Completeness, Compactness, Effective Dimensions

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Abstract

We investigate a directed metric on the space of infinite binary sequences defined by

$$d(Y \to X) = \limsup_n \frac{C(X \upharpoonright n \mid Y \upharpoonright n)}{n},$$

where $C(X \upharpoonright n \mid Y \upharpoonright n)$ is the Kolmogorov complexity of $X \upharpoonright n$ given $Y \upharpoonright n$. In particular we focus on the topological aspects of the associated metric space - proving that it is complete though very far from being compact.

This is a continuation of earlier work by the author investigating other geometrical and toplogical aspects of this metric.

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1 Introduction

The results in this paper fit into the general framework of the study of degree structures in information and computability theory. This area of research involves in large part the comparison of infinite binary sequences (or sets of infinite binary sequences) in terms of their information content. There are various ways of conceiving of information content - computational strength and descriptive complexity being two - and often a general structure is defined on the class of all binary sequences once a particular method of comparison is determined to be of interest. Usually the method of comparison consists of a pre-order relation defined on this class (or the class of *sets* of binary sequences) as in the case of Turing reducibility, many-one reducibility (or Muchnik and Medvedev reducibility), and the resulting structure is a partial order defined on the set of equivalence classes induced by the pre-order (Turing degrees, many-one degrees, Muchnik degrees and so on).

Often these structures will have natural operations defined on them. For example the Turing degrees have the *jump* unary operator and a least upper bound binary operator - the latter turning the structure into an upper semi-lattice. The Muchnik degrees have both a least upper bound and a greatest lower bound under which the partial order becomes a distributive lattice. To a large extent these operations focus the research into the structures and help to develop an understanding of the underlying reducibility concepts.

In this paper we look at a different notion of comparison between infinite binary sequences (usually called *reals*). This comparison tries to capture an intuitive notion of information density rather than computational strength. The resulting degree structure exhibits a significantly more algebraic and geometrical nature than the Turing degrees. Indeed the resulting structure is a metric space and a natural scalar multiplication function can be defined that allows geometric notions such as angle and projection to be expressed. The fundamental metric definition is based on the Kolmogorov complexity of finite strings. If σ is a finite binary string, then $C(\sigma)$ denotes the plain[1] Kolmogorov complexity of σ. If A and B are infinite binary sequences, then the *distance from A towards B* is defined to be

$$d(A \to B) = \limsup_n \frac{C(B\restriction n \mid A\restriction n)}{n}.$$

We justify referring to this function as a *distance* by noting that it obeys the triangle inequality in the direction of the arrow (see [1]) and that for all A, $d(A \to A) = 0$. We will refer to the function $d(\cdot \to \cdot)$ as a *directed metric*. It fails the stricter definition of a metric because it is not symmetric and distinct sequences may be distance zero from one another. We can however remedy these problems rather easily by defining the *distance between A and B* to be

$$d(A, B) = \max\{d(A \to B), d(B \to A)\},$$

and by identifying reals that are distance zero from one another (writing $A \simeq_d B$ if $d(A, B) = 0$). $d(\cdot, \cdot)$ is then a metric on these $\simeq_d$-equivalence classes.

In our context these $\simeq_d$-equivalence classes play the role of the Turing degrees in pure computability theory. Indeed from this directed metric one can also describe a natural partial order on the structure. Namely

$$A \geqslant_d B \text{ if and only if } d(A \to B) = 0.$$

However there is more information to be obtained in our structure as, if $A \geqslant_d B$, we can also quantify the *extent* to which A is above B - namely $d(B \to A)$. Only minimal work has been done investigating the properties of this partial order.

[1]Although for most of the paper it is irrelevant whether we use plain complexity $C(\sigma)$ or prefix-free complexity $K(\sigma)$.

The definition of this directed metric is a natural relativisation of the definition[2] of the *effective packing dimension* of a real A:

$$\dim_p A = \limsup_n \frac{C(A \upharpoonright n)}{n}.$$

There is of course another natural relativisation of packing dimension:

$$\limsup_n \frac{C^Y(X \upharpoonright n)}{n}$$

- one that has been studied in [**?**] - and we mention here a few of the relevant differences. This second notion of relativistion leads to a very different structure to the one studied here. One difference that makes this so is the fact that if $Z \simeq_T Y$, then

$$\limsup_n \frac{C^Y(X \upharpoonright n)}{n} = \limsup_n \frac{C^Z(X \upharpoonright n)}{n}.$$

Thus if the information in Y is diluted (see definition ...) the relativised dimension remains the same. This not the case in the text definition as if Z represents a diluted form of Y, then in general

$$d(Z \to X) \neq d(Y \to X).$$

Furthermore, with the text definition we have a notion of *projection* from one real onto another. - defining the projection of X onto Y to be $\alpha \in [0,1]$ if the distance from X to a dilution of factor α of Y is 0, and if this is the case for no lesser dilution.

Indeed, if $\mathbf{0}$ denotes the infinite sequence of 0s (or in fact any computable sequence), then

$$\dim_p A = d(\mathbf{0} \to A).$$

The dual notion - the effective Hausdorff dimension - is also of interest and has been studied even more extensively than packing dimension in the literature (see [9] for an introduction). The effective Hausdorff dimension of a real A can be defined as

$$\dim_H A = \liminf_n \frac{C(A \upharpoonright n)}{n}$$

(but also see footnote 2). Some results in the area are restricted to the so-called *regular* reals. These are reals whose effective Hausdorff and packing dimensions are equal. That is, reals A for which the effective dimension

$$\dim A = \lim_n \frac{C(A \upharpoonright n)}{n}$$

[2] This definition of effective packing dimension differs from the usual one in the literature - which is more clearly an effectivisation of the classical definition. The equivalence of the text definition is a fundamental result in the area first proved by Lutz [5] continuing work by Mayordomo [6]. See also [7] for the development of the subject.

exists.

