1. A differentiation rule of the form

$$f'(x_0) = a_0 f_0 + a_1 f_1 + a_2 f_2, \quad x_k = x_0 + kh$$

is given. Find the values of a_0, a_1, and a_2 so that the rule is exact for $f \in P_3$. Find the error term.

2. Using the following data finf $f'(6.0)$, error $= O(h)$, and $f''(6.3)$, error $= O(h^2)$.

x	6.0	6.1	6.2	6.3	6.4
$f(x)$	0.1750	-0.1998	-0.2223	-0.2422	-0.2596

3. Define

$$S(h) = \frac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h}$$

(a) Show that

$$f'(x) - S(h) = c_1 h^2 + c_2 h^3 + c_3 h^4 + ...$$

and state c_1.

(b) Compute $f'(0.398)$ using the $S(h)$ and the table

x	0.398	0.399	0.400	0.401	0.402
$f(x)$	0.408591	0.409671	0.410752	0.411834	0.412915

and give an error estimate.

4. Given the fomula

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

apply the following table to estimate the error in the computed $f''(0.3)$

x	0.1	0.2	0.3	0.4	0.5
$f(x)$	17.60519	17.68164	17.75128	17.81342	17.86742

5. Use the central difference formla to compute $f'(0.3)$ from the data of the preceding exercise.

6. Use the central difference formla to estimate $f'(0.55)$ if $f(x) = e^x/(x - 2). Compare$the error of the actual error for

(a) $h = 0.1$.

(b) $h = 0.01$.

(c) $h = 0.001$.

7. To monitor the thermal pollution of a river, a biologist takes hourly tempearature T reading (in ^oF) form 9 AM to 5 PM. The results are shown in the following table.

Time of day	9	10	11	12	13	14	15	16	17
Tempearature	75.3	77.0	83.2	84.8	86.5	86.4	81.1	78.6	75.1

Use Simpson's rule to eatimate the avrage water temperature between 9 AM and 5 PM given by

$$T_{av} = \frac{1}{b-a}\int_a^b T(t)dt.$$

8. Let $h = (b-a)/3$, $x_0 = a$, $x_1 = a+h$, and $x_2 = b$. Find the error term for the qudrature formula

$$\int_a^b f(x)dx = \frac{9}{4}hf(x_1) + \frac{3}{4}hf(x_2).$$

9. We assume that the quadrature formula

$$\int_{-1}^{1} f(x)dx = c_0 f(0) + c_1 f(1) + c_2 f(2)$$

is exact for a polynomial of degree ≤ 2. Determine c_0, c_1, and c_2.

10. Find c_0, c_1, and x_1 so that the quadrature formula

$$\int_{-1}^{1} bf(x)dx = c_0 f(0) + c_1 f(x_1)$$

has a maximum degree of precision.

11. Find x_0, x_1, and c_1 so that the quadrature formula

$$\int_{-1}^{1} f(x)dx = \frac{1}{2}f(x_0) + c_1 f(x_1)$$

has a maximum degree of precision.

12. Determine the number of subintervals n required to approximate

$$\int_0^2 \exp(2x)\sin(3x)dx$$

with an error less than 10^{-4} using

(a) The composite trapezoidal rule.

(b) Simpson's composite rule.

13. Let

$$f(x) = \begin{cases} x^3 + 1 & \text{if} \quad 0 \le x \le 0.1 \\ 1.001 + 0.3(x-0.1) + 0.3(x-0.1)^2 + 2(x-0.1)^3 & \text{if} \quad 0.1 \le x \le 0.2 \\ 1.009 + 0.15(x-0.2) + 0.9(x-0.2)^2 + 2(x-0.3)^3 & \text{if} \quad 0.2 \le x \le 0.3 \end{cases}$$

   (a) Is f continuous on the interval $[0, 0.3]$.

   (b) Use the composite trapezoidal rule with $n = 6$ to approximate $\int_0^{0.3} f(x)dx$.

14. Find a_0, a_1, a_2, a_3, x_1, and x_2 so that the quadrature formula

$$\int_{-1}^{1} f(x)dx = a_0 f(-1) + a_1 f(x_1) + a_2 f(x_2) + a_3 f(1)$$

is exact for polynomials of highest degree possible.

