$$

($z \not\leqslant_T x$ as $z \in Q_i$ and $x \in Q_j$, with $i \neq j$).

Case 2. $y \in [T]$, where $[T]$ is the Medvedev complete Π^0_1 class used in the construction of the recursive meet. Then for any $i \in \omega$, there is a $z \in Q_i$ such that $y \geqslant_T z$. We choose some $i \neq j$, and then fix z. If $\bar{x} \geqslant_T y$, we would have,

$$Q_j \ni x \equiv_T \bar{x} \geqslant_T y \geqslant_T z \in Q_i, \text{ with } i \neq j,$$

contrary to construction of $\langle Q_i : i \in \omega \rangle$.

Therefore, in both cases we have $y \not\leqslant_T \bar{x}$. As y was arbitrary, $\widehat{Q}_j \not\leqslant_w \widehat{Q}(A)$. But $\widehat{Q}_j \leqslant_w \widehat{Q}(B)$ via the map $x \mapsto \left(x\right)_j$ so it must be that $\widehat{Q}(B) \not\leqslant_w \widehat{Q}(A)$ and therefore that $\widehat{Q}(B) \not\equiv_w \widehat{Q}(A)$, as required. □

Lemma 4.4. *If A and B are non-empty and recursive, then,*

$$\widehat{Q}(A \cup B) \equiv_M \widehat{Q}(A) \vee \widehat{Q}(B).$$

Proof.

$$\begin{aligned}\widehat{Q}(A\cup B) &= \{x : \forall i \in A\cup B,\ \left(x\right)_i \in \widehat{Q}_i\},\\ &= \{x : \forall i \in A,\ \left(x\right)_i \in \widehat{Q}_i\} \cap \{x : \forall i \in B,\ \left(x\right)_i \in \widehat{Q}_i\},\\ &= \widehat{Q}(A)\cap\widehat{Q}(B).\end{aligned}$$

So, $x \mapsto x\oplus x$, is a map from $\widehat{Q}(A\cup B)$ to $\widehat{Q}(A)\vee\widehat{Q}(B)$, and therefore, $\widehat{Q}(A\cup B) \geqslant_M \widehat{Q}(A)\vee\widehat{Q}(B)$. Conversely, let $x\oplus y \in \widehat{Q}(A)\vee\widehat{Q}(B)$. Define, $z \in 2^\omega$ by,

$$\left(z\right)_i = \begin{cases}\left(x\right)_i & \text{if } i \in A\\ \left(y\right)_i & \text{if } i \in \omega \setminus A.\end{cases}$$

Then $z \leqslant_T x\oplus y$ and for all $i \in A\cup B$, $\left(z\right)_i \in \widehat{Q}_i$, so $z \in \widehat{Q}(A\cup B)$. Therefore, $\widehat{Q}(A\cup B) \leqslant_M \widehat{Q}(A)\vee\widehat{Q}(B)$ as required. $\square$

Lemma 4.5. *If A and B are recursive and $A\cap B \neq \emptyset$, then,*

$$\widehat{Q}(A\cap B) \equiv_w \widehat{Q}(A)\wedge\widehat{Q}(B).$$

Proof. First, $\widehat{Q}(A\cap B) \leqslant_w \widehat{Q}(A)\wedge\widehat{Q}(B)$ (in fact, $\leqslant_M$). If $x \in \widehat{Q}(A)\wedge\widehat{Q}(B)$, then define $z \in \widehat{Q}(A\cap B)$ by,

$$\left(z\right)_i = \left(x^-\right)_i \text{ for all } i \in \omega.$$

If $\left(x\right)_i(0) = 0$, then, for all $i \in A$, $\left(z\right)_i \in \widehat{Q}_i$, and, a fortiori, for all $i \in A\cap B$, $\left(z\right)_i \in \widehat{Q}_i$. So $z \in \widehat{Q}(A\cap B)$. There is a similar argument if $\left(x\right)_i(0) = 1$.