Along with the metric structure we also have a basic algebraic operation - a *dilution*. If A is a real and $\alpha \in [0, 1]$, then αA is the infinite binary sequence that is formed by interspersing padding 0s into the bits of A. α represents the proportion of bits of A to the bits of αA. A precise definition is given in Section 3. We refer to the operation $\alpha \mapsto \alpha A$ as a *scalar multiplication* and it is associative - that is $\alpha(\beta A) \simeq_d (\alpha\beta)A$. This operation dilutes the information in A in the sense that $\dim_p(\alpha A) = \alpha \dim_p(A)$. What is true more generally however is that

$$d(\alpha X \to \alpha Y) = \alpha d(X \to Y).$$

The dilution operation allows us to define geometric notions such as angles and projections in the metric space similar to the way that a scalar operation and a norm can define a geometry in $\mathbb{R}^n$ - see [1]. It also allows us to ask questions about compressibility of information. In [2] it is shown that every regular real can be maximally compressed. That is, for every regular real X of effective dimension α, there is a real Y of effective dimension 1 such that $X \simeq_d \alpha Y$. The assumption of regularity here is necessary, as any real with Hausdorff dimension 1 is regular and any dilution of a regular real is also regular.

In [2] it was also shown that the metric space is path-connected. That is any real can be continuously deformed along a path into any other real. Or more precisely, given any two reals A and B there is a continuous map π from $[0, 1]$ into the metric space such that $\pi(0) = A$ and $\pi(1) = B$.

In this paper we continue to concentrate on the topological aspects of the structure. The main results are that the metric space is complete - every Cauchy sequence of reals converges - and that the metric space is far from being compact. In fact no neighbourhood (set with nonempty interior) is compact.

1.1 Notation

Most notation is standard. The set of all countable binary sequences (reals) is denoted $2^{\mathbb{N}}$, and the set of all finite binary strings is denoted $2^{<\mathbb{N}}$. Reals are usually represented by uppercase roman letters and strings by lower case Greek letters. If σ and τ are strings, then $\sigma\tau$ represents the concatenation of σ and τ. Where more clarity is needed we write $\sigma^\frown\tau$. $\sigma 0$ is short for $\sigma^\frown\langle 0\rangle$. If the string σ extends the string τ we write $\sigma \supseteq \tau$. And if $A \in 2^{\mathbb{N}}$ extends σ we write $A \supset \sigma$. A string consisting of n 0s is denoted 0^n. If $A \in 2^{\mathbb{N}}$, then

$$A = A(0)A(1)A(2)\dots,$$

$$A\restriction n = A(0)A(1)A(2)\dots A(n-1),$$

and

$$A[m, n] = A(m)A(m + 1)A(m + 2)\dots A(n - 1).$$

We will follow [3] and [8] in the notation with respect to Kolmogorov complexity. $C(\sigma)$ is the plain Kolmogorov complexity of the string σ and $K(\sigma)$ is its prefix-free complexity. Other notation will be defined in the relevant section.

2 Completeness

Let $\langle X_i \rangle_{i=0}^{\infty}$ be a Cauchy sequence of reals. We will construct a real X such that

$$\forall \epsilon > 0 \exists N \forall n \geqslant N d(X_n, X) < \epsilon.$$

We do this by constructing a strictly increasing sequence of positive natural numbers $\langle m_n \rangle$ and defining the real X to be

$$X_0[0, m_0]^\frown X_1[m_0, m_1]^\frown \ldots ^\frown X_n[m_{n-1}, m_n]^\frown \ldots$$

We describe m_n by recursion, letting

1. $m_0 = 1$
2. $m_{n+1} =$ the least $k > (n+1) \sum_{i=0}^{n} m_i$ such that for all $j \leqslant n+2$ and $l \geqslant k$

$$\frac{C(X_{n+2} \restriction l \mid X_j \restriction l)}{l} < d(X_j \to X_{n+2}) + 2^{-n}. \tag{1}$$

Such a k exists as

$$\limsup_{s} \frac{C(X_{n+2} \restriction s \mid X_j \restriction s)}{s} = d(X_j \to X_{n+2}).$$

For any integer $N > 0$ we will bound $d(X_N \to X)$ in terms of N. As $\langle X_i \rangle$ is a Cauchy sequence, there is a function $d : \mathbb{N} \to \mathbb{R}^+$ such that $\lim_N d(N) = 0$ and such that for all N and all $m, n \geqslant N$ $d(X_n \to X_m) < d(N)$. We will write d_N for $d(N)$.