15. Consider the quadrature formula

$$\int_0^{\pi} f(x)\sin(2x)dx = a_1 f(x_1) + a_2 f(x_2) + E(f)$$

where $E(f)$ is the truncation error.

   (a) Find a_1, a_2, x_1, x_2 so that the quadrature formula is exact when f is a polynomial of degree ≤ 3.

   (b) Show tha the formula is also exact for a polynomial of degree ≤ 4.

16. Compute

$$I_p = \int_0^1 \frac{x^p}{x^3 + 12} dx$$

for $p = 0, 1$ using Simpson composite rule with $n = 4, 6, 10$.

17. Compute

$$\int_0^{\pi/2} \frac{\cos x \log(\sin x)}{\sin^2 x + 1} dx$$

using Gaussian quadrature formula.

18. Compute

$$\int_{-\pi/2}^{\pi/2} \frac{3\cos x}{(2 + \sin x)^2} dx$$

using Simpson's composite rule with $n = 6$.

19. Compute

$$\int_{\pi/4}^{\pi/2} (\csc^2 x)\sqrt{\cot x}dx$$

using Romberg interation with $n = 6$.

20. Compute

$$\int_{0.1}^{0.8} \left(1 + \frac{\sin x}{x}\right) dx$$

using Simpson's composite rule with $n = 8$.

21. Calculate

$$\int_{0}^{1} \cos(2x)(1 - x^2)^{-1/2}dx$$

correct to 4 decimal laces using the composite trapezoidal rule.

22. Use gaussian quadrature formula with $n = 2, 3$ to calculate

$$\int_{-1}^{1} (1 - x^2)^{1/2}dx$$

23. Evaluate the following integrals

    (a) $I = \int_0^{100} e^{-5x}dx$
    (b) $I = \int_0^{100} e^{-x}\log(1 + x)dx$

    using the three point Gaussian quadrature fomula.

24. Compute

$$\int_{2}^{3} \frac{\cos 2x}{1 + \sin x}dx$$

using the three point Gaussian quadrature formula.

25. Use Romberg integration to evaluate

$$\int_{0}^{0.8} e^{-x^2}dx$$

with an absolute convegence criteria of $\epsilon = 10^{-6}$.

26. Use Romberg integration to evaluate

$$\int_{1}^{10} \sin(\ln x)dx$$

with $n = 6$.

27. Use Romberg integration to evaluate

$$\int_0^{1.127} \frac{12.128 + \ln(e^{\frac{2.4385}{x}} - 1)}{x^4(e^{\frac{2.4385}{x}} - 1)} dx$$

with $n = 6$.

Computer Assignements

1. The Fourier coefficients for $f(x) = 2x - x^2$ over the interval $(-2, 2)$ are given by

$$\begin{aligned} A_0 &= \frac{1}{2} \int_{-2}^{2} (2x - x^2) dx \\ A_n &= \frac{1}{2} \int_{-2}^{2} (2x - x^2) \cos\left(\frac{n\pi x}{2}\right) dx, \quad n = 1, 2, 3, ... \\ B_n &= \frac{1}{2} \int_{-2}^{2} (2x - x^2) \sin\left(\frac{n\pi x}{2}\right) dx \quad n = 1, 2, 3, ... \end{aligned}$$

Find A_0, A_1, A_2, B_1,and B_2 using the MATLAB function trapez.m with $n = 20$. Compare your results ith the exact values $A_0 = \frac{-8}{3}$, $A_n = \dfrac{16(-1)^{n+1}}{n^2\pi^2}$, and $B_n = \dfrac{8(-1)^{n+1}}{n\pi}$, $n = 1, 2, 3, ...$

2. Use the MATLAB function simpson.m to approximate the integral

$$I = \int_0^{2\pi} \exp\left(\frac{\sin x}{\sqrt{2}}\right) dx$$

with $n = 10$.

3. Use the MATLAB function gauss_quad.m with $n = 4$ to approximate the values of the Bessel function of order 0

$$J_0(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin\theta) d\theta$$

at $x = 1$ and $x = 2$.

4. Use the MATLAB function romberg.m with $n = 6$ to approximate the integral

$$I = \int_0^1 \frac{\sin x}{1 + x} dx$$