Next, $\widehat{Q}(A\cap B) \geqslant_w \widehat{Q}(A)\wedge\widehat{Q}(B)$. Modulo the following two claims, the argument will be:

$$\begin{aligned}\widehat{Q}(A\cap B) &= \textstyle\bigvee_{i\in A\cap B}\widehat{Q}_i,\\ &\geqslant_w \textstyle\bigvee_{i\in A}\bigvee_{j\in B}\widehat{Q}_i\wedge\widehat{Q}_j \quad \text{(in fact, } \geqslant_M\text{) } \textit{Claim 1},\\ &\geqslant_w \textstyle\bigvee_{i\in A}\widehat{Q}_i\wedge\bigvee_{j\in B}\widehat{Q}_j \quad \textit{Claim 2},\\ &= \widehat{Q}(A)\wedge\widehat{Q}(B).\end{aligned}$$

Proving the *Claims* :

Claim 1. Let $x \in \bigvee_{i \in A \cap B} \widehat{Q}_i$ and take any $k \in A \cap B$. So $\left(x\right)_k \in \widehat{Q}_k$. We define (recursively in x) $z \in \bigvee_{i \in A} \bigvee_{j \in B} \widehat{Q}_i \wedge \widehat{Q}_j$ by defining $\left(\left(z\right)_i\right)_j$ for all $i, j \in \omega$, such that,

$$\left(\left(z\right)_i\right)_j \in \widehat{Q}_i \wedge \widehat{Q}_j \text{ for all } i \in A \text{ and } j \in B.$$

To this end, let,

$$\left(\left(z\right)_i\right)_j = \begin{cases} \langle 0 \rangle ^\frown \left(x\right)_i & \text{if } i = j, \\ \langle 0 \rangle ^\frown \left(x\right)_k & \text{if } i \neq j \text{ and } \left(x\right)_k \not\supseteq \sigma_i, \\ \langle 1 \rangle ^\frown \left(x\right)_k & \text{if } i \neq j \text{ and } \left(x\right)_k \supseteq \sigma_i. \end{cases}$$

So, suppose that $i \in A$ and $j \in B$. If $i = j$, then $i \in A \cap B$ and $\left(\left(z\right)_i\right)_j = \langle 0 \rangle ^\frown \left(x\right)_i \in \widehat{Q}_i \wedge \widehat{Q}_j$. If $i \neq j$ and $\left(x\right)_k \not\supseteq \sigma_i$, then $\left(x\right)_k \in \widehat{Q}_i$, and $\left(\left(z\right)_i\right)_j = \langle 0 \rangle ^\frown \left(x\right)_k \in \widehat{Q}_i \wedge \widehat{Q}_j$. If $i \neq j$ and $\left(x\right)_k \supseteq \sigma_i$, then $\left(x\right)_k \in \widehat{Q}_j$ and $\left(\left(z\right)_i\right)_j = \langle 1 \rangle ^\frown \left(x\right)_k \in \widehat{Q}_i \wedge \widehat{Q}_j$. These three cases are exhaustive and so *Claim 1* is established. Note that the above is a uniform procedure for computing z from an arbitrary x, and so the stronger, Medvedev reducibility has been shown.

Claim 2. Let $x \in \bigvee_{i \in A} \bigvee_{j \in B} \widehat{Q}_i \wedge \widehat{Q}_j$. We will construct $z \leqslant_T x$ such that $z \in \bigvee_{i \in A} \widehat{Q}_i \wedge \bigvee_{j \in B} \widehat{Q}_j$. There are two cases.

Case 1. $\exists i \in A \smallsetminus B \ \forall j \in B \smallsetminus A \ \ \left(\left(x\right)_i\right)_j(0) = 1$.
Fix such an i, set $z(0) = 1$ and let,

$$\left(z^-\right)_k = \begin{cases} \left(\left(x\right)_i\right)_k^- & \text{if } k \notin A \cap B, \\ \left(\left(x\right)_k\right)_k^- & \text{if } k \in A \cap B. \end{cases}$$

Then, if $k \in B \smallsetminus A$, $\left(z^-\right)_k = \left(\left(x\right)_i\right)_k^- \in \widehat{Q}_k$ and if $k \in B \cap A$, $\left(z^-\right)_k = \left(\left(x\right)_k\right)_k^- \in \widehat{Q}_k$. So, for all $k \in B$, $\left(z^-\right)_k \in \widehat{Q}_k$, giving $z^- \in \bigvee_{j \in B} \widehat{Q}_j$ and $z \in \bigvee_{i \in A} \widehat{Q}_i \wedge \bigvee_{j \in B} \widehat{Q}_j$.