Fix $N \in \mathbb{N}$. We will first find an upper bound on $d(X_N \to X)$ that vanishes as $N \to \infty$, and then do the same for $d(X \to X_N)$. We calculate

$$\limsup_{s \to \infty} \frac{C(X \restriction s \mid X_N \restriction s)}{s}.$$

For large enough s (namely $s \geqslant m_N$), $X \restriction s$ can be written,

$$X \restriction m_N{}^\frown X_{N+1}[m_N, m_{N+1}]^\frown X_{N+2}[m_{N+1}, m_{N+2}]^\frown \ldots ^\frown X_{N+k}[m_{N+k-1}, m_{N+k}]^\frown \tau$$

for some unique largest $k \geqslant 0$ and $\tau \subsetneq X_{N+k+1}[m_{N+k}, m_{N+k+1}]$. To describe $X \restriction s$ given $X_N \restriction s$, we first describe $X \restriction m_N$ (using m_N bits); then we describe τ and

$X_{N+i}[m_{N+i-1}, m_{N+i}]$ for each positive $i \leqslant k$ by describing all the values m_{N+i} for $1 \leqslant i \leqslant k$ (using in the order of $\sum_{i=1}^{k} \log m_{N+i}$ bits) along with $X_{N+k+1} \upharpoonright s$ and $X_{N+i} \upharpoonright m_{N+i}$ for each $1 \leqslant i \leqslant k$. For these last strings we use $X_N \upharpoonright s$ and $X_N \upharpoonright m_{N+i}$ for each $1 \leqslant i \leqslant k$ respectively, and then leverage the fact that $d(X_N \to X_{N+j}) < d_N$ for all $j \geqslant 0$. A multiplicative factor of 2 will be used to distinguish the various descriptions, and an additive constant independent of N and k is appended. Thus

$$\begin{aligned}
C(X \upharpoonright s \mid X_N \upharpoonright s) &\leqslant 2\big[m_N + \sum_{i=1}^{k} \log m_{N+i} + \sum_{i=1}^{k} C(X_{N+i} \upharpoonright m_{N+i} \mid X_N \upharpoonright m_{N+i}) \\
&\qquad + C(X_{N+k+1} \upharpoonright s \mid X_N \upharpoonright s)\big] + \mathcal{O}(1) \\
&\leqslant 2\big[\log m_{N+k} + \sum_{i=0}^{k-1} m_{N+i} + \sum_{i=1}^{k} C(X_{N+i} \upharpoonright m_{N+i} \mid X_N \upharpoonright m_{N+i}) \\
&\qquad + C(X_{N+k+1} \upharpoonright s \mid X_N \upharpoonright s)\big] + \mathcal{O}(1) \\
&\leqslant 2\big[\log m_{N+k} + \sum_{i=0}^{N+k-1} m_i + \sum_{i=1}^{k} C(X_{N+i} \upharpoonright m_{N+i} \mid X_N \upharpoonright m_{N+i}) \\
&\qquad + C(X_{N+k+1} \upharpoonright s \mid X_N \upharpoonright s)\big] + \mathcal{O}(1)
\end{aligned}$$

Now using the fact that $d(X_N \to X_{N+j}) \leqslant d_N$ for all $j \geqslant 0$ and the definition of m_{n+1}, we have

$$C(X_{N+i} \upharpoonright m_{N+i} \mid X_N \upharpoonright m_{N+i}) < m_{N+i}(d_N + 2^{-N})$$

(in (1) above taking $n = N + i - 2$, $l = m_{N+i} > m_{N+i-1} = m_{n+1}$, and $j = N \leqslant n + 2$) and

$$C(X_{N+k+1} \upharpoonright s \mid X_N \upharpoonright s) < s(d_N + 2^{-N}),$$

(taking $n = N + k - 1$ in (1) etc.). Also by the definition of m_{n+1}, $\sum_{i=0}^{N+k-1} m_i < \frac{m_{N+k}}{N+k}$. Therefore

$$\begin{aligned}
C(X \upharpoonright s \mid X_N \upharpoonright s) &\leqslant 2\Big[\log m_{N+k} + \frac{m_{N+k}}{N+k} + (d_N + 2^{-N})(s + \sum_{i=1}^{k} m_{N+i})\Big] + \mathcal{O}(1) \\
&\leqslant 2\Big[\log m_{N+k} + \frac{m_{N+k}}{N+k} + (d_N + 2^{-N})(s + \sum_{i=0}^{N+k} m_i)\Big] + \mathcal{O}(1) \\
&\leqslant 2\Big[\log m_{N+k} + \frac{m_{N+k}}{N+k} + (d_N + 2^{-N})(s + m_{N+k}) \\
&\qquad + (d_N + 2^{-N}) \sum_{i=0}^{N+k-1} m_i\Big] + \mathcal{O}(1) \\
&\leqslant 2\Big[\log m_{N+k} + \frac{m_{N+k}}{N+k} + 2s(d_N + 2^{-N}) + (d_N + 2^{-N})\frac{m_{N+k}}{N+k}\Big] + \mathcal{O}(1).
\end{aligned}$$

Now dividing by s and using the fact that $s \geqslant m_{N+k}$:

$$\begin{aligned}\frac{C(X\restriction s \mid X_N\restriction s)}{s} &\leqslant 2\Big[\frac{\log m_{N+k}}{s} + \frac{m_{N+k}}{s(N+k)} + 2(d_N + 2^{-N}) \\ &\qquad + (d_N + 2^{-N})\frac{m_{N+k}}{s(N+k)}\Big] + \frac{\mathcal{O}(1)}{s} \\ &\leqslant 2\Big[\frac{\log m_{N+k}}{m_{N+k}} + \frac{1}{N+k} + 2(d_N + 2^{-N}) + \frac{d_N + 2^{-N}}{N+k}\Big] + \frac{\mathcal{O}(1)}{s}.\end{aligned}$$

As s goes to infinity, so does k, so

$$\begin{aligned}d(X_N \to X) &= \limsup_s \frac{C(X\restriction s \mid X_N\restriction s)}{s} \\ &\leqslant \limsup_s 2\Big[\frac{\log m_{N+k}}{m_{N+k}} + \frac{1}{N+k} + 2(d_N + 2^{-N}) + \frac{d_N + 2^{-N}}{N+k}\Big] + \frac{\mathcal{O}(1)}{s} \\ &= 4(d_N + 2^{-N})\end{aligned}$$

And thus $d(X_N \to X) \to 0$ as $N \to \infty$.

