1. Consider the IVP

$$y' = (t + y - 1)^2, \quad y(0) = 2$$

   (a) Solve the IVP in terms of elementary functions. [Set $u = t + y - 1$].
   (b) Use Euler's method with $h = 0.1$ and $h = 0.05$ to obtain approximate values of the solution of the IVP at $t = 0.5$. Compare the approximate values with the exact values using part (a).

Use Euler's method to obtain approximations to the following IVPs to the indicated value:

2. $y' = 2t - 3y$, $y(1) = 5$, $y(1.5)$ with $h = 0.05$.
3. $y' = 4t - 2y$, $y(0) = 2$, $y(0.5)$ with $h = 0.05$.
4. $y' = 1 + y^2$, $y(0) = 0$, $y(1)$ with $h = 0.1$.
5. $y' = 2t - 3y$, $y(1) = 5$, $y(1.5)$ with $h = 0.05$.
6. $y' = t^2 + y^2$, $y(0) = 1$, $y(0.5)$ with $h = 0.05$.
7. $y' = e^{-y}$, $y(0) = 0$, $y(1)$ with $h = 0.1$.
8. $y' = t + y^2$, $y(0) = 0$, $y(0.5)$ with $h = 0.05$.
9. $y' = (t + y)^2$, $y(0) = 0.5$, $y(0.5)$ with $h = 0.05$.
10. $y' = ty + \sqrt{y}$, $y(0) = 1$, $y(1)$ with $h = 0.1$.
11. $y' = ty^2 - \frac{y}{x}$, $y(1) = 1$, $y(1.5)$ with $h = 0.05$.
12. $y' = y - y^2$, $y(0) = 0.5$, $y(1)$ with $h = 0.1$.
13. Repeat exercises 2-12 using the Runge-Kutta method of order 4.
14. Repeat exercises 2-12 using the Adams-Bashforth method of order 4.
15. Let $y(t)$ be tho solution of the IVP

$$y' = t^2 + y^2, \quad y(1) = 1$$

   (a) Use an ODE solver to obtain a graph of the solution on the interval $[1, 1.4]$.
   (b) Using the step size $h = 0.1$, compare the results obtained from Euler's method with the results from the modified Euler method (RK2) in the approximation of $y(1.4)$.

16. Consider the IVP $y' = 2y$, $y(0) = 1$. The analytic solution is $y(t) = e^{2t}$.

   (a) Approximate $y(0.1)$ using one step and Euler's method.
   (b) Find an error bound for the local truncation error in y_1.

(c) Compare the actual error in y_1 with your error bound.

(d) Approximate $y(0.1)$ using two steps and Euler's method.

(e) Verify that the global truncation error for Euler's method is $O(h)$ by comparing the errors in parts (a) and (d).

17. Solve the preceding exercise using the Runge-Kutta method of order 2. Its global truncation error is $O(h^2)$.

18. Repeat exercise 16 using the IVP $y' = -2y + t$, $y(0) = 1$. The analytic solution is $y(t) = \frac{1}{2}t - \frac{1}{4} + \frac{5}{4}e^{-2t}$.

19. If air resistance is proportional to the square of the instantaneous velocity, then the velocity v of the mass m dropped from a height h is determined from

$$m\frac{dv}{dt} = mg - kv^2, \quad k > 0$$

Let $v(0) = 0$, $k = 0.125$, $m = 5$ slugs, and $g = 32$ ft$/s^2$.

(a) Use the Runge-Kutta method of order 4 with $h = 1$ to find an approximation of the velocity of the falling mass at $t = 5$ s.

20. Consider the IVP $y' = -y + 10\sin 3t$ with $y(0) = 0$.

(a) Use the Runge-Kutta method of order 4 with $h = 0.1$ to approximate the solution in the interval $[0, 2]$.

(b) Using the result in (a) obtain an interpolating function and gragh it. Find the positive roots of the interpolating function on the interval $[0, 2]$.

21. Show that when Euler's method is used to approximate the solution of the IVP

$$y' = 5y, \quad y(0) = 1$$

at $t = 1$, then the approximation with step size h is $(1 + 5h)^{1/h}$.

22. Show that when the Runge-Kutta method of order 2 is used to approximate the solution of the IVP

$$y' = y, \quad y(0) = 1$$

at $t = 1$, then the approximation with step size h is $(1 + h + \frac{h^2}{2})^{1/h}$. The exact value of the solution of the IVP at $t = 1$ is e. Prove that the error:= $e - (1 + h + \frac{h^2}{2})^{1/h}$ approaches zero as $h \to 0$. Use the L'Hopital's rule to show that

$$\lim_{h\to 0} \frac{\text{error}}{h^2} = \frac{e}{6} \approx 0.45305.$$

Derive the difference equation corresponding to Taylor's method of order two for the following IVP:

23. $y' = ty - y^2, \quad y(0) = -1.$

24. $y' = t - y, \quad y(0) = 0.$

25. $y' = t^2 + y, \quad y(0) = 0.$

26. $y' = t + 1 - y, \quad y(0) = 1.$

27. Use Taylor's method of order 2 with $h = 0.25$ to approximate the solution to the IVP
$$y' = t + 1 - y, \quad y(0) = 1$$
at $t = 1$. Compare these approximations to the actual solution $y = t + e^{-t}$ evaluated at $t = 1$.

28. Use the Runge-Kutta method of order 4 to approximate the solution of the IVP
$$y'' = t^2 + y^2, \quad y(0) = 1, \quad y'(0) = 0$$
at $t = 1$.

29. Use the Runge-Kutta method of order 4 to approximate the solution of the IVP
$$\begin{aligned} x' &= 2x - y, \quad x(0) = 0, \\ y' &= 3x + 6y, \quad y(0) = -2. \end{aligned}$$
at $t = 1$. Compare this approximation to the actual solution $x(t) = e^{5t} - e^{3t}$ and $y(t) = e^{3t} - 3e^{5t}$.

Computer Assignments

1. Use the MATLAB function euler.m to find the approximate solution of the IVP
$$y' = ty^2 - \frac{y}{t}, \quad y(1) = 1$$
over the interval $[1, 2]$ with $h = 0.1$

2. Use the MATLAB function rk2_4.m to find the maximum value over the interval $[1, 2]$ of the solution of the IVP
$$y' = \frac{1.8}{t^4} - y^2, \quad y(1) = -1.$$

3. The solution of the IVP
$$y' = \frac{2}{t^4} - y^2, \quad y(1) = -0.414$$
crosses the t-axis at a point in the interval $[1, 2]$. By experimenting with the MATLAB function rk2_4.m determine this point.

4. The solution to the IVP

$$y' = y^2 - 2e^t y + e^{2t} + e^t, \quad y(0) = 3$$

has a vertical asymptote at some point in the interval $[0, 2]$. By experimenting with the MATLAB function rk2_4.m determine this point.

5. Use the MATLAB function abash.m to find the approximate solution of the IVP

$$y' = ty^2 - \frac{y}{t}, \quad y(1) = 1$$

over the interval $[1, 2]$ with $h = 0.1$

6. Solve the preceding exercise using the MATLAB function amoulton.m

7. In the study of the nonisothermal flow of a Newtonian fluid between parallel plates, the equation

$$y'' + t^2 e^y = 0, \quad t > 0,$$

was encountered. By a series of substitutions, this equation can be transformed into the first order equation

$$\frac{dv}{du} = u\left(\frac{u}{2} + 1\right)v^3 + \left(u + \frac{5}{2}\right)v^2.$$

Use the MATLAB function rk2_4.m to approximate $v(3)$ if $v(u)$ satisfies $v(2) = 0.1$.