Case 2. $\forall i \in A \smallsetminus B \ \exists j \in B \smallsetminus A \ \ \left(\left(x\right)_i\right)_j(0) = 0$.
Let $z(0) = 0$ and define,

$$f(i) = \begin{cases} \text{the least such } j & \text{if } i \in A \smallsetminus B, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \leqslant_T x$, and $\big((x)_i\big)^-_{f(i)} \in \widehat{Q}_i$ for all $i \in A \smallsetminus B$. We can then define,

$$\big(z^-\big)_k = \begin{cases} \big((x)_k\big)^-_{f(k)} & \text{if } k \notin A \cap B, \\ \big((x)_k\big)^-_k & \text{if } k \in A \cap B. \end{cases}$$

As above we have $\big(z^-\big)_k \in \widehat{Q}_k$, if $k \in A \cap B$ and if $k \in A \smallsetminus B$ then $\big(z^-\big)_k = \big((x)_k\big)^-_{f(k)} \in \widehat{Q}_k$. So $z^- \in \bigvee_{i \in A} \widehat{Q}_i$, and $z \in \bigvee_{i \in A} \widehat{Q}_i \wedge \bigvee_{j \in B} \widehat{Q}_j$, as required.

□

We would like to improve Lemma 4.5 by showing that $\widehat{Q}(A \cap B) \equiv_M \widehat{Q}(A) \wedge \widehat{Q}(B)$, but the division into cases in the proof of *Claim 2* is non-effective and we have only been able to show the weaker result. However, we can improve the result under the stricter conditions of the following lemma.

Lemma 4.6. *If the symmetric difference of two recursive sets,*

$$A \triangle B = (A \smallsetminus B) \cup (B \smallsetminus A),$$

is finite, then,

$$\widehat{Q}(A \cap B) \equiv_M \widehat{Q}(A) \wedge \widehat{Q}(B).$$

Proof. The proof is identical with the proof of 4.5 noting that in the proof of Lemma *Claim 2* the division into two cases is now effective as both $A \smallsetminus B$ and $B \smallsetminus A$ are finite. □

We are now in a position to prove the theorem in the title of the section.

Theorem 4.7. *The free Boolean algebra on countably many generators, $FB(\omega)$, is lattice-embeddable into $\mathcal{P}_w$.*

Proof. Consider the mapping $A \mapsto \widehat{Q}(A)$. Lemmas 4.3, 4.4 and 4.5 prove that this is an embedding of the lattice of non-empty, recursive subsets of ω under $\cap$ and $\cup$ into $\mathcal{P}_w$. So to prove the theorem it is sufficient to show that $FB(\omega)$ can be represented by a collection of non-empty, recursive subsets of ω.

Let p_j be the j^{th} prime number and let $B_j = \{np_j : n \in \omega\}$. Define $\widetilde{B_j} = (\omega \smallsetminus B_j) \cup \{0\}$. The set $\{B_j : j \in \omega\}$ generates a distributive lattice under operations of intersection and union. Further, this lattice can be extended to a Boolean algebra with **1** represented by ω, **0** represented by $\{0\}$ and $\widetilde{B_j}$ the Boolean complement of B_j. It would, perhaps, seem more natural to have $\emptyset$ as the minimum element and $\omega \setminus B_j$ as the Boolean complement, however the text definition ensures that each element of the Boolean algebra is non-empty. This Boolean algebra is in fact free and therefore a representation of $FB(\omega)$. To show this it is sufficient to show (Exercise II.3.43 [**?**]) that for all finite $X, Y \subseteq \omega$,

$$\bigcap_{i \in X} B_i \subseteq \bigcup_{j \in Y} B_j \Rightarrow X \cap Y \neq \emptyset.$$

But this is easily seen as $\prod_{i \in X} p_i \in \bigcap_{i \in X} B_i$ and so, if the antecedent holds, $\prod_{i \in X} p_i \in B_j$ for some $j \in Y$. By primality, this means $p_j = p_i$ for some $i \in X$, giving $X \cap Y \neq \emptyset$. □

Corollary 4.8. *$FB(\omega)$ can be embedded into $\mathcal{P}_w$ below any given special Π^0_1 set, P.*

Proof. Let such a P be given and let $\langle Q_i : i \in \omega \rangle$ be as in Theorem 2.7. The required embedding will be,

$$A \mapsto P \wedge \widehat{Q}(A).$$

The fact that this is a homomorphism follows from the lattice theoretic identities:

$$(P \wedge \bigwedge_{i \in A} \widehat{Q}_i) \wedge (P \wedge \bigwedge_{i \in B} \widehat{Q}_i) = P \wedge (\bigwedge_{i \in A} \widehat{Q}_i \wedge \bigwedge_{i \in B} \widehat{Q}_i),$$

$$(P \wedge \bigwedge_{i \in A} \widehat{Q}_i) \vee (P \wedge \bigwedge_{i \in B} \widehat{Q}_i) = P \wedge (\bigwedge_{i \in A} \widehat{Q}_i \vee \bigwedge_{i \in B} \widehat{Q}_i),$$