To work out $d(X \to X_N)$, we first calculate $C(X_N\restriction s \mid X\restriction s)$. We claim that this is at most

$$2\big[\log m_{N+k} + C(X_{N+k+1}\restriction m_{N+k} \mid X\restriction m_{N+k})\big] + C(X_N\restriction s \mid X_{N+k+1}\restriction s) + \mathcal{O}(1).$$

To see this note that to describe $X_N\restriction s$ given $X\restriction s$, we first take a description of m_{N+k} ($\log m_{N+k}$ bits), then use m_{N+k} and $X\restriction s$ to get a description of $X\restriction m_{N+k}$ and τ. Using these and a description of $X_{N+k+1}\restriction m_{N+k}$ given $X\restriction m_{N+k}$, we can describe $X_{N+k+1}\restriction s$ by prepending $X_{N+k+1}\restriction m_{N+k}$ to τ. Finally, using a description of $X_N\restriction s$ given $X_{N+k+1}\restriction s$, we describe $X_N\restriction s$. We distinguish the three descriptions from each other by doubling the lengths of two of them in the standard way.

Dividing by s and taking the limit supremum gives that $d(X \to X_N)$ is at most

$$\limsup_s \frac{2\big[\log m_{N+k} + C(X_{N+k+1}\restriction m_{N+k} \mid X\restriction m_{N+k})\big] + C(X_N\restriction s \mid X_{N+k+1}\restriction s)}{s}$$

But the $\frac{\log m_{N+k}}{s}$ term will vanish as $s \to \infty$ (as $s \geqslant m_{N+k}$), and

$$\limsup_s \frac{C(X_N\restriction s \mid X_{N+k+1}\restriction s)}{s} = d(X_{N+k+1} \to X_N) \leqslant d_N.$$

So $d(X \to X_N)$ is at most

$$\limsup_s \frac{2C(X_{N+k+1}\restriction m_{N+k} \mid X\restriction m_{N+k})}{s} + d_N. \tag{2}$$

We will establish the result by finding a bound on $C(X_{N+k+1}\restriction m_{N+k} \mid X\restriction m_{N+k})$ with the next lemma.

Lemma 2.1. *$C(X_{N+k+1}\restriction m_{N+k} \mid X\restriction m_{N+k})$ is at most*

$$2\sum_{i=0}^{k-1} \log m_{N+i} + 2(d_N + 2^{-N}) \sum_{i=0}^{k} m_{N+i} + \mathcal{O}(k).$$

Proof. We prove this by induction. Let the constants A and B be such that for a given k

$$C(X_{N+k+1}\restriction m_{N+k} \mid X\restriction m_{N+k}) \leqslant 2\sum_{i=0}^{k-1} \log m_{N+i} + 2(d_N + 2^{-N}) \sum_{i=0}^{k} m_{N+i} + Ak + B.$$

A and B can be chosen so that the inequality also holds for the base case when $k = 1$. We now establish the inequality for $k + 1$.

To describe $X_{N+k+2}\restriction m_{N+k+1}$ given $X\restriction m_{N+k+1}$, we first describe m_{N+k} ($\log m_{N+k}$ bits) and use this to partition $X\restriction m_{N+k+1}$ to get $X\restriction m_{N+k}$ and $X[m_{N+k}, m_{N+k+1}]$. We then take a description of $X_{N+k+1}\restriction m_{N+k}$ given $X\restriction m_{N+k}$ (the length of which will be bounded by the induction hypothesis) to get $X_{N+k+1}\restriction m_{N+k}$. Now, by the definition of X, $X[m_{N+k}, m_{N+k+1}] = X_{N+k+1}[m_{N+k}, m_{N+k+1}]$ so we can prepend $X_{N+k+1}\restriction m_{N+k}$ to $X[m_{N+k}, m_{N+k+1}]$ to get $X_{N+k+1}\restriction m_{N+k+1}$. Finally we can take a description of $X_{N+k+2}\restriction m_{N+k+1}$ given $X_{N+k+1}\restriction m_{N+k+1}$ (which will be short as $d(X_{N+k+1} \to X_{N+k+2})$ is small) to furnish a description of $X_{N+k+2}\restriction m_{N+k+1}$. Doubling the length of two of the descriptions and adding a constant C independent of k, A and B gives that $C(X_{N+k+2}\restriction m_{N+k+1} \mid X\restriction m_{N+k+1})$ is bounded above by

$$2\big[\log m_{N+k} + C(X_{N+k+2}\restriction m_{N+k+1} \mid X_{N+k+1}\restriction m_{N+k+1})\big] \\ + C(X_{N+k+1}\restriction m_{N+k} \mid X\restriction m_{N+k}) + C.$$

This in turn is less than or equal to

$$2\big[\log m_{N+k} + (d_N + 2^{-N}) m_{N+k+1}\big] + C(X_{N+k+1}\restriction m_{N+k} \mid X\restriction m_{N+k}) + C.$$

By the induction hypothesis, this gives:

$$
\begin{aligned}
&C(X_{N+k+2}\restriction m_{N+k+1} \mid X\restriction m_{N+k+1}) \\
&\leqslant 2\big[\log m_{N+k} + (d_N + 2^{-N})m_{N+k+1}\big] + 2\sum_{i=0}^{k-1}\log m_{N+i} \\
&\qquad\qquad + 2(d_N + 2^{-N})\sum_{i=0}^{k} m_{N+i} + Ak + B + C \\
&\leqslant 2\sum_{i=0}^{k}\log m_{N+i} + 2(d_N + 2^{-N})\sum_{i=0}^{k+1} m_{N+i} + Ak + B + C.
\end{aligned}
$$

As C is independent of A, we can choose $A \geqslant C$, thus

$$
C(X_{N+k+2}\restriction m_{N+k+1} \mid X\restriction m_{N+k+1}) \leqslant 2\sum_{i=0}^{k}\log m_{N+i} + 2(d_N + 2^{-N})\sum_{i=0}^{k+1} m_{N+i} + A(k+1) + B,
$$

as required. □

Finally using the lemma, the fact that $k \to \infty$ as $s \to \infty$, and the fact that $s \geqslant m_{N+k} \geqslant m_k \geqslant \mathcal{O}(k^2)$, gives:

$$
\begin{aligned}
&d(X \to X_N) \\
&\leqslant \limsup_s \frac{2C(X_{N+k+1}\restriction m_{N+k} \mid X\restriction m_{N+k})}{s} + d_N \qquad \text{from (2)} \\
&\leqslant \limsup_s \frac{4\sum_{i=0}^{k-1}\log m_{N+i} + 4(d_N + 2^{-N})\sum_{i=0}^{k} m_{N+i} + \mathcal{O}(k)}{s} + d_N \\
&\leqslant \limsup_s \frac{4\sum_{i=0}^{N+k-1} m_i + 4(d_N + 2^{-N})\sum_{i=0}^{N+k} m_i}{s} + d_N \\
&\leqslant \limsup_s \frac{4m_{N+k}/(N+k) + 4(d_N + 2^{-N})\sum_{i=0}^{N+k-1} m_i + 4(d_N + 2^{-N})m_{N+k}}{s} + d_N \\
&\leqslant \limsup_s \frac{4m_{N+k}/(N+k) + 4(d_N + 2^{-N})m_{N+k}/(N+k) + 4(d_N + 2^{-N})m_{N+k}}{s} + d_N \\
&\leqslant \limsup_s \frac{4}{N+k} + \frac{4(d_N + 2^{-N})}{N+k} + 4(d_N + 2^{-N}) + d_N \\
&\leqslant 4(d_N + 2^{-N}) + d_N.
\end{aligned}
$$

And this approaches 0 as $N \to \infty$ as required.

3 Compactness

We now prove some results related to compactness under the d-metric. For a metric space, compactness and sequential compactness are equivalent. That is, a metric space is compact if and only if every sequence of points in the space has a convergent subsequence. It will be convenient for us to work with sequential compactness here.

Although it is generally a relatively simple thing in an unbounded metric space to create a sequence with no convergent subsequence, the metric space under consideration is not unbounded - for every $X, Y \in 2^{\mathbb{N}}$, $d(X, Y) \leqslant 1$. Furthermore, an unbounded metric space may still have compact neighbourhoods. For example in $\mathbb{R}^n$ every point is contained in a compact neighbourhood, that is it is *locally compact*. For a metric space local compactness is equivalent to every point's being contained in the interior of some compact (and hence closed) ball.

The space $2^{\mathbb{N}}$ under the d-metric is very far from being compact or indeed locally compact. We show in fact that *no* closed ball is compact and hence that $2^{\mathbb{N}}$ contains no compact neighbourhoods. This is essentially due to the existence of an infinite set of pairwise relatively random reals.

We will need in the proof two constructions that were introduced in [1]. The first is the *dilution* of a real. This takes a real X and intersperses padding bits (0s) into it in order to decrease its dimension (Hausdorff and packing) by a given factor.

As standard, if $x \in \mathbb{R}$, then $[x]$ is the least nearest integer to x.

Definition 3.1. If $X \in 2^{\mathbb{N}}$ and $\alpha \in [0, 1]$, then we define αX, the *dilution of X by factor α*, as follows:

$$\alpha X = \sigma_1 0^{i_1} \sigma_2 0^{i_2} \sigma_3 0^{i_3} \ldots \sigma_n 0^{i_n} \ldots$$

where

1. $\sigma_i \in 2^{<\mathbb{N}}$ for all $i \in \mathbb{N}^+$
2. $X = \sigma_1 \sigma_2 \sigma_3 \ldots$
3. $|\sigma_n 0^{i_n}| = n$
4. $|\sigma_n| = [\alpha n]$.

The second construction differs from the first only by using the bits of a second real Y instead of 0s.

Definition 3.2. If $X, Y \in 2^{\mathbb{N}}$ and $\alpha \in [0, 1]$, then we define $\alpha[XY]$ as follows:

$$\alpha[XY] = \sigma_1 \tau_1 \sigma_2 \tau_2 \sigma_3 \tau_3 \ldots \sigma_n \tau_n \ldots$$

where

1. $\sigma_i, \tau_i \in 2^{<\mathbb{N}}$ for all $i \in \mathbb{N}$

2. $X = \sigma_1\sigma_2\sigma_3\ldots$

3. $Y = \tau_1\tau_2\tau_3\ldots$

4. $|\sigma_n\tau_n| = n$

5. $|\sigma_n| = [\alpha n]$.