and the fact that $A \mapsto \widehat{Q}(A)$ describes a lattice homomorphism. To see that it's an embedding, suppose that $A \neq B$ and take $j \in B \setminus A$, $x \in Q_j$ and $\bar{x} \in \widehat{Q}(A)$ as in the proof of Lemma 4.3. Let $\bar{x}_1 = \langle 1 \rangle^\frown \bar{x} \in P \wedge \widehat{Q}(A)$. Suppose that there is a $y \in P \wedge \widehat{Q}(B)$ such that $y \leqslant_T \bar{x}_1$. By the proof of Lemma 4.3 we know that $y^- \notin \widehat{Q}(B)$ (or else $\bar{x} \equiv_T \bar{x}_1 \geqslant_T y \equiv_T y^- \in \widehat{Q}(B)$, contradiction). But, if $y^- \in P$, then,

$$P \ni y^- \leqslant_T \bar{x}_1 \equiv_T x \in Q_j,$$

contrary to the construction of $\langle Q_i : i \in \omega \rangle$. So there is no $y \in P \wedge \widehat{Q}(B)$, such that $y \leqslant_T \bar{x}_1$. Therefore, $P \wedge \widehat{Q}(B) \nleqslant_M P \wedge \widehat{Q}(A)$, as required. □

Theorem 4.9. *Every countable distributive lattice can be embedded into $\mathcal{P}_w$ below any given special Π^0_1 set.*

We show that every countable distributive lattice embeds into $FB(\omega)$ and then apply Theorem 4.7. All the lattice theoretical background can be found in [**?**] or [**?**]. Every countable distributive lattice can be embedded into a countable Boolean algebra so it is sufficient to show that every countable Boolean algebra can be embedded into $FB(\omega)$.

It is most convenient here to work with the dual space of $FB(\omega)$. Stone duality gives a contravariant functor from the category of closed subspaces of 2^ω and continuous maps to the category of Boolean Algebras and Boolean homomorphisms. Such a functor will take 2^ω to $FB(\omega)$ and continuous surjections to Boolean injections. So it is enough (in fact equivalent) to prove the following theorem (attributed to Sierpiński in [**?**] page 46):

Theorem 4.10. *For every closed subset, P, of 2^ω, there exists a continuous surjection,*

$$\psi : 2^\omega \longrightarrow P.$$

Proof. Recall that $\mathrm{Ext}(P) = \{\sigma \in 2^{<\omega} : \exists f \in T\ f \supset \sigma\}$. We will define a continuous surjection, $\phi : 2^{<\omega} \longrightarrow \mathrm{Ext}(P)$, which will then induce the required map on 2^ω. Let

$$\phi(\langle\rangle) = \langle\rangle,$$

$$\phi(\sigma^\frown\langle i\rangle) = \begin{cases} \phi(\sigma)^\frown\langle i\rangle & \text{if } \phi(\sigma)^\frown\langle i\rangle \in \mathrm{Ext}(P) \\ \phi(\sigma)^\frown\langle 1-i\rangle & \text{otherwise.} \end{cases}$$

It is straightforward to see that this is a continuous surjection. It is in fact a retract ([**?**] page 46) of 2^ω. $\square$

The next theorem is result 3 of page 2.

Theorem 4.11. *Let $\mathcal{L}_1$ ($\mathcal{L}_2$) be the lattice of finite (co-finite) subsets of ω under $\cap$ and $\cup$. Then, for any special Π^0_1 set, P, there is an embedding of $\mathcal{L}_1 \times \mathcal{L}_2$ into $\mathcal{P}_M$ below P.*

Proof. Let E be any infinite, co-infinite recursive subset of ω (for example the even numbers). Let $\mathcal{K}$ be the distributive lattice $\{X \subseteq \omega : X \triangle E$ is finite $\}$ with the operations of $\cap$ and $\cup$. Then $\mathcal{K} \simeq \mathcal{L}_1 \times \mathcal{L}_2$ (represent $\mathcal{L}_1$ by finite subsets of odd numbers and $\mathcal{L}_2$ by (relatively) co-finite sets of even numbers and the isomorphism is witnessed by $(X, Y) \mapsto X \cup Y$). The symmetric difference of any two elements of $\mathcal{K}$ is finite so Lemmas 4.3, 4.4, 4.6 and the proof of Corollary 4.8 give the result. $\square$

Corollary 4.12. *$\mathcal{L}_1$ and $\mathcal{L}_2$ are embeddable in $\mathcal{P}_M$ below any special Π^0_1 set.*

Proof. Immediate, as $\mathcal{L}_1$ and $\mathcal{L}_2$ are sublattices of $\mathcal{K}$, above. $\square$