Lemma 3.3. *For all $Y, W \in 2^{\mathbb{N}}$,*

$$\limsup_n \frac{C^W(\alpha Y \restriction n)}{n} = \alpha \limsup_n \frac{C^W(Y \restriction n)}{n}.$$

Proof. The proof is a relativisation of Lemma 3.4 in [2]. □

Lemma 3.4. *If $Y, Z \in 2^{\mathbb{N}}$, then*

$$C(\alpha[YZ] \restriction n) = C(\alpha Y \restriction n, (1-\alpha)Z \restriction n) \pm \mathcal{O}(\sqrt{n}\log n).$$

Proof. Let m be the greatest integer such that $m(m+1)/2 \leqslant n$ and let

$$\alpha[YZ] \restriction n = \gamma_1\gamma_2\gamma_3\ldots\gamma_m\mu,$$

where $\gamma_i = \sigma_i\tau_i$ as in the definition of $\alpha[YZ]$. Then $|\mu| < m+1$ and $n = m(m+1)/2+|\mu|$. To describe $\alpha[YZ] \restriction n$, it is enough to describe a certain number of bits of Y, a certain number of bits of Z, the numbers $|\sigma_i|$ for positive $i \leqslant m$, and the string μ. To describe the numbers $|\sigma_i|$ it is sufficient to use $\mathcal{O}(m \log m)$ bits, as there are m numbers and each has value at most m. The length of the string μ is at most $m + 1$ so requires at most $\mathcal{O}(m)$ bits for its description.

The bits of Y required for $\alpha[YZ]$ are exactly the bits in $\alpha Y \restriction n$. The number of bits of Z in $\alpha[YZ] \restriction n$ may differ slightly from the number of bits of Z in $(1-\alpha)Z \restriction n$, but we bound this difference now. Let k be the number of bits of Z in $\alpha[YZ] \restriction n$ and l be the number of bits of Z in $(1-\alpha)Z \restriction n$. Then

$$\sum_{i=1}^{m} i - [\alpha i] \leqslant k \leqslant |\mu| + \sum_{i=1}^{m} i - [\alpha i].$$

But $\alpha i - 1/2 \leqslant [\alpha i] \leqslant \alpha i + 1/2$ and $|\mu| < m + 1$ so

$$(1-\alpha)m(m+1)/2 - m/2 \leqslant k \leqslant (1-\alpha)m(m+1)/2 + m/2 + m + 1.$$

And by similar reasoning

$$(1-\alpha)m(m+1)/2 - m/2 \leqslant l \leqslant (1-\alpha)m(m+1)/2 + m/2 + m + 1.$$

Thus both $k-l$ and $l-k$ are $\mathcal{O}(m)$. Therefore

$$C(\alpha[YZ]\restriction n) \leqslant C(\alpha Y\restriction n, (1-\alpha)Z\restriction n) + \mathcal{O}(m\log m).$$

As $m = \mathcal{O}(\sqrt{n})$, the result follows.

The other direction is very similar. Given a description for $\alpha[YZ]$, one only needs to know the values of all the $|\sigma_i|$ and any excess bits of Z (at most $\mathcal{O}(m)$ as above) to describe $\alpha Y\restriction n$ and $(1-\alpha)Z\restriction n$. Thus

$$C(\alpha Y\restriction n, (1-\alpha)Z\restriction n) \leqslant C(\alpha[YZ]) + \mathcal{O}(\sqrt{n}\log n).$$

□

This next Lemma is standard and will be used extensively.

Lemma 3.5 (Symmetry of Information - Kolmogorov, Levin)**.**

$$C(\sigma,\tau) = C(\sigma) + C(\tau|\sigma) \pm \mathcal{O}\log C(\sigma,\tau).$$

Consequently if $|\sigma| = |\tau| = n$*, then*

$$C(\sigma,\tau) = C(\sigma) + C(\tau|\sigma) \pm \mathcal{O}\log n.$$

Proof. See [3], or [4] for standard proofs and exposition. □

We will have use for the following relatively simple extension.

Lemma 3.6.

$$C(\sigma,\tau|\mu) = C(\sigma|\mu) + C(\tau|\sigma,\mu) \pm \mathcal{O}\log C(\sigma,\tau|\mu),$$

and if $|\sigma| = |\tau| = n$*, then*

$$C(\sigma,\tau|\mu) = C(\sigma|\mu) + C(\tau|\sigma,\mu) \pm \mathcal{O}\log n.$$

Proof. This is a mechanical adaptation of the proof of Lemma 3.5. □

Lemma 3.7. *For all* $W \in 2^{\mathbb{N}}$ *and* $\sigma \in 2^{<\mathbb{N}}$*,*

$$C^W(\sigma) \leqslant K^W(\sigma) + \mathcal{O}(1) \leqslant C^W(\sigma) + 2\log(|\sigma|) + \mathcal{O}(1).$$

Proof. This is straightforward relativisation of a standard result. For a proof see for example [8] Corollary 2.4.2. □

Lemma 3.8. *If* $Y \in 2^{\mathbb{N}}$ *and* $\alpha \in [0,1]$*, then*

$$C(Y\restriction n, \alpha Y\restriction n) = C(Y\restriction n) + \mathcal{O}(\sqrt{n}\log n).$$

Proof. To describe $\alpha Y \upharpoonright n$ given a description of $Y \upharpoonright n$, it is sufficient to be given a description of all the $|\sigma_i|$ from the definition of αY. This requires $\mathcal{O}(\sqrt{n}\log n)$ bits as in the proof of Lemma 3.4. Thus

$$\begin{aligned} C(Y\upharpoonright n, \alpha Y\upharpoonright n) &= C(\alpha Y\upharpoonright n \,\big|\, Y\upharpoonright n) + C(Y\upharpoonright n) \pm \mathcal{O}\log C(Y\upharpoonright n, \alpha Y\upharpoonright n) \\ &= \mathcal{O}(\sqrt{n}\log n) + C(Y\upharpoonright n) \pm \mathcal{O}(\log n) \\ &= C(Y\upharpoonright n) + \mathcal{O}(\sqrt{n}\log n). \end{aligned}$$

□

Lemma 3.9. *If $Y, W \in 2^{\mathbb{N}}$, Y is random relative to W, and $\alpha \in [0,1]$, then*

$$\limsup_n \frac{C^W(\alpha Y\upharpoonright n)}{n} = \alpha \limsup_n \frac{C(Y\upharpoonright n)}{n}.$$

Proof. Using Lemma 3.3 above,

$$\limsup_n \frac{C^W(\alpha Y \upharpoonright n)}{n} = \alpha \limsup_n \frac{C^W(Y \upharpoonright n)}{n} \leqslant \alpha \limsup_n \frac{C(Y \upharpoonright n)}{n},$$

and for the other direction:

$$\begin{aligned} \limsup_n \frac{C^W(\alpha Y\upharpoonright n)}{n} &= \alpha \limsup_n \frac{C^W(Y\upharpoonright n)}{n} && \text{Lemma 3.3} \\ &\geqslant \alpha \limsup_n \frac{K^W(Y\upharpoonright n) - 2\log n}{n} && \text{Lemma 3.7} \\ &= \alpha \limsup_n \frac{K^W(Y\upharpoonright n)}{n} \\ &= \alpha \limsup_n \frac{K(Y\upharpoonright n)}{n} && Y \text{ random relative to } W \\ &\geqslant \alpha \limsup_n \frac{C(Y\upharpoonright n)}{n} && \text{Lemma 3.7} \end{aligned}$$

□

The next lemma also uses relative randomness. It lacks a generality that was proved in [1]. What is more generally true is that the result holds for $Y, Z \in 2^{\mathbb{N}}$ if $\angle YZ = 1$ (see [1] for the definition).

Lemma 3.10. *If $Y, Z \in 2^{\mathbb{N}}$, Z random relative to Y, and $\alpha, \beta \in [0,1]$, then*

$$\limsup_n \frac{C(\alpha Y\upharpoonright n, \beta Z\upharpoonright n)}{n} = \limsup_n \frac{\alpha C(Y\upharpoonright n) + \beta C(Z\upharpoonright n)}{n}$$

Proof. For one direction note that

$$\begin{aligned} C(\alpha Y\restriction n, \beta Z\restriction n) &\leqslant C(\alpha Y\restriction n) + C(\beta Z\restriction n) + \mathcal{O}(\log C(\beta Z\restriction n)) \\ &\leqslant C(\alpha Y\restriction n) + C(\beta Z\restriction n) + \mathcal{O}(\log n). \end{aligned}$$

For the other:

$$\begin{aligned} &\limsup_n \frac{C(\alpha Y\restriction n, \beta Z\restriction n)}{n} \\ &\geqslant \limsup_n \frac{C(\alpha Y\restriction n) + C(\beta Z\restriction n \mid \alpha Y\restriction n) - \mathcal{O}(\log n)}{n} \\ &\qquad\qquad \text{using } \mathcal{O}(\sqrt{n}\log n) \text{ bits to change from } Y\restriction n \text{ to } \alpha Y\restriction n \\ &\geqslant \limsup_n \frac{\alpha C(Y\restriction n) + C(\beta Z\restriction n \mid Y\restriction n) - \mathcal{O}(\sqrt{n}\log n)}{n} \\ &\geqslant \limsup_n \frac{\alpha C(Y\restriction n) + C^Y(\beta Z\restriction n) - \mathcal{O}(\sqrt{n}\log n)}{n} \\ &\geqslant \limsup_n \frac{\alpha C(Y\restriction n) + \beta C(Z\restriction n)}{n} \qquad \text{Lemma 3.9.} \end{aligned}$$

□

Theorem 3.11. *For every closed ball $\bar{B}$ in the metric space $\langle 2^{\mathbb{N}}, d\rangle$, there is a sequence of reals in $\bar{B}$ with no Cauchy subsequence. Hence there are no compact neighbourhoods in $2^{\mathbb{N}}$ under the d-metric.*

Proof. Let $\bar{B} = \overline{B(X,\epsilon)}$ be a closed ball. First we define a sequence $\langle X_i\rangle$ so that $X_0 = X$ and X_{i+1} is chosen to be random relative to $\oplus_{k=0}^{i} X_k$. Now define $Y_i = \epsilon[X_i X]$ for each $i > 0$. The claim is that for all distinct positive i and j, $Y_i \in \bar{B}$ and $d(Y_i \to Y_j) \geqslant \epsilon$, and thus $\langle Y_i\rangle_{i=1}^{\infty}$ can have no Cauchy (and hence no convergent) subsequence. For the first part:

$$
\begin{aligned}
& d(X \to Y_i) \\
&= \limsup_n \frac{C(Y_i \restriction n \mid X \restriction n)}{n} \\
&= \limsup_n \frac{C(Y_i \restriction n, X \restriction n) - C(X \restriction n) \pm \mathcal{O}(\log n)}{n} && \text{Lemma 3.5} \\
&= \limsup_n \frac{C(\epsilon X_i \restriction n, (1-\epsilon) X \restriction n, X \restriction n) - C(X \restriction n) \pm \mathcal{O}(\sqrt{n} \log n)}{n} && \text{Lemma 3.4} \\
&= \limsup_n \frac{C(\epsilon X_i \restriction n, X \restriction n) - C(X \restriction n) \pm \mathcal{O}(\sqrt{n} \log n)}{n} && \text{Lemma 3.8} \\
&= \limsup_n \frac{C(\epsilon X_i \restriction n \mid X \restriction n) \pm \mathcal{O}(\sqrt{n} \log n)}{n} && \text{Lemma 3.5} \\
&\leqslant \epsilon \limsup_n \frac{C(X_i \restriction n)}{n} && \text{Lemma 3.3} \\
&= \epsilon \dim_p(X_i) \\
&= \epsilon
\end{aligned}
$$

$$
\begin{aligned}
&d(Y_i \to X) \\
&= \limsup_n \frac{C(X\restriction n \mid Y_i\restriction n)}{n} \\
&= \limsup_n \frac{C(X\restriction n, Y_i\restriction n) - C(Y_i\restriction n) \pm \mathcal{O}(\sqrt{n}\log n)}{n} \\
&= \limsup_n \frac{C(X\restriction n, \epsilon X_i\restriction n, (1-\epsilon)X\restriction n) - C(\epsilon X_i\restriction n, (1-\epsilon)X\restriction n) \pm \mathcal{O}(\sqrt{n}\log n)}{n} \\
&\qquad\qquad\text{using Lemma 3.4} \\
&= \limsup_n \frac{C(X\restriction n, \epsilon X_i\restriction n) - C(\epsilon X_i\restriction n, (1-\epsilon)X\restriction n) \pm \mathcal{O}(\sqrt{n}\log n)}{n} \\
&\qquad\qquad\text{using Lemma 3.8} \\
&= \limsup_n \frac{C(X\restriction n) + \epsilon C(X_i\restriction n) - \epsilon C(X_i\restriction n) - (1-\epsilon)C(X\restriction n) \pm \mathcal{O}(\sqrt{n}\log n)}{n} \\
&\qquad\qquad\text{Lemma 3.10} \\
&= \limsup_n \frac{\epsilon C(X\restriction n) \pm \mathcal{O}(\sqrt{n}\log n)}{n} \\
&= \epsilon \dim_p X \\
&\leqslant \epsilon
\end{aligned}
$$

Thus $d(Y_i, X) \leqslant \epsilon$ and $Y_i \in \bar{B}$ for all i.

For the second part, suppose $i < j$:

$$
\begin{aligned}
& d(Y_i \to Y_j) \\
= & \limsup_n \frac{C(Y_j \upharpoonright n \mid Y_i \upharpoonright n)}{n} \\
= & \limsup_n \frac{C(Y_j \upharpoonright n, Y_i \upharpoonright n) - C(Y_i \upharpoonright n) \pm \mathcal{O}(\log n)}{n} && \text{Lemma 3.5} \\
= & \limsup_n \frac{C(\epsilon X_j \upharpoonright n, \epsilon X_i \upharpoonright n, (1-\epsilon)X \upharpoonright n) - C(\epsilon X_i \upharpoonright n, (1-\epsilon)X \upharpoonright n) \pm \mathcal{O}(\sqrt{n}\log n)}{n} \\
& && \text{Lemmas 3.4, 3.8} \\
= & \limsup_n \frac{C(\epsilon X_j \upharpoonright n, \epsilon X_i \upharpoonright n \mid (1-\epsilon)X \upharpoonright n) + C((1-\epsilon)X \upharpoonright n) - C(\epsilon X_i \upharpoonright n, (1-\epsilon)X \upharpoonright n)}{n} \\
& && \text{Lemma 3.5 and henceforth suppressing the } \mathcal{O}(\sqrt{n}\log n) \text{ term} \\
= & \limsup_n \frac{C(\epsilon X_j \upharpoonright n, \epsilon X_i \upharpoonright n \mid (1-\epsilon)X \upharpoonright n) - C(\epsilon X_i \upharpoonright n \mid (1-\epsilon)X \upharpoonright n)}{n} \\
& && \text{Lemma 3.5} \\
= & \limsup_n \frac{C(\epsilon X_j \upharpoonright n \mid \epsilon X_i \upharpoonright n, (1-\epsilon)X \upharpoonright n)}{n} && \text{Lemma 3.6} \\
\geqslant & \limsup_n \frac{C^{\oplus_{k=0}^{j-1} X_k}(\epsilon X_j \upharpoonright n)}{n} \\
= & \epsilon \limsup_n \frac{C(X_j \upharpoonright n)}{n} && \text{Lemma 3.9} \\
= & \epsilon && \text{as } X_j \text{ is random}
\end{aligned}
$$

Thus $d(Y_i, Y_j) \geqslant \epsilon$ for all $i \neq j$ and the sequence $\langle Y_k \rangle$ has no Cauchy subsequence. □

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